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APPROXIMATION AND OPTIMIZATION OF DISCRETE AND DIFFERENTIAL INCLUSIONS

ELIMHAN MAHMUDOV

Approximation and Optimization of Discrete and Differential Inclusions

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Dedication

This book is dedicated to the memory of my doctoral thesis advisor B.N. Pshenichnyi, Academician of the Academy of Sciences of Ukraine, from whom I learned the theory of extremal problems, numerical methods in optimal systems theory, optimal control, differential game theory, and much more.

This book is also dedicated to memory of my lovely son Elshad (1977−2000), who was a selfless manager for the American petrol company Exxon.

Besides this book is dedicated to the memory of Khojali massacre's martyrs (25-26.02.1992).

Preface

Give me a place to stand on, and I will move the Earth.

Archimedes

Mathematics is the language with which God has written the universe.

Galileo Galilei

As a well spent day brings happy sleep, so life well used brings happy death

Leonardo da Vinci

Scientists investigate that which already is;engineers create that which has never been

Albert Einstein

The primary goals of this book are to present the basic concepts and principles of mathematical programming in terms of set-valued analysis (Chapters 2 and 3) and on the basis of the method of approximation, to develop a comprehensive optimality theory of problems described by ordinary and partial differential inclusions (DFI) (Chapters 4−6). This book consists of six chapters divided into sections and subsections, and contains many results that have not been published in the monographic literature.

In Chapter 1, convex sets and convex functions are studied in the setting of n-dimensional Euclidean space. However, the reader familiar with functional analysis can generalize the main results to the case of infinite-dimensional functional spaces. In spite of the fact that the stated notions and results are known, they play a decisive role for obtaining the main results in the next chapters of the book. The key issues of convex analysis in finite-dimensional spaces have been addressed in the books *Convex Analysis* by Rockafellar and *Convex Analysis and Extremum Problems* by B.N. Pshenichnyi. The identifications of convex functions and their epigraphs make it easy to pass back and forth between the geometric and analytical approaches. It is shown that convex sets and functions form classes of objects preserved under numerous operations of combination; pointwise addition, pointwise supremum, and infimal convolution of convex functions are convex.

In Chapter 2, the apparatus of locally adjoint mappings (LAM) (which is new) is studied in the light of convex analysis. It is the fundamental concept in what follows, and it is used to obtain the optimality conditions for the problems posed in this book. We give the calculus of LAM on different multivalued mappings, such as the sum, composition, and inverse. We introduce the adjoint (not locally adjoint) mapping, using the recession cone, and the connection between the adjoint and LAM is established. Based on the adjoint mapping, the duality theorems for convex set-valued mappings are proved.

Chapter 3 is devoted to applications of these basic tools to the study of mathematical programming with possibly nonsmooth data. Starting with problems of mathematical programming under functional and geometric constraints, we then consider various problems of constrained optimization, minimax problems and equilibrium constraints, infimal convolution of convex functions, duality in convex programming, and duality relations. In order to formulate a necessary condition for the existence of an extremum, of course, some special condition by function taking part in the given problem is required. In particular, in some neighborhood of a point minimizing our objective function, we deal with the comparatively easily computable functions. As is known, a smooth function admits a linear approximation. On the other hand, a convex function can be approached by positively homogeneous functions. However, a nonsmooth and nonconvex function cannot be approximated in a neighborhood of a point by positively homogeneous functions. Precisely this class of functions is required for introducing the concept of convex upper approximations (CUAs). The key tools of our analysis are based on the extremal principle and its modifications together with the LAM calculus.

Chapters 4 and 5 deal mostly with optimal control problems of the Bolza type described by ordinary differential, high-order differential, delay-differential, and neutral functional-differential inclusions. The development and applications of the LAM are demonstrated in these problems with ordinary discrete and differential inclusions. In particular, for polyhedral DFI, under the corresponding condition for generality of position, the theorem of the number of switchings is proved. The corresponding results are obtained for linear optimal control problems in linear manifolds. For a nonautonomous polyhedral DFI, a special condition for generality of position is formulated. Moreover, for problems described by ordinary nonconvex DFI under the specially formulated monotonicity and t_1-transversality conditions, sufficient conditions for optimality are proved.

In Chapter 6, we continue the study of optimal control problems governed by discrete and differential inclusions with distributed parameters, which during the past 15−20 years has been a basic source of inspiration for analysis and applications. Using LAM and the discrete-approximation method in Hamiltonian and Euler−Lagrange forms, we derive necessary and sufficient optimality conditions for various boundary values (Dirichlet, Neumann, Cauchy) problems for first-order, elliptic, parabolic, and hyperbolic types of discrete and partial DFI. One of the most characteristic features of such approaches with partial DFI is peculiar to the presence of equivalents to the LAM. Such problems have essential specific features in comparison with the ordinary differential model considered in the second part of the book. For every concrete problem with partial DFI, we establish rather interesting equivalence results that shed new light on both qualitative and quantitative relationships between continuous and discrete approximation problems.

In Chapter 5 and the second part of Chapter 6, we construct the dual problem of convex problems for ordinary and partial differential inclusions of hyperbolic, parabolic, and elliptic types. We study separately the duality problems with first-order partial differential inclusions. As is known, duality problems have always been at the center of convex optimality theory and its applications. In this book, we formulate duality results and search for the conditions under which primary and dual problems are connected by such duality relations. For duality constructions of convex problems, we use the duality theorems concerning infimal convolution and the addition of convex functions.

Thus, we can list the major features of our book that make it unique:

- The introduction of a new concept of LAM and its calculus.
- The connection between LAM and adjoint (not locally) mappings defined in terms of the recession cone.
- Duality theorems for convex multivalued mappings established in terms of a Hamiltonian function.
- The basic results of mathematical programming in terms of Hamiltonian functions.
- Under a suitable condition for generality of position, the theorem of the finiteness of switching numbers for optimal control of polyhedral differential inclusions.
- Under a special t_1-transversality condition, new sufficient conditions for optimality in terms of extended Euler–Lagrange inclusions for Bolza-type problems with ordinary differential inclusions and state constraints.
- A new class of optimal control problems for higher-order differential inclusions.
- Duality relations in mathematical problems with equilibrium constraints via recession cones. Major features using the method of discrete approximation.
- Optimization of first-order discrete and partial differential inclusions. Note that one of the characteristic features of optimization of Cauchy for first-order discrete inclusions is the intrinsic presence of the infinite dimensional self-adjoint Hilbert space l_2.
- Optimization of Darboux-type partial differential inclusions and duality.
- Optimization of elliptic, hyperbolic, and parabolic types of discrete and partial differential inclusions and duality.
- Optimization of partial differential inclusions with a second-order elliptic operator.
- Equivalence results that facilitate making a bridge between discrete and corresponding discrete-approximation problems.

Throughout this book, a proof is marked with an empty square □ and its end is marked with a Halmos box, ■. Since many problems in engineering reduce to such problems, the book will be of interest to mathematicians and nonmathematician specialists whose study involves the use of ordinary and partial differential equations (inclusions) and approximation methods and its applications, as well as to undergraduate, graduate, and postgraduate students at universities and technical colleges. In other words, the book is intended for a broad audience—students of universities and colleges with comprehensive mathematical programs, engineers, economists, and mathematicians involved in the solution of extremal problems.

Basic material has also been incorporated into many lectures given by the author at various international conferences in London, UK; Zurich, Switzerland; and Leipzig, Germany, and the Banach international mathematical center in Warsaw, Poland, during recent years.

This book includes an index of symbols and notation, and a subject index. Using the subject index, the reader can easily find the page where some notion or notation is introduced. Our notation and terminology are generally consistent with those used by Rockafellar, Mordukhovich, and Pshenichnyi in their writings. For the reader's convenience, an introduction in each chapter of the book describes the contents and commentaries, and outlines a selection of material that would be appropriate for the subject. This book is also accompanied by an abundant bibliography. Parts of this book have been used by me in teaching graduate and postgraduate students on Convex Analysis, Optimal Control Theory, and Nonlinear Analysis and Its Applications at Azerbaijan State University and Istanbul Technical University.

Prof. Elimhan N. Mahmudov
Baku and Istanbul
August 2011

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About the Author

Elimhan N. Mahmudov was born in Kosali, Karayazi, and after graduating the secondary school with a gold medal for being the most successful student, he attended Azerbaijan State University as part of the Mechanical–Mathematical Department. From 1973 to 1977, he was a PhD student at the Cybernetics Institute of Academy of Sciences of Ukraine advised by Prof. B.N. Pshenichnyi—Academician, one of the most prominent authorities in the fields of theory of extremal problems, numerical methods, and game theory. E.N. Mahmudov defended his thesis in Ukraine in 1980. He then worked as a Ph.D. and senior scientific associate at the Cybernetics Institute of Academy of Sciences of Azerbaijan until 1995. He also taught part time at the Azerbaijan State University and was a permanent member of the Physical–Mathematical Doctoral Sciences Committee in Azerbaijan. On the recommendation of the Physical–Mathematical Doctoral Committee of the Taras Shevchenko's Kiev State University, in 1992 he became a Doctor of Physical–Mathematical Sciences, and he has earned both a Ph.D. and a Doctorate of Sciences from Moscow. In 1992, he received a Grant Support for Mathematics by National Science Foundation in Washington, DC, and became a member of the American Mathematical Society.

Prof. Mahmudov has devoted numerous research papers to Convex Analysis, Approximation Theory, Optimal Control Theory, Dual Problems, and Mathematical Economy, which have been published in high-level Science Citation Index (SCI) journals in the former Soviet Union, the United States, and Europe. His textbooks include *Mathematical Analysis and Its Applications*, published in 2002 by Papatya, Istanbul, and *Single Variable Differential and Integral Calculus* (in press). He has lectured on the problems of Optimal Control at International Conferences in England, France, Switzerland, Germany, Poland, Russia, and Turkey, and at the World Mathematical Banach Center in Warsaw, Poland. He is an Editor of the journal *Advances in Pure Mathematics* (APM).

Prof. Mahmudov currently teaches Nonlinear Programming, Game Theory, and applications in Economics, Nonlinear Optimization, and Numerical Analysis at Istanbul Technical University. He enjoys playing chess, playing Tar and Saz, Azerbaijan folk instruments, and traveling and is an avid painter.

1 Convex Sets and Functions

1.1 Introduction

Convexity is an attractive subject to study, for many reasons; it draws upon geometry, analysis, linear algebra, and topology, and it has a role to play in such topics as classical optimal control theory, game theory, linear programming, and convex programming. Convex sets and convex functions are studied in this chapter in the setting of n-dimensional Euclidean space $\mathbb{R}^n$. (However, if you are familiar with functional analysis, you will be able to generalize the main results to the case of infinite dimensional functional spaces.) These results play a decisive role in obtaining the main results in the next chapters of this book. In the field of convex analysis, you can consult Rockafellar [228] and Pshenichnyi [226] for related and additional material (most of the familiar results on convex analysis presented in this chapter are taken from Pshenichnyi [226]). The basic idea in convexity is that a convex function on $\mathbb{R}^n$ can be identified with a convex subset of $\mathbb{R}^{n+1}$, which is called the epigraph of the given function. This identification makes it easy to move back and forth between geometrical and analytical approaches. It is shown that pointwise addition of functions, pointwise supremum, and infimal convolution of convex functions are convex, in fact, convex sets and functions are classes of objects that are preserved under numerous operations of combination. A function is closed if its epigraph is closed. The latter is equivalent to lower semicontinuity of functions. This leads to the notion of the closure operation for proper convex functions, which corresponds to the closure operation for epigraphs.

In Section 1.2, we study some topological properties of sets and their convex hull, and consider how a convex set can be characterized by both Minkowski's method and support functions. The role of dimensionality in the generation of convex hulls is explored in Carathéodory's theorem (Theorem 1.1). It is interesting that Theorem 1.2 (Gauss–Lucas) says that the roots of the derivative of a nonconstant complex polynomial belong to the convex hull of the set of roots of the polynomial itself. The foundations for extremal theory are laid in the Separation Theorems 1.5–1.7.

In Section 1.3, we discuss the convex cone, which is one of the important concepts in convex analysis and extremal theory. The investigation of its properties is connected with the calculation of the dual cone.

The cones $K_1, \ldots, K_m$ are called separable if there exist not all zero vectors $x_i^* \in K_i^*$, such that $x_1^* + \cdots + x_m^* = 0$. By Theorem 1.12, if $K = K_1 \cap \cdots \cap K_m$, then either $K^* = K_1^* + \cdots + K_m^*$ or the cones $K_1, \ldots, K_m$ are separable. By Lemma 1.17,

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$(K_1 \cap \cdots \cap K_m)^* = \overline{K_1^* + \cdots + K_m^*}$. But since for polyhedral cones, $K_1^* + \cdots + K_m^*$ is also a polyhedral cone, this sum of cones is closed and the bar above it can be removed. One of the remarkable properties of a polyhedral set is that it can be represented as a sum of polytope (polyhedron) and polyhedral cone. Conversely, the sum of any polytope and polyhedral cone is a polyhedral set (Theorem 1.14). The recession cone of a nonempty convex set M, i.e., the set of vectors $\bar{x}$ such that $M + \bar{x} \subset M$ is denoted by $0^+ M$ and for a bounded set $0^+ M = \{0\}$.

In Section 1.4, we develop the main properties of convex functions. Recall that by Definition 1.20 a function f is said to be proper, if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x. A function that is not proper is improper. It follows from Lemma 1.24 that dom f is convex, even if f is an improper function. Besides, an improper convex function may have a finite values only at points of the relative boundary of dom f. The sum of proper convex functions f_i, $i = 1, \ldots, m$ with nonnegative coefficients is convex (Lemma 1.27).

It is known that the indicator function $\delta_M(\cdot)$ of M is useful as a correspondence between convex sets and convex functions. Then note that the sum of proper convex functions f_i, $i = 1, \ldots, m$ with nonnegative coefficients may not be a proper function (Lemma 1.27). For example, for the disjoint sets M_1 and M_2, the sum of indicator functions $\delta_{M_1} + \delta_{M_2}$ is identically $+\infty$.

We shall denote the gradient of f at x by $f'(x)$ and the Hessian matrix of f at x by $f''(x)$, whose (i,j)th element is $\partial^2 f/\partial x^i \partial x^j$. Then if f is twice differentiable, the convexity of f and the positive semidefiniteness of $f''(x)$ are equivalent. Of course, the latter is an important result not only in analysis but also in nonlinear programming.

Lemma 1.29 implies that properness of convex functions is not always preserved by infimal convolution $f_1 \oplus f_2$, which is commutative, associative, and convexity-preserving. Indeed, if f_1 and f_2 are linear functions not equal to each other, then their infimal convolution identically is $-\infty$.

The convex hull conv g of a nonconvex function g, defined as the greatest convex function majorized by g, is used in establishing the dual problem governed by polyhedral maps in Section 5.2. In Theorems 1.16 and 1.17, the continuity and Lipschitz properties of convex functions are shown. By Theorem 1.18, f, a proper convex function, is necessarily continuous on ri dom f. As is seen from this theorem, a convex function is continuous in dom f and may have a point of discontinuity only in its boundary. In order to characterize the case in which there is no such discontinuity, it is convenient to introduce the closure function concept (a function f is said to be a closure if its epigraph epi f is a closed set in $\mathbb{R}^{n+1}$). By Definition 1.26, the recession function is denoted by $f0^+$ and defined as epi $(f0^+) = 0^+$ (epi f). Obviously, if f is a proper convex function, then the recession function $f0^+$ of f is a positively homogeneous proper convex function.

Section 1.5 is devoted to the conjugate of a convex function, which is one of the basic concepts both of convex analysis and of duality theory. The definition of the conjugate of a function grows naturally out of the fact that the epigraph of a closed proper convex function on $\mathbb{R}^n$ is the intersection of the closed half-spaces in $\mathbb{R}^{n+1}$ that contain it. The function defined as $f^*(x^*) = \sup_x\{\langle x, x^* \rangle - f(x)\}$ is called the conjugate of f. It is closed and convex. It is useful to remember, in particular, that

Young's inequality $f(x)+f^*(x^*)\geq\langle x,x^*\rangle$ holds for any function. If here f is a proper convex function, then we shall refer to this relation as Fenchel's inequality. Taking conjugates clearly reverses functional inequalities: $f_1\geq f_2$ implies $f_1^*\leq f_2^*$.

The polar of a set M is denoted by M^{O} and defined as $M^{\mathrm{O}}=\{x^*: H_M(x^*)\leq 1\}$, where $H_M(x^*)$ is a support function of M. Note that if K is a convex cone, then $K^{\mathrm{O}}=\{x^*:\langle x,x^*\rangle\leq 0,\ \forall x\in K\}$ and so $K^*=-K^{\mathrm{O}}$. Then $r_M^*(x^*)=\delta_{M^{\mathrm{O}}}(x^*)$, where $r_M(\cdot)$ is Minkowski's function. If f is a closed proper convex function, then $f=f^{**}$ (Theorem 1.21). Thus, the conjugacy operation $f\rightarrow f^*$ induces a symmetric one-to-one correspondence in the class of all closed proper convex functions on $\mathbb{R}^n$. By Theorem 1.23, if f is a closed proper convex positively homogeneous function, then $f^*(x^*)=\delta_{\mathrm{dom}\ f^*}(x^*)$ and dom f^* is a closed set. The conjugate function of the support function of a closed convex set is the indicator function of this set (Theorem 1.25).

In Section 1.6, we analyze directional derivatives and subdifferentials of convex functions. The theory of subdifferentiation is a fundamental tool in the analysis of extremum problems. It is known that, in general, convex functions are not differentiable. Nevertheless, these functions have many useful differential properties, and one of them is the fact that directional derivatives exist universally. Moreover, for convex function can be defined as the notion of subgradient and the set of subgradients is the subdifferential conception. If f is a proper convex function and $x_0\in\mathrm{dom}\ f$, then the value of the directional derivative $f'(x_0,p)$, finitely or not, always exists for all p (Lemma 1.31). Obviously, the subdifferential $\partial f(x_0)$ is a closed convex set. The directional derivative $f'(x_0,p)$ is a positively homogeneous convex function of p. Theorem 1.27 asserts that $x^*\in\partial f(x_0)$ if and only if $\langle x_0,x^*\rangle - f(x_0)=f^*(x^*)$. This simple fact will be used in the next investigations. Also, if f is a closed proper convex function, then $x^*\in\partial f(x_0)$ if and only if $x_0\in\partial f^*(x^*)$ (Corollary 1.2). In Section 1.4, we will see that multiplication by positive constants of convex functions, their addition, and the pointwise supremum of convex functions are again convex. So it is important to calculate the subdifferentials of such functions. We have that for $f(x)=\alpha f_0(x)$, $\alpha>0$, $\partial f(x_0)=\alpha\partial f_0(x_0)$. Let f_1, f_2 be proper convex functions and $f=f_1+f_2$, $x_0\in\mathrm{dom}\ f_1\cap\mathrm{dom}\ f_2$ and suppose either (1) there is a point $x_1\in\mathrm{dom}\ f_1\cap\mathrm{dom}\ f_2$, where f_1 is continuous, or (2) ri dom $f_1\cap$ri dom $f_2\neq\varnothing$. Then $\partial f(x_0)=\partial f_1(x_0)+\partial f_2(x_0)$ (Moreau–Rockafellar). In Theorem 1.31, we calculate the subdifferential of functions of the form $f(x)=\sup_\alpha\{f(x,\alpha): \alpha\in A\}$.

If a set M is given by the inequality of the form $M=\{x: f(x)\leq 0\}$, where f is a convex function, then under the existence of an interior point it is proved that the dual cone is expressed by cone $\partial f(x_0)$, i.e., $[\mathrm{cone}(M-x_0)]^*=-\mathrm{cone}\ \partial f(x_0)$ (Theorem 1.33).

1.2 Some Basic Properties of Convex Sets

We consider the vector x, belonging to n-dimensional real Euclidean space $X=\mathbb{R}^n$. We also consider its dual space $X^*=\mathbb{R}^n$. In this book, the inner product of two vectors $x \in X$ and $x^* \in X^*$ is defined by $\langle x, x^* \rangle = \sum_{i=1}^{n} x^i x^{i^*}$, where x^i and x^{i^*} are the components of x and x^*, respectively. Let us explain the difference between the two spaces X and X^*. The point is that each element of X^* plays the role of a linear function defined on X. It is well known that any linear function $l(x)$ defined on X can be given by some x^* so that $l(x) = \langle x, x^* \rangle$. Thus, in order to emphasize the fact that x^* defines a linear function, we introduced such notation.

Definition 1.1. A subset M is said to be convex if

$$\lambda_1 x_1 + \lambda_2 x_2 \in M, \quad \lambda_1, \lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 = 1.$$

Whenever $x_1, x_2 \in M$, it follows that $\varnothing$ and $\mathbb{R}^n$ are convex.

Let us give some important properties of convex sets.

Lemma 1.1. Let $\{M_i : i \in I\}$ be a collection of nonempty convex sets in $\mathbb{R}^n$, where I is an arbitrary index set. Then the intersection

$$M = \bigcap_{i \in I} M_i$$

is also convex.

□ By definition, if $x_1, x_2 \in M$, then $x_1, x_2 \in M_i$, $i \in I$, and so $\lambda_1 x_1 + \lambda_2 x_2 \in M_i$ for all $i \in I$ and hence $\lambda_1 x_1 + \lambda_2 x_2 \in M$ ■

Lemma 1.2. If M_1 and M_2 are convex sets in $\mathbb{R}^n$, then so is the sum $c_1 M_1 + c_2 M_2$, where c_1, c_2 are real numbers.

□ Recall that by definition, $c_1 M_1 + c_2 M_2$ is the set of points of the form $c_1 x_1 + c_2 x_2$, where $x_1 \in M_1$, $x_2 \in M_2$. Now, the proof of the lemma follows from the definition of a convex set. ■

Lemma 1.3. If M is a convex set and $c_1 \geq 0$, $c_2 \geq 0$, then

$$(c_1 + c_2)M = c_1 M + c_2 M.$$

□ The inclusion $\subset$ would be true whether M is convex or not. The reverse inclusion $\supset$ follows from the convexity relation

$$M \supset \frac{c_1}{c_1 + c_2} M + \frac{c_2}{c_1 + c_2} M$$

on multiplying through by $c_1 + c_2$, provided $c_1 + c_2 > 0$. If $c_1 = c_2 = 0$, the assertion of the theorem is trivial. ■

Definition 1.2. A point x is called a convex combination of points $x_1, \ldots, x_m$ if there are nonnegative numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\begin{gathered} x = \lambda_1 x_1 + \cdots + \lambda_m x_m \\ \lambda_i \geq 0, \quad i = 1, \ldots, m, \quad \lambda_1 + \cdots + \lambda_m = 1. \end{gathered} \tag{1.1}$$

Lemma 1.4. If M is a convex set and $x_i \in M$, $i = 1, \ldots, m$, then the set M contains all the convex combinations of elements $x_1, \ldots, x_m$.
□ It is convenient to use mathematical induction; for $m = 2$, the assertion follows from the definition. Make the induction hypothesis that M contains all the convex combinations of elements $x_1, \ldots, x_m$, where $m \leq k$. We show that

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \lambda_{k+1} x_{k+1} \in M$$

for all λ_i, such that $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_{k+1} \geq 1$. For convenience, suppose $\lambda_i > 0$. Thus, $1 - \lambda_{k+1} = \lambda_1 + \cdots + \lambda_k > 0$. Since

$$\sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_{k+1}} = \frac{1}{1 - \lambda_{k+1}} \sum_{i=1}^{k} \lambda_i = 1,$$

by the induction hypothesis

$$y = \frac{\lambda_1}{1 - \lambda_{k+1}} x_1 + \cdots + \frac{\lambda_k}{1 - \lambda_{k+1}} x_k \in M.$$

Therefore, by the definition of convex set

$$x = (1 - \lambda_{k+1}) y + \lambda_{k+1} x_{k+1} \in M. \blacksquare$$

Note that in many situations in which convex combinations occur in applied mathematics, $\lambda_1, \ldots, \lambda_m$ can be interpreted as proportions or probabilities. For example, if m particles with masses $\alpha_1, \ldots, \alpha_m$ are located at points $x_1, \ldots, x_m$ of $\mathbb{R}^3$, the center of gravity of the system is the point $x = \lambda_1 x_1 + \cdots + \lambda_m x_m$, where $\lambda_i = \alpha_i / (\alpha_1 + \cdots + \alpha_m)$. In this convex combination, λ_i is the proportion of the weight, which is at x_i.

Definition 1.3. The intersection of all convex sets containing M is called the convex hull of a set M and denoted by conv M.

We show that conv M is the smallest convex set containing M, characterizing it as the set of all convex combinations of points of M. Using this characterization, we prove two combinatorial results, one due to Carathéodory, the other to Gauss–Lucas on the location of the roots of the derivative of complex polynomial.

According to Lemma 1.1, conv M is a convex set. Since $M \subseteq \text{conv } M$, by Lemma 1.4 it contains all the convex combinations of elements $x_1, \ldots, x_m$:

$$\begin{aligned} x &= \lambda_1 x_1 + \cdots + \lambda_m x_m, \quad x_i \in M, \quad i = 1, \ldots, m \\ \lambda_i &\geq 0, \quad i = 1, \ldots, m, \quad \lambda_1 + \cdots + \lambda_m = 1. \end{aligned}$$

Moreover, it is not hard to see that the set of such points itself is convex. Thus, on the one hand, conv M contains all the convex combinations (Eq. 1.1) of elements $x_1, \ldots, x_m$; on the other hand, by Definition 1.3, conv $M \subseteq M$.

Thus, we have obtained the following result.

Lemma 1.5. Let M be a set in $X = \mathbb{R}^n$. Then conv M is the set of all convex combinations of points of M.

Carathéodory's theorem, which is proved below, states that each point of the convex hull conv M can be expressed as a convex combination of $n+1$ or fewer points of the set. Thus, a point in the convex hull of a set in $\mathbb{R}^3$ is either a point of the set or belongs to a line segment, a triangle, or a tetrahedron with vertices in the set.

Theorem 1.1. (Carathéodory) Let M be any set in $X = \mathbb{R}^n$ and x be any point in conv M. Then x can be expressed as a convex combination of $n+1$ or fewer points of M; i.e., for every $x \in \operatorname{conv} M$, there are points $x_1, \ldots, x_r \in M$, such that

$$\begin{gathered} x = \lambda_1 x_1 + \cdots + \lambda_r x_r, \\ \lambda_1 + \cdots + \lambda_r = 1; \quad \lambda_1 \geq 0, \ldots, \lambda_r \geq 0, \end{gathered} \tag{1.2}$$

where $r \leq n+1$.

□ In fact, we must show that $r \leq n+1$. Take a point of the form of Eq. (1.1) and show that the number of nonzero terms in Eq. (1.1) can be reduced if $m > n+1$. It is sufficient to assume that $\lambda_i > 0$. Consider the $(n+1)$-dimensional vectors $(x_i, 1)$, $i = 1, \ldots, m$. Since by assumption, the number of such vectors is $m > n+1$, there is a linear dependence. Hence, there exist scalars α_i, $i = 1, \ldots, m$, not all zero such that

$$\sum_{i=1}^{m} \alpha_i x_i = 0 \tag{1.3}$$

and

$$\sum_{i=1}^{m} \alpha_i = 0. \tag{1.4}$$

Of course, by Eq. (1.4) there are necessarily positive α_i. Let us denote

$$\varepsilon_0 = \min\left\{ \frac{\lambda_i}{\alpha_i} : \alpha_i > 0, \quad i = 1, \ldots, m \right\}.$$

Suppose the minimum is attained at $i = i_0$. Then it is clear that for any α_i, $i = 1, \ldots, m$

$$\overline{\lambda_i} = \lambda_i - \varepsilon_0 \alpha_i \geq 0.$$

Obviously by (1.3) and (1.4),

$$\begin{gathered} \sum_{i=1}^{m} \overline{\lambda_i} = \sum_{i=1}^{m} \lambda_i x_i - \varepsilon_0 \left(\sum_{i=1}^{m} \alpha_i x_i \right) = x \\ \sum_{i=1}^{m} \overline{\lambda_i} = \sum_{i=1}^{m} \lambda_i - \varepsilon_0 \left(\sum_{i=1}^{m} \alpha_i \right) = 1. \end{gathered}$$

By the choice of ε_0, the new coefficients $\overline{\lambda_i}$ are nonnegative, and at least one of them is 0. We therefore have an expression of x as a nonnegative linear combination of fewer than m elements. This reduction is possible until $m > n + 1$. ■

Theorem 1.2. (Gauss−Lucas) The roots of the derivative of a nonconstant complex polynomial belong to the convex hull of the set of roots of the polynomial itself.
□ Let P be the complex polynomial defined by the equation

$$P(z) = a_0 z^n + \cdots + a_{n-1} z + a_n,$$

where $n \geq 1$ and $a_0, \ldots, a_n$ are complex numbers with $a_0 \neq 0$. Then

$$P(z) = a_0(z - z_0)\cdots(z - z_n),$$

where $z_1, \ldots, z_n$ are the roots of P, each being repeated according to its multiplicity. A routine verification shows that for $z \neq z_1, \ldots, z_n$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \cdots \frac{1}{z - z_n} = \frac{\overline{z} - \overline{z_1}}{|z - z_1|^2} + \cdots + \frac{\overline{z} - \overline{z_n}}{|z - z_n|^2}, \tag{1.5}$$

where $\overline{z}$ is the conjugate of z. Now, let z be a root of P'. We establish the theorem by showing z as a convex combination of $z_1, \ldots, z_n$. This can be done trivially if z is one of $z_1, \ldots, z_n$, so we assume that this is not the case. Setting $P'(z) = 0$ in Eq. (1.5), we find that

$$z = \frac{\dfrac{1}{|z - z_1|^2} z_1 + \cdots + \dfrac{1}{|z - z_n|^2} z_n}{\dfrac{1}{|z - z_1|^2} + \cdots + \dfrac{1}{|z - z_n|^2}},$$

which expresses z as a convex combination of $z_1, \ldots, z_n$. Here we have used the fact that if $P'(z) = 0$, then $P'(\overline{z}) = 0$. ■

Note that the open unit ball B in $\mathbb{R}^n$ centered at the origin and radius 1 is the set of all points of $\mathbb{R}^n$ whose distance from the origin is less than 1; i.e.,

$$B = \{x : \ \|x\| < 1\}, \quad \|x\| = \langle x, x \rangle^{1/2}.$$

Definition 1.4. A point x of a set M in $\mathbb{R}^n$ is said to be an interior point of M, if there exists a number $\varepsilon > 0$ such that $x + \varepsilon B \subseteq M$.

The set of interior points of M is called the interior of M and denoted by int M. Clearly, int $M_1 \subseteq$ int M_2 when $M_1 \subseteq M_2$. A set in $\mathbb{R}^n$, each of whose points is an interior point of the set, is said to be open. Since int $M \subseteq M$ is always true, M is open if and only if int $M = M$. Clearly, the sets $\varnothing$ and $\mathbb{R}^n$ are open.

Definition 1.5. A point x is said to be a limit point of M if there exists a sequence $x_k \in M$ that converges to x. The set of all limit points of M is called the closure of M and is denoted by $\overline{M}$.

Lemma 1.6. The closure and interior of a convex set M are convex sets.
$\square$ If M is convex, then $x_1 \in \text{int } M$ and $x_2 \in \text{int } M$ imply that $x_1 + \varepsilon_1 B \subseteq M$, $x_2 + \varepsilon_2 B \subseteq M$. Let $\lambda_1 x_1 + \lambda_2 x_2$ be the convex combination of points x_1 and x_2. Thus,

$$\lambda_1(x_1 + \varepsilon_1 B) + \lambda_2(x_2 + \varepsilon_2 B) = \lambda_1 x_1 + \lambda_2 x_2 + (\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2) B \subseteq M;$$

i.e., $\lambda_1 x_1 + \lambda_2 x_2$ is a point of int M.

Now, if $x_1, x_2 \in \overline{M}$, then by definition of a closure set there exist sequences x_{1k}, $x_{2k} \in M$, such that $x_{1k} \to x_1$, $x_{2k} \to x_2$. Let $\lambda_1 x_{1k} + \lambda_2 x_{2k}$ be the convex combination of points x_{1k}, x_{2k}. Therefore, since $\lambda_1 x_{1k} + \lambda_2 x_{2k} \in M$, we have $\lambda_1 x_1 + \lambda_2 x_2 = \lim_{k \to \infty} (\lambda_1 x_{1k} + \lambda_1 x_{2k}) \in \overline{M}$. $\blacksquare$

Definition 1.6. Let M be a convex set in $\mathbb{R}^n$ and x_0 be any point of M. Then the intersection of all possible underlying (carrier) subspaces containing $M - x_0$ is called the linear hull for a set M and denoted by Lin M. In other words, Lin M is the smallest subspace of $\mathbb{R}^n$ that contains M. The affine hull Aff M of a set M is the intersection of all affine sets (synonyms used by other authors for *affine set* include *affine manifold*, *affine variety*, *linear variety*, *flat*) in $\mathbb{R}^n$ containing M.

Thus, we may refer to Aff M as the smallest affine set containing M. Then we can define the affine hull as Aff $M = x_0 + \text{Lin } M$.

Definition 1.7. The relative interior of a convex set M in $\mathbb{R}^n$, which we denote by ri M, is defined as the interior that results when M is regarded as a subset of its affine hull Aff M. That is, ri M consists of the points $x_1 \in \text{Aff } M$ for which there exists an $\varepsilon > 0$, such that $x_2 \in M$ whenever $x_2 \in \text{Aff } M$ and $\|x_1 - x_2\| \leq \varepsilon$; i.e.,

$$\text{ri } M = \{x \in \text{aff } M : \exists \varepsilon > 0, (x + \varepsilon \overline{B}) \cap (\text{aff } M) \subset M\},$$

where $\overline{B}$ is a closed unit ball. Observe that ri $M \subset M \subset \overline{M}$. Naturally, M is said to be relatively open if ri $M = M$.

Definition 1.8. The dimension of a convex set M is the dimension of Lin M and is denoted by dim M.

A set of $m + 1$ points $x_0, x_1, \dots, x_m$ is said to be affinely independent if Aff $\{x_0, x_1, \dots, x_m\}$ is m-dimensional. If $\{x_0, x_1, \dots, x_m\}$ is affinely independent, its convex hull is called an m-dimensional simplex, and $x_0, x_1, \dots, x_m$ are called the vertices of the simplex.

For an n-dimensional convex set, Aff $M = \mathbb{R}^n$ by definition, so ri $M = \text{int } M$.

It should be pointed out that although the inclusion $M_1 \subset M_2$ implies $\overline{M_1} \subset \overline{M_2}$ and int $M_1 \subset \text{int } M_2$, it does not generally imply ri $M_1 \subset \text{ri } M_2$. For example, if M_2 is a square in $\mathbb{R}^2$ and M_1 is one of the sides of M_2, then ri M_1 and ri M_2 are both nonempty but disjoint.

Lemma 1.7. ri $\overline{M}$ = ri M.

□ Observe that Lin M is a closed set containing M; i.e., Lin $M \supseteq \overline{M}$. On the other hand, Lin M = Lin $\overline{M}$. It is also clear that ri $M \subseteq$ ri $\overline{M}$. Let us show the reverse inclusion. Let $x \in$ ri $\overline{M}$ and suppose $e_1, \ldots, e_r$ is a basis for Lin M. Then, for small ε we have

$$y_k = x + \varepsilon\left(e_k - \frac{1}{r+1}e\right) \in \overline{M}, \quad k = 1, \ldots, r,$$
$$y_0 = x - \frac{\varepsilon}{r+1}e \in \overline{M},$$

where $e = e_1 + \cdots + e_r$. The vectors $y_k - y_0 = \varepsilon e_k$ are linearly independent and

$$x = \frac{1}{r+1}y_0 + \cdots + \frac{1}{r+1}y_r.$$

This equality means that x is an interior point of an r-dimensional simplex with vertices $y_0, \ldots, y_r$. Taking points $\overline{y_k} \in M$ sufficiently close to y_k, we see that x is an interior point of a simplex with vertices $\overline{y_k}$. It follows that x is also an interior point of M. Thus, ri $M \supseteq$ ri $\overline{M}$. ■

Definition 1.9. The intersection of all closed convex sets containing M is called the closed convex hull of a set M and denoted by $\overline{\text{conv}}\, M$.

Lemma 1.8. $\overline{\text{conv}}\, M = \overline{\text{conv}\, M}$.

□ Since conv M is the intersection of all convex sets containing M, we have the inclusion $\overline{\text{conv}}\, M \supseteq \text{conv}\, M$. It follows that $\overline{\text{conv}}\, M \supseteq \overline{\text{conv}\, M}$. Conversely, $\overline{\text{conv}\, M}$ is a closed convex set. Hence, $\overline{\text{conv}}\, M \subseteq \overline{\text{conv}\, M}$. This ends the proof of lemma. ■

Theorem 1.3. The convex hull of a compact set is itself compact.

□ Remember that a set in $\mathbb{R}^n$ is compact if and only if it is both closed and bounded (or, equivalently, if each sequence of its points contains some subsequence that converges to a point of the given set).

If M is compact and $x \in$ conv M, then by Theorem 1.1 we can write:

$$x = \sum_{i=1}^{n+1} \lambda_i x_i, \quad x_i \in M, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

Therefore,

$$\|x\| \leq \sum_{i=1}^{n+1} \lambda_i \|x_i\| \leq c,$$

where c is a constant such that $\|x\| \le c$ for all $x \in M$. Thus, conv M is a bounded set. We show that M is a closed set. Let

$$x_k = \sum_{i=1}^{n+1} \lambda_{ik} x_{ik}, \quad x_{ik} \in M, \quad \sum_{i=1}^{n+1} \lambda_{ik} = 1. \tag{1.6}$$

Since the sequences λ_{ik} and x_{ik} are bounded, they contain convergent subsequences. Without loss of generality, assume that $\lambda_{ik} \to \lambda_{i0}$ and $x_{ik} \to x_{i0} \in M$ (remember that M is compact). Then by passing to the limit in Eq. (1.6), we have

$$x_0 = \sum_{i=1}^{n+1} \lambda_{i0} x_{i0}, \quad x_{i0} \in M, \quad \sum_{i=1}^{n+1} \lambda_{i0} = 1.$$

It follows that $x_0 \in \text{conv}\, M$, so conv M is closed. ■

The following property of closures and relative interiors of convex sets is fundamental.

Theorem 1.4. Let M be a convex set in $\mathbb{R}^n$. If $x_1 \in \overline{M}, x_1 \in \text{ri}\, M$, then $(1-\lambda) x_1 + \lambda x_2 \in \text{ri}\, M$ for all λ in (0,1). In addition, $\overline{M} = \overline{\text{ri}\, M}$.

□ If $x_2 \in \text{ri}\, M$, then $x_2 + \text{Lin}\, M \cap (\varepsilon B) \subseteq M$. Hence, it follows that

$$(1-\lambda)x_1 + \lambda(x_2 + \text{Lin}\;\; M(\varepsilon B)) = (1-\lambda)x_1 + \lambda x_2 + \text{Lin}\;\; M(\lambda \varepsilon B) \subseteq \overline{M}.$$

That is, since $\text{ri}\, M = \text{ri}\, \overline{M}$ by Lemma 1.7, we have $(1-\lambda)x_1 + \lambda x_2 \in \text{ri}\, M$. Now, let $x_0 \in \overline{M}$, $x_k \in M$, $x_k \to x_0$, $\lambda_k \to 0$. Thus, $(1-\lambda_k)x_k + \lambda_k y \in \text{ri}\, M$ if $y \in \text{ri}\, M$.

Therefore, x_0 is a limit point for ri M. Consequently, $\overline{M} \subseteq \overline{\text{ri}\, M}$. The reverse inclusion is trivial. ■

Lemma 1.9. Let M be a convex set and $x_0 \in \overline{M}$, but $x_0 \notin M$. Then in any neighborhood of x_0 there are points not contained in $\overline{M}$.

□ Take a point $y \in \text{ri}\, M$. Then the points of the ray $y + \lambda(x_0 - y)$, $\lambda > 1$, do not belong to $\overline{M}$. We argue by contradiction. Suppose that if $\lambda > 1$, then $x_1 = y + \lambda(x_0 - y) \in \overline{M}$. Thus, by Theorem 1.4, we deduce that

$$x_0 = \frac{1}{\lambda} x_1 + \left(1 - \frac{1}{\lambda}\right) y \in \text{ri}\, M.$$

But by hypothesis, $x_0 \notin M$. This contradiction proves the lemma. ■

The results on the separation of convex sets that we establish below are among the most important ones in convexity. They are based on the fact that a hyperplane in $\mathbb{R}^n$ divides $\mathbb{R}^n$ into two, in the sense that the complement of the hyperplane is the union of two disjoint open convex sets—the open half-spaces associated with the hyperplane. A hyperplane H is said to separate the sets M_1 and M_2 if M_1 lies in one of the closed half-spaces determined by H, and M_2 lies in other. Some kinds (properly, strongly) of separation the reader can find, for example, in Refs. [111, 226]. Obviously, if a hyperplane separates two sets, then it also separates their convex hulls; for this reason, we consider only the separation of convex sets.

Theorem 1.5. Let M be a nonempty convex set and assume that $x_0 \notin \overline{M}$. Then there are x^* and $\varepsilon > 0$, such that

$$\langle x, x^* \rangle \le \langle x_0, x^* \rangle - \varepsilon \quad \forall x \in M.$$

□ Let y be the point of $\overline{M}$ closest to x_0 (since $\overline{M}$ is closed, a point y exists) such that

$$\|x - x_0\| \ge \|y - x_0\|, \quad \forall x \in \overline{M}.$$

The convexity of M implies that $\lambda x + (1 - \lambda)y = y + \lambda(x - y) \in \overline{M}$ for all $\lambda \in [0,1]$. Thus,

$$\begin{aligned}&\|\lambda x + (1 - \lambda)y - x_0\|^2 = \langle y - x_0 + \lambda(x - y), y - x_0 + \lambda(x - y)\rangle \\ &\quad = \|y - x_0\|^2 + 2\lambda\langle x - y, y - x_0\rangle + \lambda^2\|x - y\|^2 \ge \|y - x_0\|^2.\end{aligned}$$

As a result of a simple transformation we obtain

$$2\langle x - y, y - x_0\rangle + \lambda\|x - y\|^2 \ge 0, \quad \lambda \in [0, 1].$$

In particular, if $\lambda = 0$ it follows that

$$\langle x - y, y - x_0\rangle \ge 0. \tag{1.7}$$

Let $x^* = x_0 - y$, $\varepsilon = \|x^*\|^2$. Here, $y \ne x_0$, because $x_0 \notin \overline{M}$ by hypothesis.

This implies that $x^* \ne 0$ and so $\varepsilon > 0$. Now, the inequality in Eq. (1.7) can be rewritten as follows:

$$\langle x, x^* \rangle \le \langle y, x^* \rangle = \langle x_0, x^* \rangle - \langle x^*, x^* \rangle = \langle x_0, x^* \rangle - \varepsilon.$$

for arbitrary $x \in M$. ■

Theorem 1.6. Let M be a convex set not containing x_0. Then, there is a point $x^* \ne 0$ such that

$$\langle x, x^* \rangle \le \langle x_0, x^* \rangle \quad \forall x \in M.$$

□ The assertion follows from the previous theorem if $x_0 \notin \overline{M}$. So assume that $x_0 \notin \overline{M}$. Then, by Lemma 1.9, there exists a sequence $x_k \to x_0$, $x_k \notin \overline{M}$. Then, according to Theorem 1.5,

$$\langle x, x_k^* \rangle \le \langle x_k, x_k^* \rangle - \varepsilon_k, \quad \varepsilon_k > 0.$$

Since $\varepsilon_k > 0$, it follows that $x_k^* \neq 0$. Taking into account that ε_k is arbitrarily small and $x_k^* \neq 0$, we can write

$$\left\langle x, \frac{x_k^*}{\|x_k^*\|} \right\rangle \leq \left\langle x_k, \frac{x_k^*}{\|x_k^*\|} \right\rangle. \tag{1.8}$$

The sequence $x_k^*/\|x_k^*\|$ is bounded and so contains a convergent subsequence. Without loss of generality, suppose that

$$\frac{x_k^*}{\|x_k^*\|} \to x_0^*, \quad \|x_0^*\| = 1. \tag{1.9}$$

Thus, by passing to the limit in Eq. (1.8), we have $\langle x, x_0^* \rangle \leq \langle x_0, x_0^* \rangle$. ■

Theorem 1.7. If M_1 and M_2 are nonempty disjoint convex sets, then there exists $x^* \neq 0$ such that

$$\langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle$$

for all $x_1 \in M_1$, $x_2 \in M_2$.

□ Consider the convex set $M = M_1 - M_2$. Since M_1 and M_2 are nonempty disjoint convex sets, the point $x_0 = 0$ does not belong to M. Then by the preceding theorem we obtain the existence of $x^* \neq 0$ such that

$$\langle x, x^* \rangle \leq \langle 0, x^* \rangle \tag{1.10}$$

for all $x \in M$; i.e., for $x = x_1 - x_2$, $x_1 \in M_1$, $x_2 \in M_2$. Then by substituting $x = x_1 - x_2$ into Eq. (1.10), we obtain

$$\langle x_1 - x_2, x^* \rangle \leq 0, \quad x_1 \in M_1, \ x_2 \in M_2. \blacksquare$$

Theorem 1.8. Let M_1 and M_2 be nonempty disjoint closed convex sets, one of which is compact. Then there exist $x^* \neq 0$ and $\varepsilon > 0$ such that

$$\langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle - \varepsilon \quad \forall x_1 \in M_1, \ x_2 \in M_2.$$

□ Observe that, as in the proof of the preceding theorem, the point $x_0 = 0$ does not belong to $M = M_1 - M_2$. But since M_1 and M_2 are closed and one of them is compact, it is not hard to see that M is a closed set. Thus, $0 \notin M$. Applying the preceding theorem ends the proof of this theorem. ■

Definition 1.10. Let M be a convex set and $0 \in \operatorname{int} M$. Then a function

$$r_M(x) = \inf\left\{\alpha : \alpha > 0, \frac{x}{\alpha} \in M\right\}$$

defined on $X = \mathbb{R}^n$ is called a Minkowski (or gauge) function.

Since $0 \in \operatorname{int} M$, $\alpha^{-1}x \in M$ for any x, when α is sufficiently large and $r_M(x)$ is always a finite number. Besides, by definition, $r_M(x) \geq 0$.

Lemma 1.10. Let M be a convex set and $0 \in \operatorname{int} M$. Then:

a. $r_M(\lambda x) = \lambda r_M(x)$ if $\lambda \geq 0$;
b. $r_M(x) \leq 1$, if $x \in M$ and $r_M(x) \geq 1$, if $x \notin M$;
c. $r_M(x + y) \leq r_M(x) + r_M(y)$.
□ **a.** Let $\lambda\alpha_1 = \alpha$. We have

$$r_M(\lambda x) = \inf_{\alpha}\left\{\alpha : \alpha > 0, \frac{\lambda x}{\alpha} \in M\right\} = \inf_{\alpha_1}\{\lambda\alpha_1 : \alpha_1 > 0, \alpha_1^{-1}x \in M\}$$

$$= \lambda \inf_{\alpha_1}\{\alpha_1 : \alpha_1 > 0, \alpha_1^{-1}x \in M\} = \lambda r_M(x).$$

b. If $x \in M$, then $x/1 \in M$, so that $r_M(x) \leq 1$. Now, let $x \notin M$. On the contrary, if $r_M(x) < 1$, then there is $\alpha < 1$ such that $\alpha^{-1}x \in M$. Because $0 \in M$ and M is convex, we can write

$$x = (1 - \alpha)0 + \alpha(\alpha^{-1}x) \in M,$$

which contradicts the fact that $x \notin M$. Thus, $r_M(x) \geq 1$ for all $x \notin M$.

c. Let $\gamma > r_M(x) + r_M(y)$. Then there are α and β so that $\gamma = \alpha + \beta$, $\alpha > r_M(x)$, $\beta > r_M(y)$. The inequality $\alpha > r_M(x)$ implies that $\alpha^{-1}x \in M$. Indeed, by definition of $r_M(x)$, there is a number α_1 such that $\alpha > \alpha_1 > r_M(x)$ and $\alpha_1^{-1}x \in M$. Because M is convex and $0 \in M$, it follows that $\alpha^{-1}x = (1 - \alpha^{-1}\alpha_1)0 + \alpha^{-1}\alpha_1(\alpha_1^{-1}x) \in M$. Similarly, we can prove that $\beta^{-1}y \in M$. Thus, we have

$$\frac{x+y}{\gamma} = \frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}(\alpha^{-1}x) + \frac{\beta}{\alpha+\beta}(\beta^{-1}y) \in M.$$

Consequently, $r_M(x + y) \leq \gamma$. But since γ is an arbitrary number satisfying $\gamma > r_M(x) + r_M(y)$, finally we obtain

$$r_M(x + y) \leq r_M(x) + r_M(y).$$

We now associate with each nonempty convex set a convex function known as its support function. ■

Definition 1.11. The support function H_M of a nonempty set M in $\mathbb{R}^n$ is defined by:

$$H_M(x^*) = \sup_x\{\langle x, x^*\rangle : x \in M\}.$$

Theorem 1.9. Let M be a closed convex set. Then $x \in M$ if and only if

$$\langle x, x^*\rangle \leq H_M(x^*) \tag{1.11}$$

for all x^*.

□ If $x \in M$, then according to the definition of support function, the inequality in Eq. (1.11) is satisfied.

Conversely, let Eq. (1.11) be satisfied for x_0 and $x_0 \notin M$. Then by Theorem 1.5, there exist x^* and $\varepsilon > 0$ such that

$$\langle x, x^* \rangle \leq \langle x_0, x^* \rangle - \varepsilon$$

for all $x \in M$. Thus, the least upper bound (supremum) of the left-hand side of this inequality is $H_M(x^*)$; i.e.,

$$H_M(x^*) \leq \langle x_0, x^* \rangle - \varepsilon,$$

which contradicts Eq. (1.11). ■

Thus, a convex set can be characterized by both Minkowski's function and support functions.

By way of an example, we find the support function H_M of the set M defined as

$$M = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \leq 1\}.$$

Let $x^* = (x_1^*, \ldots, x_n^*)$. Then

$$\begin{aligned} H_M(x^*) = \sup_{x \in M} \langle x^*, x \rangle &= \sup_x \{x_1 x_1^* + \cdots + x_n x_n^* : |x_1| + \cdots + |x_n| \leq 1\} \\ &\leq \sup_x \{|x_1^*||x_1| + \cdots + |x_n^*||x_n| : |x_1| + \cdots + |x_n| \leq 1\} \\ &\leq \sup_x \{[\max(|x_1^*|, \ldots, |x_n^*|)](|x_1| + \cdots + |x_n|) : |x_1| + \cdots + |x_n| \leq 1\} \\ &= \max\{|x_1^*|, \ldots, |x_n^*|\}. \end{aligned}$$

Let $k \in \{1, \ldots, n\}$ be such that $|x_k^*| = \max\{|x_1^*|, \ldots, |x_n^*|\}$. Define a point $x = (x_1, \ldots, x_n)$ of M by the conditions $x_i = 0$ when $i \neq k$ and x_k^* is 1 or -1 according to whether x_k^* is nonnegative or negative. Then

$$\langle x^*, x \rangle = |x_k^*| = \max\{|x_1^*|, \ldots, |x_n^*|\},$$

and, therefore, $H_M(x^*) \geq \max\{|x_1^*|, \ldots, |x_n^*|\}$. We have thus shown that

$$H_M(x^*) = \max\{|x_1^*|, \ldots, |x_n^*|\}.$$

1.3 Convex Cones and Dual Cones

A convex cone is one of the important concepts in the theory of extremal problems. To investigate its properties, first we must calculate the dual cone.

Definition 1.12. A convex set K is said to be a cone if $x \in K$ and $\lambda > 0$ imply that $\lambda x \in K$.

Lemma 1.11. If $x_1, \ldots, x_m \in K$, $\lambda_1 > 0, \ldots, \lambda_m > 0$, then

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m \in K.$$

□ The proof of the lemma follows immediately from the formula:

$$\lambda_1 x_1 + \cdots + \lambda_m x_m = \lambda\left(\frac{\lambda_1}{\lambda} x_1 + \cdots + \frac{\lambda_m}{\lambda} x_m\right),$$

where $\lambda = \lambda_1 + \cdots + \lambda_m$, which is Lemma 1.4, and the definition of a cone. ■
Thus, a convex cone (or, briefly, a cone) contains any linear combination with positive coefficients of its elements.

We assume, unless otherwise stated, that the cones being considered are convex.

Definition 1.13. The set of vectors $x^* \in X^* = \mathbb{R}^n$ for which $\langle x, x^* \rangle \geq 0$ holds for all $x \in K$ is called the dual cone to the cone K and is denoted by K^*. Briefly, it can be written as follows:

$$K^* = \{x^* \in X^* : \langle x, x^* \rangle \geq 0 \quad \forall x \in K\}.$$

Obviously, K^* is a convex cone.

Some important properties of cones follow.

Lemma 1.12. K^* is a closed cone.
□ If $x_k^* \in K^*$, then $\langle x_k^*, x \rangle \geq 0$ for all $x \in K$. By passing to the limit in this inequality as $x_k^* \to x_0^*$, $k \to \infty$, we have

$$\langle x, x_0^* \rangle \geq 0 \quad \forall x \in K,$$

which means that $x_0^* \in K^*$. ■

Lemma 1.13. K and $\overline{K}$ have the same dual cones; i.e., $K^* = (\overline{K})^*$.
□ If $x^* \in (\overline{K})^*$, then $\langle x, x^* \rangle \geq 0$ for all $x \in \overline{K}$ and so for all $x \in K$. Hence, $x^* \in K^*$. Conversely, if $x^* \in K^*$, then by continuity, $\langle x, x^* \rangle$ is nonnegative, because elements of $\overline{K}$ are the limit points of K. ■

Lemma 1.14. If K is a closed cone and $\langle x, x^* \rangle \geq 0$ for all $x^* \in K^*$, then $x \in K$.
□ We argue by contradiction. Suppose $x_0 \notin K$, but $\langle x_0, x^* \rangle \geq 0$ for all $x^* \in K^*$. Since $x_0 \notin K$, by Theorem 1.8, there exist $\overline{x}^*$ and $\varepsilon > 0$ such that

$$\langle x_0, \overline{x}^* \rangle \leq \langle x, \overline{x}^* \rangle - \varepsilon \tag{1.12}$$

for all $x \in K$. We show that

$$\inf_{x \in K} \langle x, \overline{x}^* \rangle = 0 \tag{1.13}$$

Indeed, since by definition $x \in K$ implies $\lambda x \in K$, $\lambda > 0$, any cone K contains elements sufficiently close to zero. In particular, a closed cone always contains the zero point, so the infimum of $\langle x, \overline{x}^* \rangle$ cannot be greater than zero. On the other hand, according to Eq. (1.12), the inner product $\langle x, \overline{x}^* \rangle$ is bounded below. We prove that $\langle x, \overline{x}^* \rangle \geq 0$. On the contrary, suppose that there exists a point $x_1 \in K$ such that $\langle x, \overline{x}^* \rangle < 0$. Then denoting $x = \lambda x_1$ and by passing to the limit in $\langle x, \overline{x}^* \rangle$, as $\lambda \to +\infty$, we observe that $\langle x, \overline{x}^* \rangle \to -\infty$. This contradiction proves that $\langle x, \overline{x}^* \rangle \geq 0$. It follows that the formula in Eq. (1.13) is true. Thus, $\overline{x}^* \in K^*$. Setting $x = 0$ in the inequality in Eq. (1.12), we have $\langle x, \overline{x}^* \rangle < -\varepsilon$, which contradicts that $\langle x_0, x^* \rangle \geq 0$ for all $x^* \in K^*$. Consequently, $x_0 \in K$. ■

Naturally, later on we denote $K^{**} = (K^*)^*$.

Lemma 1.15. If K is a closed cone, then $K^{**} = K$.

□ According to Definition 1.13, we can write

$$K^{**} = \{x : \langle x, x^* \rangle \geq 0, \quad x^* \in K^*\}.$$

And if $x \in K$, then $\langle x, x^* \rangle \geq 0$ for $x^* \in K^*$. Conversely, if K is closed, then by the previous lemma, from the inequality, $\langle x, x^* \rangle \geq 0$ for all $x^* \in K^*$, it follows that $x \in K$. ■

Note that in the more general case for any convex cone, $K^{**} = \overline{K}$. This follows from Lemma 1.13 because $K^{**} = (\overline{K})^*$ and so $K^{**} = (\overline{K})^{**} = \overline{K}$.

Lemma 1.16. Let K_1 and K_2 be convex cones. Then $K_1 + K_2$ is also a convex cone and

$$(K_1 + K_2)^* = K_1^* \cap K_2^*.$$

□ Note that the convexity of $K_1 + K_2$ follows immediately from Lemma 1.2. On the other hand, it is easy to see that if $K_1 + K_2$ contains a point $x = x_1 + x_2$, $x_1 \in K_1$, $x_2 \in K_2$, then it also contains an element $\lambda x = \lambda x_1 + \lambda x_2$, $\lambda > 0$. Moreover, $x^* \in (K_1 + K_2)^*$ if and only if

$$\langle x_1 + x_2, x^* \rangle \geq 0, \quad x_1 \in K_1, \quad x_2 \in K_2.$$

Since x_1 and x_2 change independently and one of them can tend to zero, if necessary, the latter inequality is equivalent to the inequalities

$$\begin{aligned} &\langle x_1, x^* \rangle \geq 0, \quad x_1 \in K_1, \\ &\langle x_2, x^* \rangle \geq 0, \quad x_2 \in K_2; \end{aligned}$$

i.e., $x^* \in K_1^*$ and $x^* \in K_2^*$. Thus, it is proved that $x^* \in (K_1 + K_2)^*$ if and only if $x^* \in K_1^* \cap K_2^*$. ■

Lemma 1.17. If the cones K_1 and K_2 are closed, then

$$(K_1 \cap K_2)^* = \overline{K_1^* + K_2^*}. \tag{1.14}$$

□ In fact, by using the previous results, we can write

$$(K_1 \cap K_2)^* = (K_1^{**} \cap K_2^{**})^* = ((K_1^* + K_2^*)^*)^* = (K_1^* + K_2^*)^{**} = \overline{K_1^* + K_2^*}. \blacksquare$$

Remark 1.1. We mention that a bar denotes the closure of a set. It should be pointed out that, in general, the formula in Eq. (1.14) is not true without the closure: the left-hand side of Eq. (1.14) is always closed, but the right-hand side may or may not be closed.

Lemma 1.18. If $\langle x, x^* \rangle$ is bounded below for all $x \in K$, then $x^* \in K^*$. Moreover, if $x \in \text{int } K$, then

$$\langle x, x^* \rangle > 0$$

for all $x^* \in K^*$, $x^* \neq 0$.
□ In fact, the first assertion is proved in Lemma 1.14. We show the second assertion. If $x \in \text{int } K$, then there is an $\varepsilon > 0$ such that $x + \varepsilon B \subseteq K$, where B is the Euclidean unit ball. Hence,

$$\langle x + \varepsilon z, x^* \rangle \geq 0$$

for all $x^* \in K^*$ and $z \in B$. Then it follows that

$$\langle x, x^* \rangle \geq \varepsilon \operatorname*{sub}_{z \in B} \langle -z, x^* \rangle \geq \varepsilon \left\langle \frac{x^*}{\|x^*\|}, x^* \right\rangle = \varepsilon \|x^*\| > 0;$$

i.e., $\langle x, x^* \rangle > 0$ for all $x^* \in K^*$, $x^* \neq 0$. ■

Theorem 1.10. Let $K_1, \ldots, K_m$ be convex cones. In order that their intersection be empty, it is necessary and sufficient that there exist $x_i^* \in K_i^*$, $i = 1, \ldots, m$ not all zero, such that

$$x_1^* + \cdots + x_m^* = 0.$$

□ Consider the Cartesian product $X^m = X \times X \times \cdots \times X = \mathbb{R}^{mn}$. The elements of X^m have the form $(x_1, x_2, \ldots, x_m)$, where $x_i \in X (i = 1, \ldots, m)$. Obviously, the scalar product of vectors $(x_1, \ldots, x_m)$ and $(x_1^*, \ldots, x_m^*)$ can be written in the form of the sum:

$$\langle x_1, x_1^* \rangle + \langle x_2, x_2^* \rangle + \cdots + \langle x_m, x_m^* \rangle.$$

Consider two cones in X^m of the form

$$K = K_1 \times \cdots \times K_m = \{(x_1, \ldots, x_m) : x_1 \in K_1, \ldots, x_m \in K_m\},$$
$$P = \{(x, \ldots, x) : x \in X\}.$$

Since $\cap_{i=1}^m K_i = \varnothing$, it is not hard to see that the cones K and P are disjoint, i.e., $K \cap P = \varnothing$. Applying Theorem 1.6, we determine that there is a vector $(x_1^*, \ldots, x_m^*)$ such that

$$\langle x, x_1^* \rangle + \cdots + \langle x, x_m^* \rangle \leq \langle x_1, x_1^* \rangle + \cdots + \langle x_m, x_m^* \rangle \tag{1.15}$$

for all $x \in X$, $x_1 \in K_1, \ldots, x_m \in K_m$.

The inequality in Eq. (1.15) implies that $\langle x_i, x_i^* \rangle$ is bounded on K_i, $i = 1, 2, \ldots, m$. By Lemma 1.18, it follows that

$$x_i^* \in K_i^*, \quad i = 1, \ldots, m \tag{1.16}$$

By Eq. (1.15), $\langle x, x_1^* + \cdots + x_m^* \rangle$ is bounded above for all $x \in X$. Since a nonzero linear function takes infinitely large values, the boundedness of $\langle x, x_1^* + \cdots + x_m^* \rangle$ implies that

$$x_1^* + \cdots + x_m^* = 0. \tag{1.17}$$

Now taking into account Eqs. (1.16) and (1.17), we have the desired result. ■

Theorem 1.11. Let $K_1, K_2, \ldots, K_m$ be convex cones. If $K = K_1 \cap K_2 \cap \cdots \cap K_m$ and $K_1 \cap \operatorname{int} K_2 \cap \cdots \cap \operatorname{int} K_m \neq \varnothing$, then

$$K^* = K_1^* + \cdots + K_m^*.$$

□ The validity of the inclusion

$$K^* \supseteq K_1^* + \cdots + K_m^*$$

is verified immediately, starting from the definition of a dual cone. We prove the inverse inclusion. Let $x^* \in K^*$, $x^* \neq 0$ denote

$$K_0 = \{x : \langle x, x^* \rangle < 0\}.$$

We show that the cones K_0 and K are disjoint. On the contrary, suppose that there exists an element $x_1 \in K_0 \cap K \neq \varnothing$. Since $x_1 \in K_0$, we have:

$$\langle x_1, x^* \rangle < 0.$$

On the other hand, since $x_1 \in K$, $x^* \in K^*$, by definition of dual cone,

$$\langle x_1, x^* \rangle \geq 0.$$

Therefore, the last two inequalities contradict each other, so $K_0 \cap K \neq \varnothing$. Consider the dual cone K_0^*. Suppose that $\langle x, x^* \rangle < 0$ implies that $\langle x, y^* \rangle \geq 0$; i.e., $y^* \in K_0^*$, $y^* \neq 0$. Then x^* and y^* are linearly dependent vectors. This means that there are numbers α_1 and α_2 (at least one of them is nonzero) such that

$$\alpha_1 x^* - \alpha_2 y^* = 0.$$

Because $x^* \neq 0$ and $y^* \neq 0$, we have $\alpha_2 \neq 0$ and

$$y^* = \lambda x^*, \quad \lambda = \alpha_1/\alpha_2.$$

Moreover,

$$0 \leq \langle x, y^* \rangle = \lambda \langle x, x^* \rangle$$

for all $x \in K_0$. Since $\langle x, x^* \rangle < 0$, it follows from the last inequality that $\lambda < 0$. If $y^* = 0$, we put $y^* = 0 \cdot x^*$. Thus, we find that

$$K_0^* = \{y^* : y^* = \lambda x^*, \ \ \lambda \leq 0\}.$$

Because the cones K_0 and K are disjoint (i.e., $K_0, K_1, \ldots, K_m$ are disjoint), according to preceding theorems, there exist vectors $y^* \in K_0^*$, $x_i^* \in K_i^*$, $i = 1, \ldots, m$ such that

$$y^* + x_1^* + \cdots + x_m^* = 0, \tag{1.18}$$

where not all terms are zero. By using $y^* = \lambda x^*$ and $\lambda \leq 0$, we rewrite Eq. (1.18) in the form

$$-\lambda x^* = x_1^* + \cdots + x_m^*. \tag{1.19}$$

Observe that if $\lambda < 0$, then

$$x^* = \left(-\frac{1}{\lambda}\right) x_1^* + \left(-\frac{1}{\lambda}\right) x_2^* + \cdots + \left(-\frac{1}{\lambda}\right) x_m^* \in K_1^* + \cdots + K_m^*.$$

Here, note that $x_i^* \in K_i^*$, $i = 1, \ldots, m$.

Next, we show that $\lambda \neq 0$. On the contrary, suppose that $\lambda = 0$. Then, it follows from Eq. (1.19) that

$$x_1^* + \cdots + x_m^* = 0 \tag{1.20}$$

Here, not all x_i^* are zero and so at least two of them, say x_1^* and x_2^*, are nonzero. By the hypotheses of the theorem, there exists a point $x_0 \in K_1 \cap \text{int } K_2 \cdots \cap \text{int } K_m$ so that by Lemma 1.18

$$\langle x_0, x_2^* \rangle > 0$$

and $\langle x_0, x_i^* \rangle \geq 0$ for $i \neq 2$. Then taking into account Eq. (1.20), we establish the following contradiction:

$$0 = \langle x_0, x_1^* + x_2^* + \cdots + x_m^* \rangle = \langle x_0, x_1^* \rangle + \cdots + \langle x_0, x_m^* \rangle > 0;$$

i.e., $0 > 0$. ■

Theorem 1.12. Let $K_1, \ldots, K_m$ be convex cones and $K = K_1 \cap \cdots \cap K_m$. Then either

$$K^* = K_1^* + \cdots + K_m^* \tag{1.21}$$

or there exist $x_i^* \in K_i^*$, not all zero, such that

$$x_1^* + \cdots + x_m^* = 0.$$

□ If for all $x^* \in K^*$ it occurs that $\lambda < 0$ in Eq. (1.19), then x^* can be represented as a sum of $x_i^* \in K_i^*$. Clearly, this representation implies Eq. (1.21). If $\lambda = 0$ for some x^*, then the formula in Eq. (1.20) is satisfied. ■

As is shown in Theorem 1.11, if the intersection of cones is empty, then the equality in Eq. (1.20) is true.

Definition 1.14. The cones $K_1, \ldots, K_m$ are called separable if there exist vectors $x_i^* \in K_i^*$, not all zero, such that

$$x_1^* + \cdots + x_m^* = 0.$$

Definition 1.15. Let M be an arbitrary nonempty set in $\mathbb{R}^n$. The cone defined by

$$\text{cone } M = \{x : x = \lambda x_1,\ x_1 \in M,\ \lambda > 0\}$$

is called the cone generated by M.

Lemma 1.19. If M is a convex set, then cone M is a convex cone.
□ Take $x_1 \in \text{cone } M$ and $x_2 \in \text{cone } M$. Then

$$x_1 = \lambda \overline{x_1},\ x_2 = \lambda \overline{x_2},\ \overline{x_1},\ \overline{x_2} \in M,\ \ \lambda_1,\ \lambda_2 > 0.$$

Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be arbitrary numbers. Then we can write:

$$\begin{aligned}\alpha_1 x_1 + \alpha_2 x_2 &= \alpha_1 \lambda_1 \overline{x_1} + \alpha_2 \lambda_2 \overline{x_2} \\ &= (\alpha_1\lambda_1 + \alpha_2\lambda_2)\left[\frac{\alpha_1\lambda_1}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\overline{x_1} + \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\overline{x_2}\right] \in \text{cone } M.\end{aligned}$$

Here, we have used the fact that M is convex, and so

$$\left[\frac{\alpha_1\lambda_1}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\overline{x_1} + \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1 + \alpha_2\lambda_2}\overline{x_2}\right] \in M. \blacksquare$$

Lemma 1.20. Let M be a convex set. Then $x^* \in (\text{cone } M)^*$ if and only if $\langle x, x^* \rangle \geq 0$ for all $x \in M$.

☐ In fact, $x^* \in (\text{cone } M)^*$ if and only if

$$\langle \lambda x, x^* \rangle \geq 0$$

for all $\lambda \geq 0$ and all $x \in M$. Since $\lambda \geq 0$, it follows that $\langle x, x^* \rangle \geq 0$ for all $x \in M$. ■

Definition 1.16. Let M be a nonempty convex set in $\mathbb{R}^n$. The recession cone of M is denoted by $0^+ M$ and is defined as

$$0^+ M = \{\bar{x} \neq 0 : x + \lambda\bar{x} \in M,\ \forall x \in M,\ \forall \lambda \geq 0\} \cup \{0\}.$$

Observe that if M is bounded, it certainly contains no half-lines, so that $0^+ M = \{0\}$.

As examples of recession cones in $\mathbb{R}^2$, for

$$\begin{aligned}&M_1 = \left\{(x_1, x_2) : x_1 > 0,\ x_2 \geq \frac{1}{x_1}\right\}, \\ &M_2 = \{(x_1, x_2) : x_1 > 0,\ x_2 > 0\} \cup \{(0,0)\},\end{aligned}$$

one has

$$\begin{aligned}&0^+ M_1 = \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0\}, \\ &0^+ M_2 = \{(x_1, x_2) : x_1 > 0,\ x_2 > 0\} \cup \{(0,0)\} = M_2.\end{aligned}$$

Moreover, if

$$M = \{x : \langle x, x_k^* \rangle \leq \beta_k, k \in I\},$$

where $x_k^* \in \mathbb{R}^n$, $\beta_k \in \mathbb{R}^1$ and I is an arbitrary index set, then

$$0^+ M = \{x : \langle x, x_k^* \rangle \leq 0, k \in I\}.$$

Definition 1.17. A polyhedral convex set in $\mathbb{R}^n$ is a set that can be expressed as the intersection of some finite family of closed half-spaces, that is, as the set of solutions to some finite system of inequalities of the form

$$\langle x, x_k^* \rangle \leq \beta_k, \ k = 1, \ldots, l. \tag{1.22}$$

The definition of a polyhedral set immediately makes it clear why such sets play a leading role in linear programming.

In particular, if the finite system of inequalities in Eq. (1.22) is homogeneous ($\beta_k = 0$, $k = 1, \ldots, l$); i.e.,

$$\langle x, x_k^* \rangle \leq 0, \ k = 1, \ldots, l, \tag{1.23}$$

then the set of solutions to this finite system of inequalities is called a polyhedral convex cone. A bounded polyhedral convex set is a polytope (polyhedron), which is the convex hull of finitely many points. Clearly, a polyhedral set is closed.

Theorem 1.13. (Farkas) If K is a polyhedral cone, then the dual cone K^* is polyhedral and consists of the elements of the form

$$x^* = \sum_{k=1}^{l} \gamma_k x_k^*, \quad \gamma_k \geq 0.$$

□ Consider a polyhedral cone

$$\widetilde{K} = \left\{ x^* : x^* = \sum_{k=1}^{l} \gamma_k x_k^*, \quad \gamma_k \geq 0 \right\}.$$

By the definition of a dual cone, $x \in (\widetilde{K})^*$, if

$$\left\langle x, \sum_{k=1}^{l} \gamma_k x_k^* \right\rangle \geq 0, \quad \gamma_k \geq 0, \ k = 1, \ldots, l.$$

It follows that

$$\langle x, x_k^* \rangle \geq 0, \quad k = 1, \ldots, l;$$

i.e., $x \in K$. Thus, $K = (\tilde{K})^*$. Consequently, since K is closed as a polyhedral cone, we have

$$K^* = (\tilde{K})^{**} = \tilde{K}. \blacksquare$$

Lemma 1.21. The sum of polyhedral cones is a polyhedral cone.
□ Let K_1 and K_2 be polyhedral cones. Then there exists a collection of vectors x_{i1}, $i = 1, \dots, m$ and x_{j2}, $j = 1, \dots, l$ such that all $x_1 \in K_1$, $x_2 \in K_2$ can be represented as

$$x_1 = \sum_{i=1}^{m} \lambda_i x_{i1}, \quad \lambda_i \geq 0, \quad x_2 = \sum_{j=1}^{l} \gamma_j x_{j2}, \quad \gamma_j \geq 0.$$

If now $x \in K_1 + K_2$, then

$$x = x_1 + x_2 = \sum_{i=1}^{m} \lambda_i x_{i1} + \sum_{j=1}^{l} \gamma_j x_{j2}, \quad \lambda_i \geq 0, \quad \gamma_j \geq 0;$$

i.e., $K_1 + K_2$ is a polyhedral cone. ■

Lemma 1.22. If $K_1, \dots, K_m$ are polyhedral cones, then

$$(K_1 \cap \cdots \cap K_m)^* = K_1^* + \cdots + K_m^*.$$

□ According to Lemma 1.17, we have

$$(K_1 \cap \cdots \cap K_m)^* = \overline{K_1^* + \cdots + K_m^*}.$$

Since K_i^*, $i = 1, \dots, m$ are polyhedral cones, by Lemma 1.21, $K_1^* + \cdots + K_m^*$ is also a polyhedral cone. Thus, this sum of cones is closed and the bar above it can be removed. ■

Theorem 1.14. A polyhedral set can be represented as a sum of a polytope and a polyhedral cone. Conversely, the sum of any polytope and a polyhedral cone is a polyhedral set.
□ Let us introduce an additional coordinate x^0 and consider a system of homogeneous inequalities

$$\langle x, x_k^* \rangle - x^0 \beta_k \leq 0, \quad k = 1, \dots, l; \quad x^0 \geq 0 \tag{1.24}$$

Clearly, the set of solutions to this system is a polyhedral cone in $\mathbb{R}^{n+1}$, the elements of which can be represented as follows:

$$\begin{pmatrix} x^0 \\ x \end{pmatrix} = \sum_{j=1}^{m} \lambda_j \begin{pmatrix} x_j^0 \\ x_j \end{pmatrix}, \quad \lambda_j \geq 0, \tag{1.25}$$

where $\begin{pmatrix} x^0 \\ x \end{pmatrix}$ is an $(n+1)$-dimensional vector in $\mathbb{R}^{n+1}$. Observe that $\lambda_j \geq 0$ is arbitrary. In particular, taking $\lambda_j = 1$, $\lambda_i = 0$, $i \neq j$, the inequality in Eq. (1.25) implies that $x_j^0 \geq 0$.

Let

$$I^0 = \{j : x_j^0 = 0, \ j = 1, \ldots, m\}$$

and

$$I^+ = \{j : x_j^0 > 0, \ j = 1, \ldots, m\}.$$

Denoting $y_j = x_j / x_j^0$, $\gamma_j = \lambda_j x_j^0$, $j \in I^+$, we rewrite the inequality in Eq. (1.25) in the forms:

$$x^0 = \sum_{j \in I^+} \lambda_j x_j^0 = \sum_{j \in I^+} \gamma_j, \quad \gamma_j \geq 0 \tag{1.26}$$

and

$$x = \sum_{j \in I^0} \lambda_j x_j + \sum_{j \in I^+} \lambda_j x_j^0 \left(\frac{x_j}{x_j^0} \right) = \sum_{j \in I^0} \lambda_j x_j + \sum_{j \in I^+} \gamma_j y_j, \quad \gamma_j \geq 0$$

Thus, every solution of Eq. (1.24) can be represented in the form of Eq. (1.26). Remember that setting $x^0 = 1$, a solution to Eq. (1.22) is obtained from a solution to Eq. (1.24). Consequently, every solution of Eq. (1.22) can be represented in the form

$$x = \sum_{j \in I^0} \lambda_j x_j + \sum_{j \in I^+} \gamma_j y_j, \quad \lambda_j \geq 0 \tag{1.27}$$

and

$$\sum_{j \in I^+} \gamma_j = 1, \quad \gamma_j \geq 0, \ j \in I^+. \tag{1.28}$$

Note that the first term on the right-hand side of Eq. (1.27) represents a point of a polyhedral cone, while the second term is a point of some polytope. Similarly, taking into account Eqs. (1.27) and (1.28) in reverse direction, it can be shown that the sum of any polytope and polyhedral cone is a polyhedral set. ■

1.4 The Main Properties of Convex Functions

Let f be a function whose values are real or $\pm\infty$ and whose domain is a subset of $\mathbb{R}^n$. The effective domain of a function f (dom f) is the set of points of $\mathbb{R}^n$ on which the values of f are real or $-\infty$:

$$\operatorname{dom} f = \{x : f(x) < +\infty\}.$$

The epigraph of f is the set of pairs $x_0 \in \mathbb{R}^1$ and $x \in X = \mathbb{R}^n$ such that $x^0 \geq f(x)$:

$$\operatorname{epi} f = \{(x^0, x) : x^0 \geq f(x)\}.$$

It should be noted that a point $(x_0, x) \in \mathbb{R}^{n+1}$ belongs to epi f, if $x \in \operatorname{dom} f$, because in the case $f(x) = +\infty$, there is no real number x^0 such that $x^0 \geq f(x)$. It is not hard to see that the function f can be defined by epi f as follows:

$$f(x) = \inf_{x^0}\{x^0 : (x^0, x) \in \operatorname{epi} f\}. \tag{1.29}$$

Thus, there is a close connection between subsets of $\mathbb{R}^{n+1}$ and functions defined on $\mathbb{R}^n$.

Definition 1.18. A function f is said to be convex if epi f is a convex set.

Definition 1.19. A function f is said to be proper if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x. A function that is not proper is improper.
Thus, f is proper if and only if $\operatorname{dom} f \neq \emptyset$ and $f(x)$ is finite for $x \in \operatorname{dom} f$.

Lemma 1.23. Let f be a proper function. Then f is convex if and only if

$$\begin{aligned} &f(\lambda x_1 + \lambda x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2), \\ &\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 = 1. \end{aligned} \tag{1.30}$$

□ If f is convex, then, by Definition 1.18, for all $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$,

$$\lambda_1(x_1^0, x_1) + \lambda_2(x_2^0, x_2) = (\lambda_1 x_1^0 + \lambda_2 x_2^0, \lambda_1 x_1 + \lambda_2 x_2) \in \operatorname{epi} f$$

if $(x_1^0, x_1) \in \operatorname{epi} f$, $(x_2^0, x_2) \in \operatorname{epi} f$ or, equivalently,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 x_1^0 + \lambda_2 x_2^0.$$

In particular, if $x_1^0 = f(x_1)$, $x_2^0 = f(x_2)$, then

$$\begin{aligned} &f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2), \\ &\lambda_1 + \lambda_2 = 1, \quad \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

Conversely, it is not hard to see that if f is a proper function satisfying Eq. (1.30), then f is convex. ■

Lemma 1.24. If f is convex, then its dom f is a convex set.
□ Let $x_1, x_2 \in \operatorname{dom} f$. Then there exist x_1^0, x_2^0 such that $f(x_1) \le x_1^0$, $f(x_2) \le x_2^0$. Thus, $(x_1^0, x_1) \in \operatorname{epi} f$, $(x_2^0, x_2) \in \operatorname{epi} f$, and so

$$\begin{aligned}
&\lambda_1(x_1^0, x_1) + \lambda_2(x_2^0, x_2) \in \operatorname{epi} f,\\
&\lambda_1 + \lambda_2 = 1, \quad \lambda_1, \lambda_2 \ge 0
\end{aligned}$$

or

$$f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 x_1^0 + \lambda_2 x_2^0.$$

Consequently, $f(\lambda_1 x_1 + \lambda_2 x_2)$ is not equal to $+\infty$, and so $\lambda_1 x_1 + \lambda_2 x_2 \in \operatorname{dom} f$. ■

It follows from Lemma 1.24 that dom f is convex, even if f is an improper function. Consider such a function. Let $y \in \operatorname{dom} f$, for which $f(y) = -\infty$, but $x \in \operatorname{ri} \operatorname{dom} f$. Then $(x - y) \in \operatorname{Lin} \operatorname{dom} f$ and so $x_1 = x + \varepsilon(x - y) \in \operatorname{dom} f$ for a sufficiently small $\varepsilon > 0$. It is clear that

$$x = \frac{1}{1+\varepsilon} x_1 + \frac{\varepsilon}{1+\varepsilon} y.$$

If y^0 is any number and $x_1^0 \ge f(x_1)$, then $(y^0, y) \in \operatorname{epi} f$ [remember that $f(y) = -\infty$], $(x_1^0, x_1) \in \operatorname{epi} f$, and so by virtue of the convexity of epi f,

$$\begin{aligned}
&\left(\frac{1}{1+\varepsilon} x_1^0 + \frac{\varepsilon}{1+\varepsilon} y^0, \frac{1}{1+\varepsilon} x_1 + \frac{\varepsilon}{1+\varepsilon} y\right) \in \operatorname{epi} f,\\
&f(x) = f\left(\frac{1}{1+\varepsilon} x_1 + \frac{\varepsilon}{1+\varepsilon} y\right) \le \frac{1}{1+\varepsilon} x_1^0 + \frac{\varepsilon}{1+\varepsilon} y^0.
\end{aligned}$$

In the latter inequality, by passing to the limit as $y^0 \to -\infty$, we have $f(x) = -\infty$. Therefore, if f is an improper convex function, then $f(x) = -\infty$ for every $x \in \operatorname{ri} \operatorname{dom} f$. It follows that an improper convex function may take finite values only at points of the relative boundary of dom f.

Lemma 1.25. Let f_i, $i \in I$ be a collection of convex functions, where I is an arbitrary index set. Then their pointwise supremum

$$f(x) = \sup_{i \in I} f_i(x)$$

is a convex function.

☐ It is not hard to see that

$$\operatorname{epi} f = \bigcap_{i \in I} \operatorname{epi} f_i,$$

which is a convex set by Lemma 1.1. ■

Lemma 1.26. (Jensen's inequality) Let f be a proper convex function. Then

$$f(\lambda_1 x_1 + \cdots + \lambda_m x_m) \leq \lambda_1 f(x_1) + \cdots + \lambda_m f(x_m),$$
$$\lambda_i \geq 0, \quad i = 1, \ldots, m, \quad \lambda_1 + \cdots + \lambda_m = 1.$$

☐ Without loss of generality, let $\lambda_i > 0$, $i \in I$. If $x_i \notin \operatorname{dom} f$, then $f(x_i) = +\infty$, $\lambda_i f(x_i) = +\infty$ and the inequality is fulfilled trivially, because its right-hand side is equal to $+\infty$.
Now, let $x_i \in \operatorname{dom} f$, $i = 1, \ldots, m$. Since $\operatorname{epi} f$ is a convex set, by Lemma 1.4, the inclusion $(f(x_i), x_i) \in \operatorname{epi} f$ implies that

$$(\lambda_1 f(x_1) + \cdots + \lambda_m f(x_m), \lambda_1 x_1 + \cdots + \lambda_m x_m) \in \operatorname{epi} f.$$

Thus,

$$f(\lambda_1 x_1 + \cdots + \lambda_m x_m) \leq \lambda_1 f(x_1) + \cdots + \lambda_m f(x_m). \blacksquare$$

Lemma 1.27. The sum of proper convex functions f_i, $i = 1, \ldots, m$ with nonnegative coefficients is convex.
☐ Since f_i, $i = 1, \ldots, m$ are proper convex functions, by Lemma 1.23,

$$f_i(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f_i(x_1) + \lambda_2 f_i(x_2),$$

where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\lambda_1 + \lambda_2 = 1$. Thus, multiplying these inequalities by $\alpha_i \geq 0$ and then adding them, we have

$$\sum_{i=1}^{m} \alpha_i f_i(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 \sum_{i=1}^{m} \alpha_i f_i(x_1) + \lambda_2 \sum_{i=1}^{m} \alpha_i f_i(x_2). \blacksquare$$

There are some useful correspondences between convex sets and convex functions. The simplest associates with each set M in $\mathbb{R}^n$ the indicator function $\delta_M(\cdot)$ of M, where

$$\delta_M(x) = \begin{cases} 0, & x \in M, \\ +\infty, & x \notin M. \end{cases}$$

Clearly, M is convex if and only if $\delta_M(\cdot)$ is a convex function on $\mathbb{R}^n$.

Note that the resulting function of Lemma 1.27 may not be a proper function. For example, for the disjoint sets M_1 and M_2, the sum of indicator functions $\delta_{M_1} + \delta_{M_2}$ is identically $+\infty$.

Now, we show that on the basis of Lemmas 1.25–1.27 a new class of convex functions can be constructed. Let f be a proper function. Choose a point $p \in \mathbb{R}^n$ and construct a function

$$g_{x,p}(\alpha) = f(x + \alpha p).$$

Lemma 1.28. A function f is convex if and only if $g_{x,p}(\alpha)$ is convex in α for all x and p.

☐ If f is convex, then for all $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned} g_{x,p}(\lambda_1\alpha_1 + \lambda_2\alpha_2) &= f(x + (\lambda_1\alpha_1 + \lambda_2\alpha_2)p) \\ &= f(\lambda_1(x + \alpha_1 p) + \lambda_2(x + \alpha_2 p)) \\ &\leq \lambda_1 f(x + \alpha_1 p) + \lambda_2(x + \alpha_2 p) \\ &= \lambda_1 g_{x,p}(\alpha_1) + \lambda_2 g_{x,p}(\alpha_2). \end{aligned}$$

Conversely, let $g_{x,p}(\alpha)$ be convex on α for all x and p. Then

$$\begin{aligned} f(\lambda_1 x_1 + \lambda_2 x_2) &= f(x_2 + (\lambda_1 \cdot 1 + \lambda_2 \cdot 0)(x_1 - x_2)) \\ &= g_{x_2, x_1 - x_2}(\lambda_1 \cdot 1 + \lambda_2 \cdot 0) \\ &\leq \lambda_1 g_{x_2, x_1 - x_2}(1) + \lambda_2 g_{x_2, x_1 - x_2}(0) \\ &= \lambda_1 f(x_1) + \lambda_2 f(x_2). \blacksquare \end{aligned}$$

We shall denote the gradient of f at point x by $f'(x)$ and the Hessian matrix of f at x whose (i,j)th element is $\partial^2 f/\partial x^i \partial x^j$ by $f''(x)$. Thus, $f'(x)$ is a column vector with components $\partial f(x)/\partial x^i$, $i = 1, \ldots, n$, and

$$f''(x) = \left\{ \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \right\}, \quad i = 1, \ldots, n, \;\; j = 1, \ldots, n.$$

Let denote $g(\alpha) = f(x + \alpha p)$. It is not hard to verify that

$$\begin{aligned} g'(\alpha) &= \langle p, f'(x + \alpha p) \rangle, \\ g''(\alpha) &= \langle p, f''(x + \alpha p) p \rangle. \end{aligned}$$

Theorem 1.15. Let f be a differentiable function. Then the following assertions are equivalent:

1. f is convex.
2. $f(x_2) - f(x_1) \geq \langle x_2 - x_1, f'(x_1) \rangle$ for all x_1 and x_2.
3. $\langle p, f'(x + \alpha p) \rangle$ is nondecreasing function on α for all x and p.
4. If f is twice differentiable, then $f''(x)$ is positive semidefinite matrix.

□ We show that (1) implies (2). In fact, since

$$f((1-\lambda)x_1+\lambda x_2)\le(1-\lambda)f(x_1)+\lambda f(x_2),\quad 0<\lambda<1,$$

then

$$\frac{f(x_1+\lambda(x_2-x_1))-f(x_1)}{\lambda}\le f(x_2)-f(x_1).$$

By passing to the limit as $\lambda\to 0$, we have:

$$\langle x_2-x_1,f'(x_1)\rangle\le f(x_2)-f(x_1). \tag{1.31}$$

Now we show that (2)→(3). Taking $x_1=x+\alpha_1 p$, $x_2=x+\alpha_2 p$ in Eq. (1.31), we get

$$g_{x,p}(\alpha_2)-g_{x,p}(\alpha_1)\ge(\alpha_2-\alpha_1)\langle p,f'(x+\alpha_1 p)\rangle.$$

Similarly,

$$g_{x,p}(\alpha_1)-g_{x,p}(\alpha_2)\ge(\alpha_1-\alpha_2)\langle p,f'(x+\alpha_2 p)\rangle$$

if $x_1=x+\alpha_2 p$, $x_2=x+\alpha_1 p$. From the last two inequalities, it follows that (say $\alpha_2>\alpha_1$)

$$\langle p,f'(x+\alpha_1 p)\rangle\le\frac{g_{x,p}(\alpha_2)-g_{x,p}(\alpha_1)}{\alpha_2-\alpha_1}\le\langle p,f'(x+\alpha_2 p)\rangle.$$

We show that (3)→(1). Let $g'_{x,p}(\alpha)=\langle p,f'(x+\alpha p)\rangle$ be a nondecreasing function of α. Hence, $g'_{x,p}(\alpha_1)\le g'_{x,p}(\alpha_2)$ if $\alpha_2\ge\alpha_1$.

Suppose that $0<\mu<1$. Then

$$\begin{aligned}0&\le\mu(\alpha_2-\alpha_1)\int_0^1[g'_{x,p}(\alpha_1+\tau(\alpha_2-\alpha_1))-g'_{x,p}(\alpha_1+\tau\mu(\alpha_2-\alpha_1))]d\tau\\&=(1-\mu)g_{x,p}(\alpha_1)+\mu g_{x,p}(\alpha_2)-g_{x,p}((1-\mu)\alpha_1+\mu\alpha_2);\end{aligned}$$

i.e., $g_{x,p}(\alpha)$ is convex. On the basis of Lemma 1.28, we conclude that f is convex.

Let f be a twice differentiable function. We show that (4)⇄(3). Since $g'_{x,p}(\alpha)$ is nondecreasing,

$$g''_{x,p}(\alpha)=\langle p,f''(x+\alpha p)p\rangle\ge 0, \tag{1.32}$$

and so the matrix $f''(x)$ is positive semidefinite. Conversely, if Eq. (1.32) is satisfied, then $g''_{x,p}(\alpha)$ is nonnegative and hence $g''_{x,p}(\alpha)=\langle p,f''(x+\alpha p)p\rangle$ is nondecreasing.

Consequently, we have that (1)→(2)→(3)→(1) and (4)⇄(3). ■

It is not hard to see that a quadratic function

$$f(x)=\frac{1}{2}\langle x,Ax\rangle+\langle x,b\rangle,$$

where A is a symmetric $n\times n$ matrix, is convex on $\mathbb{R}^n$ if and only if A is positive semidefinite; i.e.,

$$\langle z, Az\rangle \geq 0 \text{ for every } z\in\mathbb{R}^n.$$

Lemma 1.29. Let $f_1, f_2, \dots, f_m$ be proper convex functions on $\mathbb{R}^n$, and let

$$f(x) = \inf\left\{f_1(x_1) + f_2(x_2) + \cdots + f_m(x_m) : x_m \in \mathbb{R}^n, \sum_{i=1}^{m} x_i = x\right\}.$$

Then f is a convex function on $\mathbb{R}^n$.

□ Let $E_i = \operatorname{epi} f_i$ and $E = E_1 + \cdots + E_m$. Then E is a convex set in $\mathbb{R}^{n+1}$. By definition, $(x,x^0)\in E$ if and only if there exist $x_i\in\mathbb{R}^n$, $x_i^0\in\mathbb{R}$ so that $x_i^0 \geq f(x_i)$, $x^0 = x_1^0 + \cdots + x_m^0$ and $x = x_1 + \cdots + x_m$. Thus, the f defined in the theorem is the convex function obtained from E by the construction in formula in Eq. (1.29). ■

Definition 1.20. The function f in Lemma 1.29 will be denoted by $f_1 \oplus f_2 \oplus \cdots \oplus f_m$ and the operation $\oplus$ is called infimal convolution. The terminology arises from the fact that in the case of two functions,

$$(f_1 \oplus f_2)(x) = \inf_y \{f_1(x-y) + f_2(y)\}$$

and this is analogous to the classical formula for integral convolution.

Let f_1 be the Euclidean norm and f_2 be the indicator function of a convex set M. Then we get

$$(f_1 \oplus f_2)(x) = \inf_y \{\|x-y\| + \delta_M(y)\} = \inf_{y\in M} \|x-y\| = d_M(\cdot),$$

which establishes the convexity of the distance function $d_M(\cdot)$.

If $f_2 = \delta_M$, $M = \{b\}$ for some fixed point $b\in\mathbb{R}^n$, then $(f_1 \oplus f_2)(x) = f_1(x-b)$. Hence, $f_1 \oplus \delta_b$ is the function whose graph is obtained by translating the graph of f_1 horizontally by b.

Properness of convex functions is not always preserved by infimal convolution, because the infimum in the formula in Lemma 1.29 may be $-\infty$. Indeed, if f_1 and f_2 are linear functions not equal to each other, then their infimal convolution is identically $-\infty$. Note that $f_1 \oplus f_2$ can be defined for any functions f_1, f_2: $\mathbb{R}^n \to [-\infty, +\infty]$ in terms of addition of epigraphs:

$$(f_1 \oplus f_2)(x) = \inf\{x^0 : (x^0, x) \in (\operatorname{epi} f_1 + \operatorname{epi} f_2)\}.$$

As an operation on the family of all functions from $\mathbb{R}^n$ to $[-\infty, +\infty]$, infimal convolution is commutative, associative, and convexity-preserving.

Definition 1.21. The convex hull of a nonconvex function g is the function $f = \text{conv } g$, defined by

$$f(x) = \inf_{x^0}\{x^0 : (x^0, x) \in \text{conv}(\text{epi } g)\}.$$

It is the greatest convex function majorized by g.

The convex hull of an arbitrary collection of functions $\{f_i\colon i \in I\}$ on $\mathbb{R}^n$, denoted by conv $\{f_i\colon i \in I\}$, is the convex hull of the pointwise infimum of the collection:

$$\text{conv}\{f_i : i \in I\}(x) = \inf\left\{x^0 \in \mathbb{R} : (x^0, x) \in \text{conv}\left(\bigcup_{i \in I} \text{epi } f_i\right)\right\}.$$

Lemma 1.30. Let g be a convex function and $\alpha_0 < \alpha_1 < \alpha_2$; α_0, α_1, $\alpha_2 \in \text{dom } g$. Then

$$\frac{g(\alpha_2) - g(\alpha_0)}{\alpha_2 - \alpha_0} \geq \frac{g(\alpha_1) - g(\alpha_0)}{\alpha_1 - \alpha_0}; \quad \frac{g(\alpha_1) - g(\alpha_0)}{\alpha_1 - \alpha_0} \leq \frac{g(\alpha_2) - g(\alpha_1)}{\alpha_2 - \alpha_1}.$$

□ Setting

$$\lambda_1 = \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}, \lambda_2 = 1 - \lambda_1 = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0},$$
$$\lambda_1\alpha_2 + \lambda_2\alpha_0 = \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}\alpha_2 + \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0}\alpha_0 = \alpha_1.$$

Hence,

$$g(\alpha_1) = g(\lambda_1\alpha_2 + \lambda_2\alpha_0) \leq \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}g(\alpha_2) + \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0}g(\alpha_0).$$

Subtracting $g(\alpha_0)$ from this inequality, we have

$$g(\alpha_1) - g(\alpha_0) \leq \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}(g(\alpha_2) - g(\alpha_0)).$$

Deleting $\alpha_1 - \alpha_0$ from this inequality, we obtain the first inequality of the lemma. On the other hand, since $\lambda_1 + \lambda_2 = 1$,

$$\frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}g(\alpha_1) + \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0}g(\alpha_1) \leq \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}g(\alpha_2) + \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0}g(\alpha_0),$$

or

$$\frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_0}[g(\alpha_1) - g(\alpha_0)] \leq \frac{\alpha_1 - \alpha_0}{\alpha_2 - \alpha_0}[g(\alpha_2) - g(\alpha_1)].$$

This is the desired inequality. ■

Note that from the first inequality it can be deduced that $(g(\alpha)-g(\alpha_0))/(\alpha-\alpha_0)$ is a monotonically nondecreasing function in α, $\alpha>\alpha_0$.

It seems that the convexity of a function is closely connected with its continuity.

Theorem 1.16. Let a proper convex function f be bounded above in a neighborhood of some point $x_0 \in \operatorname{dom} f$. Then f is continuous at x_0.

□ Without loss of generality, suppose that $x_0=0$. Let Ω be an open ball with the center the origin such that $f(x) \leq c_1$ for all $x \in \Omega$. Consider a function $g(\alpha)=f(\alpha x)$ for fixed $x \in \Omega$. Taking in the first inequality of Lemma 1.30, $\alpha_0=0$, $\alpha_1=\alpha>0$, $\alpha_2=1$, we have

$$\frac{g(\alpha)-g(0)}{\alpha} \leq \frac{g(1)-g(0)}{1}.$$

Because

$$g(1)=f(x) \leq c_1, \quad g(0)=f(0) \leq c_1,$$

we also have

$$f(\alpha x)-f(0) \leq 2c_1\alpha. \tag{1.33}$$

Moreover, setting $\alpha_0=-1$, $\alpha_1=0$, $\alpha_2=\alpha$ in the second inequality of Lemma 1.30, we have

$$\frac{g(0)-g(-1)}{0-(-1)} \leq \frac{g(\alpha)-g(0)}{\alpha},$$

whence

$$-2c_1\alpha \leq f(\alpha x)-f(0). \tag{1.34}$$

Thus, it follows from the inequalities (1.33) and (1.34) that

$$\left|f(\alpha x)-f(0)\right| \leq 2c_1\alpha. \tag{1.35}$$

Take $\varepsilon>0$ and set $\delta=\varepsilon/(2c_1)<1$, $\Omega_\delta=\delta\Omega$. Let $y \in \Omega_\delta$. Then there is $x \in \Omega$ such that $y=\delta x$. Thus, according to Eq. (1.35), we have

$$\left|f(y)-f(0)\right| = \left|f(\delta x)-f(0)\right| \leq 2\delta c_1 = \varepsilon;$$

i.e., f is continuous at $x_0=0$. ■

Theorem 1.17. If a convex function f is continuous at x_0, then f satisfies the Lipschitz condition for neighborhood points of x_0:

$$\left|f(x)-f(x_0)\right| = L\|x-x_0\|,$$

where L is some constant.

□ Suppose that $x_0 = 0$. Let Ω be an open ball of radius r centered at the origin.

Take $y \in \Omega$, satisfying the inequality $\|y\| < r/2$, and set $x = (r/2)(y/\|y\|)$. By using Eq. (1.35), we obtain

$$\left|f(y) - f(0)\right| = \left|f\left(\frac{2\|y\|}{r}x\right) - f(0)\right| \leq L\|y\|,$$

where $L = 4c/r$. ■

Theorem 1.18. Let f be a proper convex function. Then f is continuous on ri dom f.
□ Any point, $x_0 \in$ ri dom f can be made an interior point of some simplex with vertices $y_0, \ldots, y_k \in$ dom f, where $k =$ dim dom f. Every point of this simplex has the form:

$$x = \lambda_0 y_0 + \cdots + \lambda_k y_k, \quad \lambda_i \geq 0, \quad i = 0, \ 1, \ldots, k,$$
$$\lambda_0 + \lambda_1 + \cdots + \lambda_k = 1.$$

Thus,

$$f(x) \leq \lambda_0 f(y_0) + \cdots + \lambda_k f(y_k) \leq \max_i f(y_i);$$

i.e., f is bounded in some neighborhood of x_0. Consider a convex function of $y = x - x_0$:

$$g(y) = f(y + x_0), y \in \text{Lin dom} f.$$

Now apply Theorem 1.15. ■

As is seen from this theorem, a convex function is continuous in dom f and may have a point of discontinuity only in its boundary. In order to characterize the case in which there is no such discontinuity, it is convenient to introduce the closure function concept.

Definition 1.22. A function f is said to be a *closure* if its epigraph epi f is a closed set.

Theorem 1.19. The following three conditions are equivalent:

1. f is a closure.
2. The level sets $C_\alpha = \{x: f(x) \leq \alpha\}$ are convex for every $\alpha \in \mathbb{R}^1$.
3. f is lower semicontinuous throughout $\mathbb{R}^n$.

□ We show that (1)→(2). Let $x_k \to x_0$ and $f(x_k) \leq \alpha$. Without loss of generality, we suppose that $f(x_k) \to \mu \leq \alpha$, where μ is either finite or $-\infty$.

We show that $f(x_0) = -\infty$ if $\mu = -\infty$. In fact, if $f(x_0) = \mu_0$ is finite, then $f(x_k) < \mu_0 - \varepsilon$ for large k. Consider the sequence of points $(\mu_0 - \varepsilon, x_k) \in$ epi f. Since

$(\mu_0-\varepsilon, x_k)\to(\mu_0-\varepsilon, x_0)$ and epi f is closed, then $(\mu_0-\varepsilon, x_0)\in$ epi f; i.e., $\mu_0-\varepsilon\geq f(x_0)=\mu_0$. This contradiction proves that $f(x_0)=-\infty$, i.e., $x_0\in C_\alpha$.

If μ is finite, then $(f(x_k), x_k)\to(\mu, x_0)\in$ epi f, so $\alpha\geq\mu\geq f(x_0)$; i.e., again $x_0\in C_\alpha$. Thus, C_α is closed.

Suppose now that (2) is satisfied. If $x_k\to x_0$ and $f(x_k)\to\alpha$, then for all $\varepsilon>0$ and sufficiently large k we have $f(x_k)\leq\alpha+\varepsilon$ and hence $f(x_0)\leq\alpha+\varepsilon$ ($C_{\alpha+\varepsilon}$ is closed). Since ε is arbitrarily small, then $f(x_0)\leq\alpha$. Thus, (2)→(3).

Finally we show that (3)→(1). Remember that f is lower semicontinuous at a point x_0 if for every sequence $x_k\to x_0$,

$$\liminf_k f(x_k)\geq f(x_0).$$

Thus, if $x_k^0\geq f(x_k), f(x_k^0, x_k)\to(x^0, x_0)$, then

$$x^0\geq\liminf_k f(x_k)\geq f(x_0);$$

i.e., $(x^0, x_0)\in$ epi f, which implies that epi f is closed.■

Theorem 1.20. If f is a closed convex function and $f(x_0)$ is finite at x_0, then $f(x)>-\infty$ everywhere.

□ Above, it was shown that if f takes the value $-\infty$, then $f(x)=-\infty$ for all $x\in$ ri dom f. But by Theorem 1.4, if $x\in$ dom f then $(1-\lambda)x+\lambda x_1\in$ ri dom f for all λ and $x_1\in$ ri dom f. Because f is closed,

$$f(x)\leq\lim_{\lambda\downarrow 0} f((1-\lambda)x+\lambda x_1)=-\infty;$$

i.e., $f(x)=-\infty$ for all $x\in$ dom f. Thus, if f is finite at even one point, then $f(x)>-\infty$ for all x.■

Definition 1.23. If f is a convex function, then the function $\bar{f}$ defined by

$$\begin{aligned}&\operatorname{epi}\bar{f}=\overline{\operatorname{epi} f}\\&\bar{f}(x)=\inf_{x^0}\{x^0 : (x^0, x)\in\overline{\operatorname{epi} f}\}\end{aligned}$$

is said to be *the closure of f*. Since the closure of a convex set is convex (Lemma 1.6), it follows that $\bar{f}$ is a convex function.

A convex function is said to be *closed* if $\bar{f}=f$.

Definition 1.24. A function f on $\mathbb{R}^n$ is said to be positively homogeneous (of degree one) if for every x one has

$$f(\lambda x)=\lambda f(x),\quad \lambda>0.$$

Clearly, positive homogeneity is equivalent to the epigraph's being a cone in $\mathbb{R}^{n+1}$. An example of a positively homogeneous convex function that is not simply a linear function is $f(x) = \|x\|$.
Furthermore, if f is convex, then

$$f(x_1 + x_2) = f\left(2\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)\right) = 2f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq f(x_1) + f(x_2)$$

or, more generally,

$$f(x_1 + \cdots + x_m) \leq f(x_1) + \cdots + f(x_m).$$

Setting in the previous inequality $x_1 = 0$, $x_2 = x$, we obtain $f(x) \leq f(0) + f(x)$; i.e., $f(0) \geq 0$.
Let f be a positively homogeneous closed proper convex function. By virtue of the closure of f,

$$\lim_{\lambda \downarrow 0} f(\lambda x) = \lim_{\lambda \downarrow 0} \lambda f(x) = 0 \geq f(0).$$

Thus, $f(0) = 0$ for homogeneous closed proper convex functions.

Now, let f be an arbitrary convex function on $\mathbb{R}^n$ not identically $+\infty$. The epigraph of f as a nonempty convex set in $\mathbb{R}^{n+1}$ has a recession cone $0^+(\text{epi } f)$. By definition, $(y^0, y) \in 0^+(\text{epi } f)$ if and only if

$$(x^0, x) + \lambda(y^0, y) = (x^0 + \lambda y^0, x + \lambda y) \in \text{epi } f$$

for every (x^0,x) epi f and $\lambda \geq 0$. This means that

$$f(x + \lambda y) \leq f(x) + \lambda y^0$$

for every x and $\lambda \geq 0$. By definition, the latter inequality holds for every x and every $\lambda \geq 0$ as long as it holds for every x with $\lambda = 1$. In any case, for a given y, the values of y^0 for which $(y^0, y) \in 0^+$ (epi f) will form either a closed interval of $\mathbb{R}$ unbounded above or the empty interval. Hence, 0^+ (epi f) is the epigraph of some function.

Definition 1.25. The recession function is denoted by $f0^+$ and defined as

$$\text{epi } (f0^+) = 0^+ (\text{epi } f).$$

Obviously, if f is a proper convex function, then the recession function $f0^+$ of f is a positively homogeneous proper convex function.

1.5 Conjugate of Convex Function

The definition of the conjugate of a function grows naturally out of the fact that the epigraph of a closed proper convex function on $\mathbb{R}^n$ is the intersection of the closed half-spaces in $\mathbb{R}^{n+1}$ that contain it. Remember that a closed convex set can be characterized completely by its support function

$$H_M(x^*) = \sup_x \{\langle x, x^* \rangle : x \in M\}$$

We now calculate the support function of the epigraph of a closed convex function f. Taking into account that $\operatorname{epi} f \subset \mathbb{R}^{n+1}$, we can write

$$H_{\operatorname{epi} f}(x^{0^*}, x^*) = \sup_{(x^0, x)} \{\langle x, x^* \rangle + x^0 x^{0^*} : (x^0, x) \in \operatorname{epi} f\}.$$

If $x^{0^*} > 0$, then for a given x the number x^0 may be an arbitrary number, larger than $f(x)$. Hence $H_{\operatorname{epi} f}(x^{0^*}, x^*) = +\infty$. It remains to calculate $H_{\operatorname{epi} f}$ when $x^{0^*} \leq 0$. Below, we will show that for the evaluation of $H_{\operatorname{epi} f}$, $x^{0^*} < 0$ when f is a closed convex function, it is sufficient to find $H_{\operatorname{epi} f}(-1, x^*)$. If $x^{0^*} = -1$, then

$$H_{\operatorname{epi} f}(-1, x^*) = \sup_{(x^0, x)} \{\langle x, x^* \rangle x^0 : x^0 \geq f(x)\} = \sup_x \{\langle x, x^* \rangle - f(x)\}.$$

Definition 1.26. The function

$$f^*(x^*) = \sup_x \{\langle x, x^* \rangle - f(x)\}$$

is called the conjugate of f.

Lemma 1.31. The conjugate function is closed and convex.
□ For a fixed x the function $\langle x, x^* \rangle - f(x)$ is linear in x^*, so it is closed and convex in x^*. The epigraph of $f^*(x^*)$ is the intersection of the epigraphs of the closed convex functions $\langle x, x^* \rangle - f(x)$. Therefore, $\operatorname{epi} f^*$ is closed and convex. ■

For example, consider the function $f(x) = (1/2)x^2$, $x \in \mathbb{R}$. Then, by definition,

$$f^*(x^*) = \sup_{x^*} \left\{ x \cdot x^* - \frac{1}{2} x^2 \right\} = \sup_{x^*} \left\{ \frac{1}{2} x^{*2} - \frac{1}{2} (x - x^*)^2 \right\};$$

i.e.,

$$f^*(x^*) = \frac{1}{2} x^{*2}, \quad x^* \in \mathbb{R}.$$

It is useful to remember, in particular, that Young's inequality,

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle,$$

holds for any function. If here, f is proper and convex, then we shall refer to this relation as Fenchel's inequality.
Taking conjugates clearly reverses inequalities: $f_1 \geq f_2$ implies $f_1{}^* \leq f_2{}^*$.

The polar of a set M is denoted by M^{O} and defined as

$$M^{\mathrm{O}} = \{x^* : H_M(x^*) \leq 1\}.$$

Note that if K is a convex cone, then

$$K^{\mathrm{O}} = \{x^* : \langle x, x^* \rangle \leq 0 \quad \forall x \in K\},$$

and so $K^* = -K^{\mathrm{O}}$. Let us calculate the conjugate of Minkowski's function $r_M(\cdot)$:

$$\begin{aligned} r_M^*(x^*) &= \sup\{\langle x,x^*\rangle - r_M(x)\} = \sup\{\langle x,x^*\rangle - \inf[\alpha>0 : \alpha^{-1}x \in M]\} \\ &= \sup\{\langle x,x^*\rangle - \alpha : \alpha>0, \alpha^{-1}x \in M\} = \sup\{\sup[\langle x,x^*\rangle : x \in \alpha M] - \alpha : \alpha>0\} \\ &= \sup_{\alpha>0}\{\alpha[\sup_{x\in M}\langle x,x^*\rangle - 1]\} = \delta_{M^{\mathrm{O}}}(x^*), \end{aligned}$$

where $\delta_{M^{\mathrm{O}}}(\cdot)$ is the indicator function of M^{O}.

Theorem 1.21. If f is a closed proper convex function, then

$$f = f^{**},$$

where, according to Definition 1.22,

$$f^{**}(x) = \sup_{x^*}\{\langle x, x^* \rangle - f^*(x^*)\}.$$

□ By Fenchel's inequality,

$$f(x) \geq \langle x, x^* \rangle - f^*(x^*),$$

and so

$$f(x) \geq f^{**}(x).$$

In particular, it follows that $\mathrm{dom}\, f \subseteq \mathrm{dom}\, f^{**}$. Let $x_0 \in \mathrm{dom}\, f$. Choose $\gamma < f(x_0)$ and consider in $\mathbb{R}^{n+1}$ the following set:

$$P_\varepsilon = \{(x^0, x) : x - x_0 \in \varepsilon B, \quad x^0 \leq \gamma\},$$

where B is the unit ball in $\mathbb{R}^n$. Since $\mathrm{epi}\, f$ is closed, then for sufficiently small ε the sets P_ε and $\mathrm{epi}\, f$ are disjoint. We argue by contradiction. Suppose that there exists a sequence $(x_k^0, x_k) \in P_{\varepsilon_k}$, $\varepsilon_k \to 0$, $x_k^0 \leq \gamma$ such that $(x_k^0, x_k) \in \mathrm{epi}\, f$. Since $\varepsilon_k \to 0$,

$x_k \to x_0$, $\lim_{k\to\infty} x_k^0 \le \gamma$, and by virtue of the fact that epi f is closed, $\gamma \ge \lim_{k\to\infty} x_k^0 \ge f(x_0)$. This contradiction proves that epi $f \cap P_\varepsilon = \varnothing$. Thus we can separate epi f and P_ε. Then there is a nonzero vector (x^{0^*}, x^*) such that

$$\begin{aligned} x^{0^*} y^0 + \langle y, x^* \rangle &\ge x^{0^*} x^0 + \langle x, x^* \rangle \\ (y^0, y) \in P_\varepsilon, &\quad (x^0, x) \in \operatorname{epi} f. \end{aligned} \tag{1.36}$$

Since y^0 can be chosen as any number less than γ, from Eq. (1.36), it follows that $x^{0^*} \le 0$. Actually, we show that $x^{0^*} < 0$. If $x^{0^*} = 0$, then Eq. (1.36) implies that

$$\langle y, x^* \rangle \ge \langle x, x^* \rangle, \quad y - x_0 \in \varepsilon B, \; x \in \operatorname{dom} f.$$

Setting $x = x_0$, we have

$$\langle y - x_0, x^* \rangle \ge 0, \quad y - x_0 \in \varepsilon B.$$

By virtue of the arbitrariness of $y - x_0 \in \varepsilon B$, this inequality is satisfied only if $x^* = 0$. In turn, this contradicts the fact that (x^{0^*}, x^*) is nonzero.
Let now $x^{0^*} = -1$. Then Eq. (1.36) takes the form

$$\begin{aligned} -y^0 + \langle y, x^* \rangle &\ge -x^0 + \langle x, x^* \rangle, \\ y^0 \le \gamma, \quad y - x_0 &\in \varepsilon B, \quad x^0 \ge f(x). \end{aligned} \tag{1.37}$$

Let $y^0 = \gamma$, $x^0 = f(x)$, $y = x_0$. Thus, from Eq. (1.37) we obtain

$$\langle x_0, x^* \rangle - \gamma \ge \sup_x \{ \langle x, x^* \rangle - f(x) \} = f^*(x^*).$$

or

$$\gamma \le \langle x_0, x^* \rangle - f^*(x^*),$$

whence

$$\gamma \le \sup_x \{ \langle x_0, x^* \rangle - f(x^*) \} = f^{**}(x_0). \tag{1.38}$$

Since γ is an arbitrary number less than $f(x_0)$, Eq. (1.36) implies that

$$f(x_0) \le f^{**}(x_0).$$

Comparing this inequality with $f(x_0) \ge f^{**}(x_0)$, we conclude that $f(x_0) = f^{**}(x_0)$.

Now assume that $x_0 \notin \operatorname{dom} f$; i.e., $f(x_0) = +\infty$. In just the same way as before, taking arbitrary γ it can be shown that epi $f \cap P_\varepsilon = \varnothing$ for a small ε and Eq. (1.36) holds.

If for arbitrarily large γ we have $x^{0^*} < 0$, then Eq. (1.38) holds for γ, and so $f^{**}(x_0) = +\infty$.

Now let $x^{0^*} = 0$ for some γ. Then from Eq. (1.36), we obtain

$$\langle y, x^* \rangle \ge \langle x, x^* \rangle, \quad y - x_0 \in \varepsilon B, \; x \in \operatorname{dom} f,$$

or

$$\langle y - x_0, x^* \rangle \geq \langle x - x_0, x^* \rangle, \quad y - x_0 \in \varepsilon B, \ \ x \in \operatorname{dom} f.$$

Thus,

$$-\varepsilon \|x^*\| \geq \sup_{x \in \operatorname{dom} f} \langle x, x^* \rangle - \langle x_0, x^* \rangle \tag{1.39}$$

Remember that there exists a point x_1 such that $f(x_1)$ is a finite number. Hence, by substituting $\gamma < f(x_1)$, $y = x_1$ into Eq. (1.37), we derive:

$$-\gamma + \langle x_1, x^* \rangle \geq -f(x) + \langle x, x^* \rangle;$$

i.e., $f^*(x^*) \leq -\gamma + \langle x_1, x^* \rangle$. Thus, there is a vector x_1^* with finite $f(x_1^*)$. It follows that

$$f^*(x_1^* + \alpha x^*) = \sup_x \{\langle x, x_1^* + \alpha x^* \rangle - f(x)\} \leq f^*(x_1^*) + \alpha \sup_{x \in \operatorname{dom} f} \langle x, x^* \rangle$$

and hence, taking into account Eq. (1.39),

$$\begin{aligned} f(x) = f^{**}(x_0) &\geq \langle x_0, x_1^* + \alpha x^* \rangle - f^*(x_1^* + \alpha x^*) \\ &\geq \langle x_0, x_1^* \rangle - f^*(x_1^*) + \alpha[\langle x_0, x^* \rangle - \sup_{x \in \operatorname{dom} f} \langle x, x^* \rangle] \\ &\geq \langle x_0, x_1^* \rangle - f^*(x_1^*) + \alpha\varepsilon \|x^*\| \to +\infty \quad \text{as } \alpha \to +\infty. \blacksquare \end{aligned}$$

Theorem 1.22. If f is closed, proper, and convex, then f^* is closed, proper, and convex.

□ By Lemma 1.31 f^* is closed and convex. It remains to prove that it is proper. Because $f(x_1)$ is a finite number for some $x_1 \in \operatorname{dom} f$, then

$$f^*(x^*) \geq \langle x_1, x^* \rangle - f(x_1),$$

and so f^* is not equal to $-\infty$. In the proof of the preceding theorem, we saw that there exists x_1^* with $f^*(x_1^*)$ finite. This means that f is proper. ■

Corollary 1.1. If f is lower semicontinuous at $x_0 \in \operatorname{dom} f$, then $f(x_0) = f^{**}(x_0)$.
The conjugate function of f, a closed proper convex positively homogeneous function, is

$$f^*(x^*) = \sup_x \{\langle x, x^* \rangle - f(x)\} \geq -f(0) = 0.$$

If there exists a point x_1 such that

$$\langle x_1, x^* \rangle - f(x_1) > 0,$$

then

$$f^*(x^*) \geq \sup_{\lambda>0}\{\langle \lambda x_1, x^*\rangle - f(\lambda x_1)\} = \sup_{\lambda>0} \lambda[\langle x_1, x^*\rangle - f(x_1)] = +\infty.$$

Thus, $f^*(x^*)$ take only two values, 0 and $+\infty$:

$$f^*(x^*) = \delta_{\operatorname{dom} f^*}(x^*) = \begin{cases} 0, & x^* \in \operatorname{dom} f^*, \\ +\infty, & x^* \notin \operatorname{dom} f^*. \end{cases}$$

Theorem 1.23. If f is closed, proper, convex, and positively homogeneous, then

$$f^*(x^*) = \delta_{\operatorname{dom} f^*}(x^*)$$

and $\operatorname{dom} f^*$ is a closed set.
☐ It remains only to prove that $\operatorname{dom} f^*$ is closed. Actually, f^* is a closed function. Then, it is not hard to see that the indicator function of $\operatorname{dom} f^*$ is closed if and only if $\operatorname{dom} f^*$ is closed. ■

Let $f(x) = |x|$, $x \in \mathbb{R}$. This function is positively homogeneous, and by Theorem 1.22, its conjugate is $\delta_{\operatorname{dom} f^*}$. Indeed,

$$f^*(x^*) = \begin{cases} 0, & \text{if } |x^*| \leq 1, \\ +\infty, & \text{if } |x^*| > 1, \end{cases}$$

and $\operatorname{dom} f^* = \{x^* \in \mathbb{R}: |x^*| \leq 1\}$; i.e., $f^*(x^*) = \delta_{\operatorname{dom} f^*}(x^*)$.

Theorem 1.24. If f is positively homogeneous, closed, proper, and convex, then

$$f(x) = \sup_{x^*}\{\langle x, x^*\rangle : x^* \in \operatorname{dom} f^*\}.$$

☐ We apply Theorem 1.21. Indeed, we have $f(x) = f^{**}(x)$. Then, by Theorem 1.23, $f^* = \delta_{\operatorname{dom} f^*}(\cdot)$ and

$$f^{**}(x) = \sup_{x^*}\{\langle x, x^*\rangle - \delta_{\operatorname{dom} f^*}(x^*)\} = f^*(x^*) = \sup_{x^*}\{\langle x, x^*\rangle : x^* \in \operatorname{dom} f^*\}. \blacksquare$$

Theorem 1.25. The conjugate function of the support function of a closed convex set is the indicator function of this set.
☐ Let C^* be a closed convex set in $x^* = \mathbb{R}^n$ and

$$f(x) = \sup_{x^*}\{\langle x, x^*\rangle : x^* \in C^*\}. \tag{1.40}$$

Obviously, f is a closed convex positively homogeneous function. Besides, f is the support function of C^*.

Consider the indicator function $\delta_{C^*}(x^*)$ of the set C^*. Clearly, its conjugate function is

$$\sup_{x^*}\{\langle x, x^*\rangle - \delta(x^*|C^*)\} = \sup_{x^*}\{\langle x, x^*\rangle : x^* \in C^*\} = f(x).$$

Thus, f is the conjugate function of $\delta_{C^*}(x^*)$. Finally, by Theorem 1.20, we obtain

$$f^*(x^*) = \delta(x^*|C^*), \quad f^* = C^*. \blacksquare$$

1.6 Directional Derivatives and Subdifferentials

In general, convex functions are not differentiable. Nevertheless, these functions have many useful "differential" properties, and one of them is the fact that directional derivatives exist universally. Moreover, for convex functions, a notion of *subgradient* can be defined and the set of subgradients yields the *subdifferential* conception.

Recall that the directional derivative of f at x with respect to a vector $p \in X = \mathbb{R}^n$ is defined to be the limit

$$f'(x, p) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda p) - f(x)}{\lambda},$$

if it exists.

Lemma 1.32. Let f be a proper convex function and $x_0 \in \operatorname{dom} f$. Then the value $f'(x_0, p)$, finite or not, exists for all p.

$\square$ If $x_0 + \lambda p \notin \operatorname{dom} f$ for all $\lambda > 0$, then $f(x_0 + \lambda p) = +\infty$ and $f'(x_0, p) = +\infty$. If $x_0 + \lambda p \notin \operatorname{dom} f$ for a small λ, then according to Lemma 1.31 the quotient

$$\frac{f(x_0 + \lambda p) - f(x_0)}{\lambda}$$

is a nonincreasing function of λ, when $\lambda \downarrow 0$. Therefore, the limit

$$f'(x, p) = \lim_{\lambda \downarrow 0} \frac{f(x_0 + \lambda p) - f(x_0)}{\lambda}$$

exists. $\blacksquare$

Lemma 1.33. If $x_0 + \lambda p \notin \operatorname{dom} f$ for $\lambda \in [-\varepsilon, +\varepsilon]$, $\varepsilon > 0$, then the value $f'(x_0,p)$ is finite.
☐ By the second inequality of Lemma 1.31, setting $\alpha_0 = -\varepsilon$, $\alpha_1 = 0$, $\alpha_2 = \lambda$, we have

$$\frac{f(x_0) - f(x_0 - \varepsilon p)}{\varepsilon} \leq \frac{f(x_0 + \lambda p) - f(x_0)}{\lambda}.$$

Whence, by passing to the limit as $\lambda \downarrow 0$, we have

$$\frac{f(x_0 + \lambda p) - f(x_0)}{\lambda} \geq f'(x_0, p) \geq \frac{f(x_0) - f(x_0 - \varepsilon p)}{\varepsilon}. \blacksquare$$

Lemma 1.34. The directional derivative $f'(x_0,p)$ is a positively homogeneous convex function of p.
☐ By definition of a directional derivative, for $\alpha > 0$:

$$f'(x_0, \alpha p) = \lim_{\lambda \downarrow 0} \frac{f(x_0 + \lambda \alpha p) - f(x_0)}{\lambda} = \alpha \lim_{\lambda \downarrow 0} \frac{f(x_0 + \lambda \alpha) p - f(x_0)}{(\lambda \alpha)} = \alpha f'(x_0, p).$$

Furthermore, for $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned}\frac{f(x_0 + \lambda(\lambda_1 p_1 + \lambda_2 p_2)) - f(x_0)}{\lambda} &= \frac{f(\lambda_1(x_0 + \lambda p_1) + \lambda_2(x_0 + \lambda p_2)) - \lambda_1 f(x_0) - \lambda_2 f(x_0)}{\lambda} \\ &\leq \lambda_1 \frac{f(x_0 + \lambda p_1) - f(x_0)}{\lambda} + \lambda_2 \frac{f(x_0 + \lambda p_2) - f(x_0)}{\lambda}\end{aligned}$$

Thus, by passing to the limit as $\lambda \downarrow 0$, we obtain

$$f'(x_0, \lambda_1 p_1 + \lambda_2 p_2) \leq \lambda_1 f'(x_0, P_1) + \lambda_2 f'(x_0, p_2). \blacksquare$$

Definition 1.27. A vector x^* is said to be a subgradient of the proper convex function f at the point $x_0 \in \operatorname{dom} f$ if

$$f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle \quad \forall x.$$

This condition says that the graph of the affine function $\alpha(x) = f(x_0) + \langle x^*, x - x_0 \rangle$ is the nonvertical supporting hyperplane to the convex set epi f at the point $(x_0, f(x_0))$.

Lemma 1.35. Let f be a proper convex function at $x_0 \in \operatorname{dom} f$. Then x^* is a subgradient of f at x_0 if and only if

$$f'(x_0, p) \geq \langle p, x^* \rangle \quad \forall p. \tag{1.41}$$

□ If x^* is a subgradient, then

$$f(x_0 + \lambda p) - f(x_0) \geq \lambda \langle p, x^* \rangle,$$

whence

$$\frac{f(x_0 + \lambda p) - f(x_0)}{\lambda} \geq \langle p, x^* \rangle$$

and $f'(x_0, p) \geq \langle p, x^* \rangle$. Conversely, if Eq. (1.41) is satisfied, then for all $0 < \lambda < 1$,

$$\begin{aligned} f(x) - f(x_0) &= \frac{f(x_0 + 1(x - x_0)) - f(x_0)}{1} \\ &\geq \frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda} \geq f'(x_0, x - x_0) \geq \langle x - x_0, x^* \rangle; \end{aligned}$$

i.e., x^* is a subgradient. ■

Definition 1.28. The set of subgradients of f at a point x_0 is called the subdifferential of f at x_0 and is denoted $\partial f(x_0)$:

$$\partial f(x_0) = \{x^* : f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle \quad \forall x\}.$$

In general, $\partial f(x_0)$ may be empty. If $\partial f(x_0)$ is not empty, f is said to be subdifferentiable at x_0. For example, the Euclidean norm $f(x) = \|x\|$ is subdifferentiable at every x, although it is not differentiable at $x = 0$ and

$$\partial f(0) = \{x^* : \|x\| \geq \langle x^*, x \rangle \quad \forall x\}.$$

In other words, it is the Euclidean unit ball. For $x \neq 0$, the subdifferential $\partial f(x) = \{\|x\|^{-1} x\}$. In particular, the subdifferential of the function $f(x) = |x|$, $x \in \mathbb{R}$ at $x = 0$ is $\partial f(0) = [-1, +1]$.

Lemma 1.36. The subdifferential $\partial f(x_0)$ is a closed convex set.

□ The proof is obtained by direct verification of the definition. ■

It follows from Lemma 1.35 that

$$\partial f(x_0) = \partial_p f'(x_0, 0),$$

where ∂_p is a subdifferential of $f'(x_0,p)$ with respect to p. Actually, by definition, $f'(x_0,0)=0$, so Eq. (1.41) can be rewritten as follows:

$$f'(x_0,p)-f'(x_0,0)\geq\langle p,x^*\rangle,$$

so $x^*\in\partial_p f'(x_0,0)$.

Theorem 1.26. If $f'(x_0,\cdot)$ is closed, then $\partial f(x_0)$ is nonempty and

$$f'(x_0,p)=\sup_{x^*}\{\langle p,x^*\rangle : x^*\in\partial f(x_0)\}.$$

□ Since $f'(x_0,\cdot)$ is positively homogeneous, closed, and convex, by Theorem 1.24,

$$f'(x_0,p)=\sup_{x^*}\{\langle p,x^*\rangle : x^*\in\operatorname{dom}(f'(x_0,\cdot))^*\}. \tag{1.42}$$

where $(f'(x_0,\cdot))^*$ is the conjugate function of $f'(x_0,p)$ relative to p; i.e.,

$$(f'(x_0,\cdot))^*(x^*)=\sup_p\{\langle p,x^*\rangle-(f'(x_0,p)\}. \tag{1.43}$$

Now, recalling that

$$(f'(x_0,\cdot))^*(x^*)=\begin{cases}0, & x^*\in\operatorname{dom}(f'(x_0,\cdot))^*,\\ +\infty, & x^*\notin\operatorname{dom}(f'(x_0,\cdot))^*,\end{cases}$$

and so taking into account Eq. (1.43), we have that $x^*\in\operatorname{dom}(f'(x_0,\cdot))^*$ if and only if

$$0\geq\langle p,x^*\rangle-f'(x_0,p),$$

which is equivalent to Eq. (1.41). It follows that

$$\partial f(x_0)=\partial_p f'(x_0,0)=\operatorname{dom}(f'(x_0,\cdot))^*.$$

The latter formula together with Eq. (1.42) ends the proof of the theorem. ■

Theorem 1.27. $x^*\in\partial f(x_0)$ if and only if $\langle x_0,x^*\rangle-f(x_0)=f^*(x^*)$.
□ In fact, by definition, $x^*\in\partial f(x_0)$ if

$$\langle x_0,x^*\rangle-f(x_0)\geq\langle x,x^*\rangle-f(x).$$

Thus,

$$\langle x_0, x^* \rangle - f(x_0) \geq \sup\{\langle x, x^* \rangle - f(x)\}$$

and so

$$\langle x_0, x^* \rangle - f(x_0) \geq f^*(x^*).$$

On the other hand, by Fenchel's inequality, $\langle x_0,x^* \rangle \leq f^*(x^*)$, and so

$$\langle x_0, x^* \rangle - f(x_0) = f^*(x^*). \blacksquare$$

Corollary 1.2. If f is a closed proper convex function, then $x^* \in \partial f(x_0)$ if and only if $x_0 \in \partial f^*(x^*)$.

☐ By Theorem 1.26, $x_0 \in \partial f^*(x^*)$ is equivalent to $\langle x_0,x^* \rangle - f^*(x^*) = f^{**}(x_0)$, which by Theorem 1.20, $f(x_0) = f^{**}(x_0)$, coincides with the condition of Theorem 1.26. ■

Remark 1.2. It can be shown that for a convex function f, continuous at x_0, its subdifferential $\partial f(x_0)$ is a nonempty bounded set. Then, Theorem 1.26 has the form

$$f'(x_0, p) = \max_{x^*}\{\langle p, x^* \rangle : x^* \in \partial f(x_0)\}.$$

In this direction we particularly refer the reader, for example, to Ref. [224, p. 74].

☐ Actually, $x_0 \in \partial f^*(x^*)$ is equivalent to

$$\langle x_0, x^* \rangle - f^{**}(x_0) = f^*(x^*),$$

and this coincides with the condition $\langle x_0,x^* \rangle - f(x_0) = f^*(x^*)$ of Theorem 1.26, if $f(x_0) = f^{**}(x_0)$. ■

Theorem 1.28. If f is a convex function differentiable at x_0, then the subdifferential consists of the single gradient vector at this point, i.e.,

$$\partial f(x_0) = \{f'(x_0)\}.$$

☐ For a function f, differentiable at x_0, we can write:

$$f'(x_0, p) = \langle p, f'(x_0) \rangle,$$

and hence, $f'(x_0,p)$ is a closed function of p. By Theorem 1.26, we then have

$$\langle p, f'(x_0) \rangle = \sup_{x^*}\{\langle p, x^* \rangle : x^* \in \partial f(x_0)\}.$$

In turn, it is easy to see that this formula holds, if $\partial f(x_0)$ consists of the point $f'(x_0)$. ■

In Section 1.4, we have seen that multiplication by a positive constant, addition, and pointwise supremum of convex functions are again convex. So it is important to calculate the behavior of subdifferentials under these operations.

Lemma 1.37. Let f_0 be a convex function and $\alpha > 0$. Then $f(x) = \alpha f_0(x)$ is subdifferentiable and

$$\partial f(x_0) = a\partial f_0(x_0).$$

□ The proof is elementary. ■

Lemma 1.38. Let f_1, f_2 be proper convex functions and let $f = f_1 + f_2$. Then

$$\partial f(x_0) \supseteq \partial f_1(x_0) + \partial f_2(x_0).$$

□ Suppose $x_1^* \in \partial f_1(x_0)$, $x_2^* \in \partial f_2(x_0)$, or, equivalently,

$$f_1(x) - f_1(x_0) \geq \langle x - x_0, x_1^* \rangle \quad \forall x$$
$$f_2(x) - f_2(x_0) \geq \langle x - x_0, x_2^* \rangle \quad \forall x.$$

Adding these inequalities, we obtain $x_1^* + x_2^* \in \partial f(x_0)$. Thus, $\partial f \supseteq \partial f_1 + \partial f_2$. ■

Theorem 1.29. (Moreau–Rockafellar) Let f_1, f_2 be a proper convex function and $f = f_1 + f_2$, $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that either (1) there is a point $x_1 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ where f_1 is continuous or (2) $\operatorname{ri} \operatorname{dom} f_1 \cap \operatorname{ri} \operatorname{dom} f_2 \neq \varnothing$. Then

$$\partial f(x_0) = \partial f_1(x_0) + \partial f_2(x_0).$$

□ For the proof, refer to Refs. [111:59,224:78,226:78,228:223]. ■

We next calculate the subdifferential of the indicator function $\partial\delta_M(x_0)$, $x_0 \in M$. By definition $x^* \in \partial\delta_M(x_0)$ if and only if

$$\partial\delta_M(x) - \partial\delta_M(x_0) \geq \langle x - x_0, x^* \rangle \quad \forall x.$$

But $\partial\delta_M(x_0) = 0$. If $x \notin M$, then by definition of the indicator function, $\partial\delta_M(x) = +\infty$ and the last inequality is always fulfilled. For $x \in M$, we obtain

$$0 \geq \langle x - x_0, x^* \rangle. \tag{1.44}$$

Let

$$\operatorname{cone}(M - x_0) = \{p : p = \lambda(x - x_0), \quad \lambda > 0, \quad x \in M\}.$$

Then the inequality in Eq. (1.44) is equivalent to

$$-x^* \in (\mathrm{cone}(M - x_0))^*.$$

Thus,

$$\partial \delta_M(x_0) = -(\mathrm{cone}(M - x_0))^*. \tag{1.45}$$

Now, if K_i, $i = 1, \ldots, m$, are convex cones and $0 \in K_i$, then $\mathrm{cone}(K_i - 0) = K_i$, and so

$$\partial \delta_{K_i}(0) = K_i^*. \tag{1.46}$$

Observe that

$$\delta_{\bigcap\limits_{i=1}^{m} K_i}(x) = \delta_{K_1}(x) + \cdots + \delta_{K_m}(x). \tag{1.47}$$

Theorem 1.30. Suppose that $\mathrm{ri}\ K_1 \cap \cdots \cap \mathrm{ri}\ K_m \neq \varnothing$. Then

$$(K_1 \cap \cdots \cap K_m)^* = K_1^* + \cdots + K_m^*.$$

□ Clearly, $\mathrm{dom}\delta_{K_i}(\cdot) = K_i$. And so, by hypothesis,

$$\mathrm{ri\ dom}\ \ \delta(\cdot|K_i) \cap \cdots \cap \mathrm{ri\ dom}\ \ \delta(\cdot|K_m) \neq \varnothing.$$

Then taking into account Eqs. (1.46) and (1.47) by Theorem 1.29, we have

$$-\left(\bigcap_{i=1}^{m} K_i\right)^* = \partial \delta_{\bigcap\limits_{i=1}^{m} K_i}(0) = \partial \delta_{K_1}(0) + \cdots + \partial \delta_{K_m}(0) = -K_1^* - \cdots - K_m^*. \blacksquare$$

Theorem 1.31. Let M^* be a closed convex set and

$$f(x) = \sup_{x^*}\{\langle x, x^* \rangle : x^* \in M^*\}.$$

Then

$$\partial f(x_0) = \{x^* \in M^* : \langle x_0, x^* \rangle = f(x_0)\}.$$

In particular, if $x_0 = 0$, then $\partial f(0) = M^*$.
□ If $x^* \in M^*$, $\langle x_0, x^* \rangle = f(x_0)$, then

$$f(x) - f(x_0) \geq \langle x, x^* \rangle - \langle x_0, x^* \rangle = \langle x - x_0, x^* \rangle;$$

i.e., $x^* \in \partial f(x_0)$.

Conversely, let $x_0^* \in \partial f(x_0)$. We must prove that $x_0^* \in M^*$. Suppose the contrary, i.e., that $x_0^* \notin M^*$. Then by Theorem 1.5 there is vector p such that

$$\sup_{x^*}\{\langle p, x^*\rangle : x^* \in M^*\} < \langle p, x_0^*\rangle. \tag{1.48}$$

On the other hand, it is not hard to see that

$$\sup_{x^*}\{\langle x - x_0, x^*\rangle : x^* \in M^*\} \geq f(x) - f(x_0) \geq \langle x - x_0, x_0^*\rangle.$$

Taking $x = x_0 + p$ in this inequality, we obtain

$$\sup_{x^*}\{\langle p, x^*\rangle : x^* \in M^*\} \geq \langle p, x_0^*\rangle,$$

which contradicts the inequality (1.48). Hence, $x_0^* \in M^*$. By assumption $x_0^* \in \partial f(x_0)$ and so

$$f(x) - \langle x, x_0^*\rangle \geq f(x_0) - \langle x_0, x_0^*\rangle.$$

Setting $x = 0$, we have $\langle x_0, x_0^*\rangle \geq f(x_0)$. On the other hand, the hypothesis $x_0^* \in M^*$ implies that $f(x_0) \geq \langle x_0, x_0^*\rangle$. Consequently, $\langle x_0, x_0^*\rangle = f(x_0)$. Furthermore, if $x_0 = 0$, then $f(0) = \langle 0, x^*\rangle = 0$ for all $x^* \in M^*$, and so $\partial f(0) = M^*$. ■

Theorem 1.32. Let A be compact and $f(x,\alpha)$ be convex in x for every fixed $\alpha \in A$ and continuous in some neighborhood of x_0 and in $\alpha \in A$. Then

$$\partial f(x_0) = \overline{\text{conv}}\left(\bigcup_{\alpha \in A(x_0)} \partial f(x_0, \alpha)\right),$$

where

$$f(x) = \sup_{\alpha}\{f(x, \alpha) : \alpha \in A\}, \quad A(x) = \{\alpha \in A : f(x, \alpha) = f(x)\}.$$

□ For the proof, refer to Refs. [111:225,226:87,214:114]. ■

Remember that the cone generated by $M - x_0$ is defined as follows:

$$\text{cone}(M - x_0) = \{p : p = \lambda(x - x_0), \quad \lambda > 0, \quad x \in M\},$$

where $x_0 \in M$ and M is a convex set. In the theory of extremal problems, one of the important problems is to calculate the dual cone, $[\text{cone}(M - x_0)]^*$.

Theorem 1.33. Let f be a convex function and

$$M = \{x : f(x) \leq 0\}$$

and suppose that $x_1 \in M$ is such that $f(x_1) < 0$. If $f(x_0) = 0$ and $f'(x_0, \cdot)$ is a closed function, then

$$[\text{cone}(M - x_0)]^* = -\text{cone}\, \partial f(x_0).$$

☐ Since

$$0 > f(x_1) = f(x_1) - f(x_0) \geq \langle x_1 - x_0, x^* \rangle, \quad x^* \in \partial f(x_0),$$

the point zero does not belong to the closed convex set $\partial f(x_0)$. We show that

$$\text{cone}\, \partial f(x_0) = \{x^* : x^* = \lambda x_0^*, \quad \lambda \geq 0, \quad x_0^* \in \partial f(x_0)\}$$

is a closed set. Indeed, let λ_k and x_k^* be sequences such that $\lambda_k x_k^* \to x^*$, $\lambda_k > 0$, $x_k^* \in \partial f(x_0)$, $x^* \neq 0$. Then the sequence λ_k is bounded, for suppose the contrary: if $\lambda_k \to +\infty$, then $\lambda_k \|x_k^*\| \to \|x^*\|$, and so $\|x_k^*\| \to 0$. This contradiction ($0 \notin \partial f(x_0)$) proves that λ_k is bounded. By virtue of the boundedness of λ_k, assume that $\lambda_k \to \lambda_0 > 0$. Hence, x_k^* converges to $\lambda_0^{-1} x^*$ and since $\partial f(x^*)$ is closed, $\lambda_0^{-1} x^* \in \partial f(x_0)$. Thus, $x^* \in \text{cone}\, \partial f(x_0)$ and cone $\partial f(x_0)$ is closed.

By definition, $p \in [-\text{cone}\, \partial f(x_0)]^*$ if and only if

$$\langle p, -\lambda x_0^* \rangle \geq 0, \quad \lambda > 0, \quad x_0^* \in \partial f(x_0);$$

i.e.,

$$\sup_{x_0^*} \{\langle p, x_0^* \rangle : x_0^* \in \partial f(x_0)\} \leq 0. \tag{1.49}$$

Since by hypothesis, $f'(x_0, \cdot)$ is closed, then by Theorem 1.26 and Eq. (1.49), we have

$$[-\text{cone}\, \partial f(x_0)]^* = \{p : f'(x_0, p) \leq 0\}.$$

Take $p_1 = x_1 - x_0$. Then for $0 < \lambda < 1$,

$$f'(x_0, p_1) \leq \frac{f(x_0 + \lambda p_1) - f(x_0)}{\lambda} \leq f(x_1) - f(x_0) < 0,$$

and by virtue of the convexity of $f'(x_0, p)$ relative to p,

$$f'(x_0, \ \lambda p_1 + (1 - \lambda)p) \leq \lambda f'(x_0, p_1) + (1 - \lambda) f'(x_0, p) < 0 \tag{1.50}$$

if $f'(x_0, p) \leq 0$.

Now, let $p = \lambda(x - x_0)$, $x \in M$, $\lambda > 0$; i.e., $p \in \text{cone}(M - x_0)$. Then $f(x) \leq 0$ and

$$0 \geq f(x) - f(x_0) \geq \langle x - x_0, \ x^* \rangle, \quad x^* \in \partial f(x_0).$$

Therefore,

$$\sup_{x^*}\{\langle p,x^*\rangle : x^*\in\partial f(x_0)\} = \lambda \sup_{x^*}\{\langle x-x_0,x^*\rangle : \quad x^*\in\partial f(x_0)\} \le 0.$$

It follows that

$$\mathrm{cone}(M-x_0)\subseteq\{p : f'(x_0,p)\le 0\}.$$

On the other hand, if $f'(x_0, p) < 0$, then for small $\lambda > 0$,

$$f(x_0+\lambda p) = f(x_0) + \lambda f'(x_0,p) + o(\lambda) = \lambda f'(x_0,p) + \mathrm{o}(\lambda) < 0,$$

and so $x_0+\lambda p\in M$. The latter implies that $p\in\mathrm{cone}(M-x_0)$. Now, it follows from Eq. (1.50) that $\lambda p_1+(1-\lambda)p\in\mathrm{cone}(M-x_0)$ if $p\in\{p{:}f'(x_0,p)\le 0\}$. We obtain that every point of $\mathrm{cone}\{p{:}f'(x_0,p)\le 0\}$ is a limit point of $\mathrm{cone}(M-x_0)$ as λ tends to zero.

Since the $\mathrm{cone}\{p{:}(f'(x_0,\ p)\le 0)\}$ is closed, $(f'(x_0,\ p)$ is closed relative to p and Theorem 1.19 is valid), and

$$\overline{\mathrm{cone}(M-x_0)}\supseteq\{p : f'(x_0,p)\le 0\},$$

then

$$\{p : f'(x_0,p)\le 0\} = \overline{\mathrm{cone}(M-x_0)}.$$

Because

$$\{p : f'(x_0,p)\le 0\} = [-\mathrm{cone}\,\partial f(x_0)]^*,$$

we have

$$[-\mathrm{cone}\,\partial f(x_0)]^* = \overline{\mathrm{cone}(M-x_0)}.$$

Since cone $\partial f(x_0)$ is closed by Lemmas 1.13 and 1.15, we obtain

$$-\mathrm{cone}\,\partial f(x_0) = [-\mathrm{cone}\ \ \partial f(x_0)]^{**} = [\overline{\mathrm{cone}(M-x_0)}]^* = [\mathrm{cone}(M-x_0)]^*. \blacksquare$$

Theorem 1.34. Let f be a convex function, continuous at $x_0\in M$, $M=\{x{:}\ f(x)\le 0\}$, and suppose that there is a point x_1 such that $f(x_1)<0$. Then

$$[\mathrm{cone}(M-x_0)]^* = \begin{cases}\{0\}, & f(x_0)<0,\\ -\mathrm{cone}\,\partial f(x_0), & f(x_0)=0.\end{cases}$$

□ If $f(x_0) < 0$, then by virtue of the continuity of f at x_0, we derive that $x_0 + \lambda p \in M$ for every P and sufficiently small $\lambda > 0$. Thus, $\operatorname{cone}(M - x_0) = X = \mathbb{R}^n$. Hence the dual cone of cone $(M - x_0)$ consists of a single point, which is the first assertion of the theorem. The second assertion is a consequence of the previous theorem and Theorem 1.26. ■

2 Multivalued Locally Adjoint Mappings

2.1 Introduction

In this chapter, we first introduce the basic definitions and properties of multivalued (set-valued) mappings (functions) $F: X \to P(Y)$, where X and Y are finite dimensional Euclidean spaces. The reader can find more details and discussions on major achievements in this direction in Refs. [9,14,18,22,38–40,85,87,93,141,191, 231,238,243,265]. In this simple case, although $X = X^*$, $Y = Y^*$, $Z = Z^*$, in general, we denote the elements of the adjoint spaces by x^*, y^*, z^*, respectively. We prove some of their basic topological properties such as their boundedness and the Lipschitz conditions they satisfy (in the Hausdorff metric). In particular, for quasi-superlinear mappings, we formulate a boundedness condition.

Local properties of differentiable functions are well described by the concept of the derivative and the gradient function. For convex functions, instead of the gradient, we use subdifferentials. In the case of multivalued functions, a similar role is played by an important concept of this book: the locally adjoint mappings.

For a convex multivalued mapping F, the multivalued mapping F^* from Y^* to X^* defined by $F^*(y^*; z) = \{x^* : (x^*, -y^*) \in K^*_{\text{gph } F}(x, y)\}$ is called the locally adjoint mapping (LAM) to F at the point $(x, y) \in \text{gph } F$, where $K^*_{\text{gph } F}(x, y)$ is the dual to the cone $K_{\text{gph } F}(z) = \text{cone}(\text{gph } F - z)$. The adjoint mapping (AM) F^* defined by the recession cone 0^+ gph F is bounded if and only if dom $F = X$.

By Theorem 2.1, all sets $F^*(y^*;(x, y))$ for every $y \in F(x; y^*)$ are the same and equal to $\partial_x H(x, y^*)$, where $H(x, y^*) = \sup_y \{\langle y, y^* \rangle : y \in F(x)\}$ is the Hamiltonian function, which is concave in x and convex in y^*, and $F(x; y^*)$ is an Argmaximum set. The LAM, $F^*(y^*; z)$, for a closed bounded convex multivalued mapping F is upper semicontinuous and uniformly bounded (Theorems 2.2 and 2.3). By Lemma 2.6, x^* is an element of the LAM F^* if and only if $M(x^*, y^*) = \langle x, x^* \rangle - H(x, y^*)$, where

$$M(x^*, y^*) = \inf_{(x,y)} \{\langle x, x^* \rangle - \langle y, y^* \rangle : (x, y) \in \text{gph } F\}.$$

By Theorem 2.5, for a closed convex multivalued mapping F, its AM is the closure of union of locally adjoint mappings over $z \in \text{gph } F$.

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If F is a convex multivalued mapping and $\alpha \neq 0$ is any real number, then

$$G^*[y^*;(x,\alpha y)] = |\alpha| F^*[y^* \operatorname{sgn} \alpha;(x,y)],$$

where $G(x) = \alpha F(x)$ (Lemma 2.7).

In Section 2.3, we prepare the calculus of LAM on different multivalued functions that arise in optimization problems. Note that in Refs. [193,194,198], general calculus rules for normal cones to nonconvex sets and coderivatives of nonsmooth multivalued mappings under natural and verifiable assumptions are obtained. We refer the reader to Refs. [52,54,60,65,77,132,194,209,230] for various constructions of this type. For the inverse mapping, F^{-1}, we have

$$(F^{-1})^*[x^*,(y,x)] = -(F^*)^{-1}[-x^*,(x,y)] \quad \text{(Theorem 2.6)}$$

Then, we compute the LAM for the sum $F(x) \equiv F_1(x) + \cdots + F_N(x)$, intersection $F(x) = \left(\cap_{i=1}^N F_i\right)(x) \equiv \cap_{i=1}^N F_i(x)$, Cartesian product $F = F_1 \times F_2 \times \cdots \times F_N$, as well as composition $F_N \circ F_{N-1} \circ \cdots \circ F_1$, of given convex mappings $F_i : X \to P(Y)$, $i = 1, \ldots, N$. Now, let $F : X^1 \to P(X^2)$ be a convex multivalued function, and let A and B be linear transformations from X^0 to X^1 and from X^2 to X^3, respectively. Construct a new mapping $G : X^0 \to P(X^3)$ defined by $G(x) = BF(Ax)$. In Section 2.2, using the LAM to the composition mapping $F \circ A$, we compute the LAM to G.

In Section 2.4, we compute the LAM of different concrete maps, usually ones given by inequality constraints. First, calculus results in the related LAM setting were obtained by Pshenichnyi [226]. If $F : X \to P(Y)$ is a bounded closed convex multivalued mapping, and φ is a proper convex function such that $\operatorname{dom} \varphi = Z$, then $\partial f(x_0) = \{x^* - F^*(-y^*;z_0) : (x^*,y^*) \in \partial_z \varphi(z_0)\}$, where $f(x) = \min_y \{\varphi(x,y) : y \in F(x)\}$ (Theorem 2.11). For a map defined by $F(x) = \{y : \varphi(x,y) \leq 0\}$, then under the existence of an interior point,

$$F^*(y^*;z_0) = \{-\lambda x_0^* : y^* = \lambda y_0^*, (x_0^*,y_0^*) \in \partial_z \varphi(z_0),\ \lambda \geq 0\}.$$

From the applied point of view, the polyhedral mapping plays an important role. The latter notion was explicitly introduced by Mahmudov and Pshenichnyi [140,141] and Mahmudov [143]. Let $X = Y = \mathbb{R}^n$, A, B be $m \times n$ matrices, and d be an m-dimensional column vector. A polyhedral mapping is defined, the graph of which is the following polyhedral set in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\operatorname{gph} F = \{(x,y) : Ax - By \leq d\}, \quad F(x) = \{y : Ax - By \leq d\}.$$

By Theorem 2.16, the Hamiltonian function $H(\cdot, y^*)$ for a polyhedral mapping is closed and its LAM is the step function in the argument z_0:

$$F^*(y^*;z_0) = \{-A^*\lambda : y^* = -B^*\lambda, \lambda \geq 0,\ \langle Ax_0 - By_0 - d,\ \lambda \rangle = 0\}.$$

A polyhedral mapping F is bounded if and only if $B^*\mathbb{R}^m_+ = \mathbb{R}^n$, where $\mathbb{R}^m_+$ is a positive orthant of $\mathbb{R}^m$. In addition, $x \in \text{dom } F$ if and only if $\langle u, d - Ax \rangle \geq 0$, $\forall u \in N = \{u \in \mathbb{R}^m : B^*u = 0,\ u \geq 0\}$ (Theorem 2.17).

Moreover, a polyhedral mapping F is bounded and dom $F = \mathbb{R}^n$ if and only if $F(x) = C_+ x + F(0)$ for some matrix C_+ (Theorem 2.18).

In Section 2.5, we will prove a result, which can be considered as the duality theorem for a convex multivalued mapping. As will be seen in the next chapters, this result implies a lot of other theorems of convex analysis and theories of extremal problems.

By Theorem 2.19, if F is a convex mapping and its Hamiltonian function H, regarded as a function of x, is closed, and further, if there is a point x_1 for which $H(x_1, y^*)$ is finite, then

$$\inf_{x^*}\{\langle x, x^* \rangle - M(x^*, y^*)\} = H(x, y^*).$$

Note that the following result is important for duality relations: for a convex mapping F,

$$\text{dom } M = \{(x^*, y^*) : M(x^*, y^*) > -\infty\} \subseteq (0^+ \text{ gph } F)^*.$$

If, in addition, F is quasisuperlinear, then we have equality. Furthermore, if $H(\cdot, y^*)$ is a closed proper convex function, then the duality relation

$$\inf_{x^* \in F^*(y^*)}\{\langle x, x^* \rangle - M(x^*, y^*)\} = \sup_{y \in F(x)} \langle y, y^* \rangle$$

holds (Theorem 2.20). In particular, if gph F is a cone, then $\inf_{x^* \in F^*(y^*)} \langle x, x^* \rangle = \sup_{y \in F(x)} \langle y, y^* \rangle$. Let M be a closed convex subset in X and N be a convex subset in Y. Further, let $f(x, y)$ be convex in x and concave in y. Then, by computing the Hamiltonian and support function of an auxiliary multivalued mapping, a minimax relation can be established:

$$\inf_{x \in M} \sup_{y \in N} f(x, y) = \sup_{y \in N} \inf_{x \in M} f(x, y).$$

2.2 Locally Adjoint Mappings to Convex Multivalued Mappings

In this chapter, we first introduce the basic definitions and properties of multivalued (set-valued) mappings (functions) $F : X \to P(Y)$, where X and Y are finite dimensional Euclidean space and $P(Y)$ denotes the family of all subsets of Y.

Let us denote the Cartesian product $X \times Y$ by Z, so that if $x \in X$, $y \in Y$, then a pair (x, y) denotes some point z of the space Z. As in Chapter 1, $\langle x, y \rangle$ is the scalar (inner) product of the elements x, y. Even in this finite dimensional case, to which we confine ourselves at first for simplicity, although $X = X^*$, $Y = Y^*$, $Z = Z^*$, in general we denote elements of the adjoint spaces by x^*, y^*, z^*, respectively.

Let M be an arbitrary subset of $Z = X \times Y$. This defines the multivalued mapping F by the formula

$$F(x) = \{y : (x, y) \in M\}.$$

The set M is called the graph of multivalued function F and is denoted by gph F:

$$\text{gph } F = \{(x, y) : y \in F(x)\}$$

Later on, we set

$$\text{dom } F = \{x : F(x) \neq \varnothing\}, \quad \|F(x)\| = \sup_y \{\|y\| : y \in F(x)\}.$$

By definition, $\|\varnothing\| = 0$.

Definition 2.1. A multivalued mapping F is convex if its graph is convex.

Definition 2.2. A multivalued mapping F is convex valued if $F(x)$ is convex in Y.

Definition 2.3. A multivalued mapping F is closed if its graph is closed.

Definition 2.4. A multivalued mapping F is bounded if there exists a constant $C > 0$, such that

$$\|F(x)\| \leq C(1 + \|x\|).$$

Definition 2.5. A multivalued mapping F is said to be upper semicontinuous at x_0 if for any neighborhood U of zero in Y there exists a neighborhood V of zero in X such that

$$F(x) \subseteq F(x_0) + U \quad \forall x \in x_0 + V.$$

Definition 2.6. A multivalued mapping F is said to be lower semicontinuous at a point x_0 if for any neighborhood U of zero in Y there exists a neighborhood V of zero in X such that

$$F(x_0) \subseteq F(x) + U \quad \forall x \in x_0 + V.$$

Definition 2.7. A multivalued mapping F is said to be continuous at a point x_0, if it is both upper and lower semicontinuous at a point x_0. It is said to be continuous if it is continuous at every $x \in X$.

Definition 2.8. We say that a multivalued mapping F is Lipschitzian in the set Ω if

$$\rho(a(x_1),\ a(x_2)) \leq L\|x_1 - x_2\|,$$

where ρ is the Hausdorff metric; i.e.,

$$\rho(A, B) = \max\left\{ \sup_{y_1 \in A} \inf_{y_2 \in B} \|y_1 - y_2\|, \quad \sup_{y_2 \in B} \inf_{y_1 \in A} \|y_1 - y_2\| \right\}$$

and L is called its Lipschitz constant.

Lemma 2.1. Let F be a closed convex multivalued mapping and suppose that at some $x_0 \in \operatorname{dom} F$ the set $F(x_0)$ is bounded. Then F is bounded.
□ Assuming that F is not bounded, there is a sequence $z_k = (x_k, y_k)$, $y_k \in F(x_k)$ such that $\|y_k\|/(1 + \|x_k\|) \to +\infty$. Let $\|x_k\| \leq r$. Setting $\lambda_k = \|x_k\|/(1 + \|y_k\|)$, we see that $\lambda_k \to 0$. Take $y_0 \in F(x_0)$ and consider the points

$$\bar{x}_k = \lambda_k x_k + (1 - \lambda_k)x_0, \quad \bar{y}_k = \lambda_k y_k + (1 - \lambda_k)y_0.$$

Since $0 < \lambda_k < 1$ for large k and F is convex, $\bar{y}_k \in F(\bar{x}_k)$ for large k. Observes that $\bar{x}_k \to x_0$. Moreover,

$$\bar{y}_k = (1 + \|x_k\|)\frac{y_k}{\|y_k\|} + (1 - \lambda_k)y_0.$$

There is no loss of generality if we assume that $y_k/\|y_k\| \to w$, $\|w\| = 1$. Also, by virtue of the boundedness of $\|x_k\|$, we obtain $\|x_k\| \to \alpha$. Thus, $\bar{y}_k \to (1 + \alpha)w + y_0$. Since F is closed, $(1 + \alpha)w + y_0 \in F(x_0)$. On the other hand, since y_0 is an arbitrary point of $F(x_0)$, we have

$$(1 + \alpha)w + F(x_0) \subseteq F(x_0)$$

which contradicts the fact that the set $F(x_0)$ is bounded. Let now $\|x_k\| \to +\infty$. In this case, we put $\lambda_k = (1 + \|x_k\|)/(\|x_k\|\,\|y_k\|)$. Then taking into account

$$\bar{y}_k = \frac{1 + \|x_k\|}{\|x_k\|}\frac{y_k}{\|y_k\|} + (1 - \lambda_k)y_0$$

and that $(1 + \|x_k\|)/\|x_k\| \to 1$, repeating the previous argument we again derive a contradiction. ■

Lemma 2.2. Let graph $F = K$ and K be a closed convex cone. Then, F is bounded if and only if $F(0) = \{0\}$.
☐ If $F(0) = \{0\}$, then F is bounded, by the previous lemma. On the contrary, if F is bounded, then $y_0 \neq 0$, $y_0 \in F(0)$ does not exist, because $\lambda(0, y_0) = (0, \lambda y_0) \in K$ for all $\lambda > 0$; i.e., $\lambda y_0 \in F(0)$, $\lambda > 0$, and we obtain that $F(0)$ is not bounded, which is a contradiction. ■

Remark 2.1. If a multivalued mapping F is quasisuperlinear—i.e., its graph is represented as gph $F = \Omega + K$, where Ω is a convex compactum and K is a closed convex cone—then in a similar way it can be shown that for the boundedness of such mappings, the hypothesis of the lemma consists of $K \cap (0 \times Y) = \{0\}$.

Definition 2.9. Let 0^+ gph F be the recession cone (see Definition 1.16) of a convex mapping F; i.e.,

$$0^+ \text{ gph } F = \{(\bar{x}, \bar{y}) : (x + \lambda\bar{x}, y + \lambda\bar{x}) \in \text{gph } F, \quad \lambda \geq 0 \ \ \forall (x, y) \in \text{ gph } F\}.$$

Then, we call the multivalued mapping F^* defined by

$$F^*(y^*) = \{x^* : (x^*, \ -y^*) \in (0^+ \text{ gph } F)^*\}$$

the adjoint (AM) to the convex F.

Lemma 2.3. The AM F^* is bounded if and only if dom $F = X$.
☐ Observe that $(0^+ \text{ gph } F)^*$ is always closed, so F^* is a closed convex mapping. According to the previous lemma it is enough to show that $F^*(0) = \{0\}$ if and only if dom $F = X$. Let $x^* \in F^*(0)$; i.e.,

$$\langle x, x^* \rangle - \langle y, 0 \rangle \geq 0, \ \ (x, y) \in 0^+ \text{ gph } F.$$

Then

$$\langle x, x^* \rangle \geq 0, \ \ x \in \text{dom } F. \tag{2.1}$$

If dom $F = X$, then Eq. (2.1) implies that $x^* = 0$, i.e., $F^*(0) = \{0\}$. Conversely, if $x^* \neq 0$, then dom F lies in the half-space defined by Eq. (2.1). Thus, $F^*(0)$ contains a nonzero element, that is not bounded, if dom $F \neq X$. ■

Lemma 2.4. Let F be a closed convex mapping, bounded at $x_0 \in \text{int(dom } F)$. Then, in some neighborhood of x_0, the mapping F is Lipschitzean.
☐ Since $x_0 \in \text{int(dom } F)$, there is $r > 0$ such that $x_0 + rB \subseteq \text{dom } F$, where B is the unit ball centered at the origin. Set

$$x' = x_0 + r\frac{x - x_0}{\|x - x_0\|} = \left(1 - \frac{r}{\|x - x_0\|}\right)x_0 + \frac{r}{\|x - x_0\|}x$$

$$x'' = x_0 - r\frac{x - x_0}{\|x - x_0\|} = \left(1 + \frac{r}{\|x - x_0\|}\right)x_0 - \frac{r}{\|x - x_0\|}x.$$

Then

$$x = \frac{\|x - x_0\|}{r}x' + \left(1 - \frac{\|x - x_0\|}{r}\right)x_0, \quad x_0 = \frac{\|x - x_0\|}{\|x - x_0\| + r}x'' + \frac{r}{\|x - x_0\| + r}x.$$

Since $x \in x_0 + rB$, then $\|x - x_0\| \le r$, and it follows from the preceding formulas that x and x_0 are convex combinations of x', x_0 and x'', x, respectively.

By virtue of the convexity of F,

$$F(x) \supseteq \frac{\|x - x_0\|}{r}F(x') + \left(1 - \frac{\|x - x_0\|}{r}\right)F(x_0)$$

and

$$F(x_0) \supseteq \frac{\|x - x_0\|}{\|x - x_0\| + r}F(x'') + \frac{r}{\|x - x_0\| + r}F(x). \tag{2.2}$$

Let $y' \in F(x')$, $y'' \in F(x'')$, $y_0 \in F(x_0)$, $y \in F(x)$ be points from the corresponding sets. Then from the first inclusion of Eq. (2.2), we obtain

$$F(x) \supseteq \frac{\|x - x_0\|}{r}(y' - y_0) + y_0$$

or

$$y_0 \in F(x) - \frac{\|x - x_0\|}{r}(y' - y_0). \tag{2.3}$$

Note that by Lemma 2.1 the set $F(x)$ is bounded. Hence,

$$\begin{aligned} &\|y'\| \le C(1 + \|x'\|), \quad \|y_0\| \le C(1 + \|x_0\|), \\ &\|y' - y_0\| \le C(1 + \|x'\|) + C(1 + \|x_0\|). \end{aligned}$$

Since $\|x'\| \le \|x_0\| + r$, finally we have

$$\|y' - y_0\| \le C(2 + 2\|x_0\| + r).$$

Now, it follows from Eq. (2.3) that

$$y_0 \in F(x) + \frac{C(2 + 2\|x_0\| + r)}{r}\|x - x_0\|B$$

or, since y_0 is an arbitrary point of $F(x_0)$,

$$F(x_0) \subseteq F(x) + \frac{C(2 + 2\|x_0\| + r)}{r} \|x - x_0\| B.$$

Taking into account the second inclusion of Eq. (2.2), by analogy it can be shown that

$$F(x) \subseteq F(x_0) + \frac{2C(1 + \|x_0\| + r)}{r} \|x - x_0\| B.$$

In turn, from these inclusions, we have

$$\rho(F(x), F(x_0)) \leq \frac{2C(1 + \|x_0\| + r)}{r} \|x - x_0\|. \blacksquare$$

Corollary 2.1. Let $\Omega \subseteq \text{int}(\text{dom } F)$ be a compactum, Then, a closed convex mapping F bounded at a point $\bar{x}$ satisfies a Lipschitz condition on Ω.
$\square$ By virtue of the compactness of Ω, there is $r > 0$ such that $\Omega + rB \subseteq \text{int}(\text{dom } F)$. Therefore, x_0 can be taken arbitrarily from Ω and r then chosen suitably, as in the proof of the previous lemma. $\blacksquare$

Local properties of differentiable functions are well described by the concept of the derivative and the gradient function. For convex functions, instead of the gradient, we use subdifferentials. In the case of multivalued functions, a similar role is played by an important concept of this book: the locally adjoint mappings (LAMs).

Let us introduce some important notations. The function defined as

$$H(x, y^*) = \sup_y \{\langle y, y^* \rangle : y \in F(x)\}, \; y^* \in Y^* = \mathbb{R}^n \tag{2.4}$$

is said to be the Hamiltonian function. Moreover, let us denote

$$M(x^*, y^*) = \inf_{(x,y)} \{\langle x, x^* \rangle - \langle y, y^* \rangle : (x, y) \in \text{gph } F\}. \tag{2.5}$$

We put $H(x, y^*) = -\infty$ if $F(x) = \varnothing$.

It is clear that

$$M(x^*, y^*) = \inf_x \{\langle x, x^* \rangle - H(x, y^*)\}; \tag{2.6}$$

i.e.,

$$M(x^*, y^*) \leq \langle x, x^* \rangle - H(x, y^*) \quad \forall x.$$

Furthermore, it follows from Eq. (2.6) that

$$M(x^*, y^*) = -(-H(\cdot, y^*))^*(x^*).$$

Lemma 2.5. Let F be a convex multivalued mapping. Then

$$F(\lambda_1 x_1 + \lambda_2 x_2) \supseteq \lambda_1 F(x_1) + \lambda_2 F(x_2),\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1 \tag{2.7}$$

and $H(\cdot, y^*)$ is a concave function.

☐ If $(x_1, y_1) \in \operatorname{gph} F$, $(x_2, y_2) \in \operatorname{gph} F$, then since $\operatorname{gph} F$ is convex,

$$(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \in \operatorname{gph} F.$$

Thus, $\lambda_1 y_1 + \lambda_2 y_2 \in F(\lambda_1 x_1 + \lambda_2 x_2)$. Observe that y_1 and y_2 are arbitrary points of the sets $F(x_1)$ and $F(x_2)$, respectively, and so the first assertion is proved.

Moreover, by virtue of Eq. (2.7),

$$\begin{aligned} H(\lambda_1 x_1 + \lambda_2 x_2, y^*) &= \sup_y \{\langle y, y^* \rangle : y \in F(\lambda_1 x_1 + \lambda_2 x_2)\} \\ &\geq \sup_{y_1, y_2} \{\langle \lambda_1 y_1 + \lambda_2 y_2, y^* \rangle : y_1 \in F(x_1), y_2 \in F(x_2)\} \\ &= \lambda_1 H(x_1, y^*) + \lambda_2 H(x_2, y^*). \blacksquare \end{aligned}$$

Take a point $z \in \operatorname{gph} F$ and denote

$$K_{\operatorname{gph} F}(z) = \operatorname{cone}(\operatorname{gph} F - z); \tag{2.8}$$

i.e.,

$$K_{\operatorname{gph} F}(z) = \{\bar{z} : \bar{z} = \lambda(z_1 - z), \quad \lambda > 0,\ z_1 \in \operatorname{gph} F\}.$$

It is not hard to see that

$$K_{\operatorname{gph} F}(z) = \{\bar{z} : z + \lambda z \in \operatorname{gph} F \text{ for sufficiently small } \lambda > 0\}.$$

The Argmaximum set is defined as

$$F(x; y^*) = \{y \in F(x) : \ \langle y, y^* \rangle = H(x, y^*)\}. \tag{2.9}$$

The multivalued mapping

$$F_z(\bar{x}) = \{\bar{y} : (\bar{x},\ \bar{y}) \in K_{\operatorname{gph} F}(z)\}$$

is defined completely by the cone $K_{\operatorname{gph} F}(z)$ and $\operatorname{gph} F_z = K_{\operatorname{gph} F}(z)$.

Definition 2.10. For a convex multivalued mapping F, a multivalued mapping F^* from Y^* into X^* defined by

$$F^*(y^*; z) = \{x^* : (x^*, -y^*) \in K^*_{\text{gph } F}(x, y)\}$$

is called the LAM to F at the point $(x, y) \in \text{gph } F$, where $K^*_{\text{gph } F}(x, y)$ is the dual to the cone $K_{\text{gph } F}(x, y)$.

Theorem 2.1. Let $F : X \to P(Y)$ be a convex multivalued mapping. Then

$$F^*(y^*; (x, y)) = \begin{cases} \partial_x H(x, y^*), & y \in F(x, y^*), \\ \varnothing, & y \notin F(x, y^*), \end{cases}$$

where

$$\partial_x H(x, y^*) = -\partial_x[-H(x, y^*)].$$

□ Obviously, $\partial_x H(x, y^*)$ is the set of x^* such that

$$H(x_1, y^*) - H(x, y^*) \leq \langle x_1 - x, \; x^* \rangle \quad \forall x_1 \in X \tag{2.10}$$

Let $x^* \in F^*(y^*; z)$, $z = (x, y)$. By the definition of the dual cone $K^*_{\text{gph } F}(z)$ this means that

$$\langle \bar{x}, x^* \rangle - \langle \bar{y}, y^* \rangle \geq 0, \quad (\bar{x}, \bar{y}) \in K_{\text{gph } F}(z)$$

or

$$\langle x_1 - x, x^* \rangle - \langle y_1 - y, y^* \rangle \geq 0, \quad (x_1, y_1) \in \text{gph } F. \tag{2.11}$$

If $x_1 = x$, $y_1 \in F(x)$, this inequality implies that

$$\langle y, y^* \rangle \geq \langle y_1, y^* \rangle, \quad y_1 \in F(x);$$

i.e., $y \in F(x, y^*)$ and

$$\langle y, y^* \rangle = H\langle x, y^* \rangle.$$

Then, it follows from Eq. (2.11) that

$$\langle y_1, y^* \rangle - H(x, y^*) \leq \langle x_1 - x, x^* \rangle.$$

Taking the supremum over $y_1 \in F(x_1)$ gives us the inequality

$$H(x_1, y^*) - H(x, y^*) \leq \langle x_1 - x, x^* \rangle,$$

or $x^* \in \partial_x H(x, y^*)$. Now let $x^* \in \partial_x H(x, y^*)$, $y \in F(x; y^*)$. Then, by going in the reverse direction, it is not hard to see that $x^* \in F^*(y^*;(x, y))$. ■

Remark 2.2. At the same time it was shown that all sets $F^*(y^*;(x, y))$ for every $y \in F(x; y^*)$ are the same and are equal to $\partial_x H(x, y^*)$.

Theorem 2.2. Let F be closed bounded convex and P be a compact set in Z such that its projection on X lies entirely in int(dom F). Then, $F^*(y^*; z)$ is uniformly bounded with respect to $z \in P$; i.e., there exists a constant C such that

$$\|F^*(y^*, z)\| \le C\|y^*\|$$

for every $z \in P$.

□ Since P is a compact set and its projection on X lies entirely in int(dom F), there is a number $r > 0$ such that for all $z = (x, y) \in P$, a ball with radius r and center at x belongs to int(dom F).

We proceed by a contradiction argument. Then, there is a sequence $(x_k^*, -y_k^*) \in K^*_{\mathrm{gph}\ F}(z_k)$, $z_k \in P$ such that $\|x_k^*\|\ \|x_k^*\|/\|y_k^*\| \to \infty$. Suppose that $z_k \to z_0 \in \mathrm{int}(\mathrm{dom}\ F)$ and $y_k^*/\|y_k^*\| \to y_0^*$. By Definition 2.10, $(x_k^*, -y_k^*) \in K^*_{\mathrm{gph}\ F}(z_k)$ is equivalent to

$$\langle x - x_k, x_k^* \rangle - \langle y - y_k, y_k^* \rangle \ge 0, \quad (x, y) \in \mathrm{gph}\ F,$$

or

$$\left\langle x - x_k, \frac{x_k^*}{\|y_k^*\|} \right\rangle - \left\langle y - y_k, \frac{y_k^*}{\|y_k^*\|} \right\rangle \ge 0.$$

Set $\tilde{x}_k = x_k - r\, x_k^*/\|x_k^*\|$ and take an arbitrary point $\tilde{y}_k \in F(\tilde{x}_k)$. By virtue of the boundedness of F, all $\tilde{y}_k$ are bounded. Then by substituting $(\tilde{x}_k, \tilde{y}_k)$ into the latter inequality, we deduce

$$-r\frac{\|x_k^*\|}{\|y_k^*\|} - \left\langle \tilde{y} - y_k, \frac{y_k^*}{\|y_k^*\|} \right\rangle \ge 0,$$

which is impossible for large k because $\|x_k^*\|/\|y_k^*\| \to \infty$. ■

Theorem 2.3. Let F be a closed bounded convex multivalued mapping. Then, its LAM $F^*(y^*; z)$ is upper semicontinuous for all $y^* \in Y^*$ and $z = (x, y)$ such that $x \in$ int (dom F).

□ As in the previous theorem, we proceed by contradiction; suppose that for some y_0^* and $z_0 = (x_0, y_0)$, where $x_0 \in \mathrm{int}(\mathrm{dom}\ F)$, the LAM $F^*(y^*, z)$ is not upper semicontinuous. Then, for $\varepsilon > 0$, there exist sequences $y_k^* \to y_0^*, z_k \to z_0, x_k^* \in F^*(y_k^*;\ z_k)$ such that

$$x_k^* \notin F^*(y_0^*; z_0) + \varepsilon B, \tag{2.12}$$

where B is the unit ball with center zero. By the previous theorem, all the mappings $F^*(y^*; z_k)$ are uniformly bounded, so we assume that all x_k^* are bounded. Thus, there is no loss of generality if we suppose that $x_k^* \to x_0^*$. By definition of LAM, we can write

$$\langle x - x_k, x_k^* \rangle - \langle y - y_k, y_k^* \rangle \geq 0, \quad (x, y) \in \operatorname{gph} F,$$

where, by passing to the limit, we derive

$$\langle x - x_0, x_0^* \rangle - \langle y - y_0, y_0^* \rangle \geq 0;$$

i.e., $x_0^* \in F^*(y_0^*; z_0)$, which contradicts Eq. (2.12). ■

Lemma 2.6. The x^* is an element of the LAM F^* if and only if

$$M(x^*, y^*) = \langle x, x^* \rangle - H(x, y^*).$$

□ By Theorem 2.1, $x^* \in F^*(y^*; z)$, $z = (x, y)$ is equivalent to

$$x^* \in \partial_x H(x, y^*), y \in (x; y^*);$$

i.e.,

$$H(x_1, y^*) - H(x, \ y^*) \leq \langle x_1 - x, x^* \rangle \quad \forall x_1 \in X.$$

Rewriting the latter inequality as

$$\langle x, x^* \rangle - H(x, y^*) \leq \langle x_1, x^* \rangle - H(x_1, y^*)$$

and taking the infimum over x_1, we have

$$M(x^*, y^*) \geq \langle x, \ x^* \rangle - H(x, y^*).$$

Now compare this inequality with the reverse inequality following from Eq. (2.6). ■

2.3 The Calculus of Locally Adjoint Mappings

We investigate the connection between adjoint mappings and LAM. For a convex set $Q \subseteq Z = X \times Y$ at a point $z \in Q$, let

$$K_Q(z) = \operatorname{cone}(Q - z) = \{\bar{z} : \bar{z} = \lambda(z_1 - z), \ \ \lambda > 0 \ \ \forall z_1 \in Q\}. \tag{2.13}$$

First, we establish the following supplementary result.

Theorem 2.4. Let $0^+ Q$ be the recession cone of the closed convex set Q. Then,

$$\bigcap_{z\in Q} K_Q(z) = 0^+ Q.$$

□ We first show that

$$Q = \bigcap_{z\in Q} (z + K_Q(z)). \tag{2.14}$$

Indeed, take an arbitrary $z_0 \in Q$. Clearly, all vectors of the form $\bar{z} = z_0 - z$ belong to the cone in Eq. (2.13) (with $\lambda = 1$); i.e., $z_0 \in z + K_Q(z)$, $z \in Q$, which implies that $z_0 \in \cap_{z\in Q}(z + K_Q(z))$. Conversely, if we have the latter inclusion, then $z_0 \in z + K_Q(z)$, $z \in Q$ and there is $z_1 \in Q$ and $\gamma > 0$ such that $z_0 - z = \gamma(z_1 - z) \in K_Q(z)$. Thus, $z_0 = \gamma z_1 + (1-\gamma)z \in Q$. This yields Eq. (2.14).

On the other hand, it is easy to see that

$$0^+ \left[\bigcap_{z\in Q} \left(z + K_Q(z)\right)\right] = \bigcap_{z\in Q} \left[0^+ (z + K_Q(z))\right].$$

In fact, if z is an arbitrary point of $Q = \cap_{z\in Q}(z + K_Q(z))$, then by Definition 1.16 of a recession cone, the half-line $z + \lambda\bar{z}$, $\lambda \geq 0$ is contained in every cone $z + K_Q(z)$, $z \in Q$. This means that

$$\bar{z} \in \bigcap_{z\in Q} \left[0^+ \left(z + K_Q(z)\right)\right].$$

Therefore,

$$0^+ Q = 0^+ \left[\bigcap_{z\in Q} \left(z + K_Q(z)\right)\right] = \bigcap_{z\in Q} \left[0^+ \left(z + K_Q(z)\right)\right] = \bigcap_{z\in Q} K_Q(z). \blacksquare$$

Remark 2.3. In the formulation of the theorem, the closure condition is essential. For example, let $Q = \{(x, y): x > 0,\ y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$. Obviously, $0^+ Q = Q$. The set Q has all the points of the form $(x_0, y_0) + \lambda(0, y_0)$, where $x_0 > 0$, $y_0 > 0$ are fixed. But $(0, y_0) \notin 0^+ Q$.

Theorem 2.4. Let Q be a closed convex set and $K_Q^*(z)$ be the dual cone to the cone $K_Q(z)$, $z \in Q$. Then

$$\overline{\bigcup_{z\in Q} K_Q^*(z)} = (0^+ Q)^*.$$

□ It is sufficient to show that

$$\overline{\bigcup_{z\in Q} K_Q^*(z)} = \left(\bigcup_{z\in Q} K_Q(z)\right)^*. \tag{2.15}$$

For an arbitrary fixed $z_0^* \in \overline{\cup_{z\in Q} K_Q^*(z)}$, there exists a sequence $z_n^* \in \overline{\cup_{z\in Q} K_Q^*(z)}$ that converges to z_0^*. Then $z_n^* \in K_Q^*(z_n)$ for some sequence $z_n \in Q$.

On the other hand, since $K_Q(z_n) \supseteq \cap_{z\in Q} K_Q(z)$, it follows that $K_Q^*(z_n) \subseteq \left(\cap_{z\in Q} K_Q(z)\right)^*$. Thus, $z_n^* \subseteq \left(\cap_{z\in Q} K_Q(z)\right)^*$ and so $z_0^* \subseteq \left(\cap_{z\in Q} K_Q(z)\right)^*$. Consequently, we have $\overline{\cup_{z\in Q} K_Q^*(z)} \subseteq \left(\cap_{z\in Q} K_Q(z)\right)^*$. Now we prove the reverse inclusion. Take an arbitrary $z_1^* \in \left(\cap_{z\in Q} K_Q(z)\right)^*$. On the contrary, suppose that $z_1^* \notin \overline{\cup_{z\in Q} K_Q^*(z)}$. This implies that $z_1^* \in K_Q^*(z)$ for every $z \in Q$. In other words, there is a vector $\bar{z}_1 \neq 0$ such that

$$\left\langle z_1^*, \bar{z}_1 \right\rangle < 0, \quad \forall \bar{z}_1 \in K_Q(z) \quad \forall z \in Q$$

or

$$\langle z_1^*, \bar{z}_1 \rangle < 0, \quad \bar{z}_1 \in \bigcap_{z\in Q} K_Q(z);$$

i.e., $z_1^* \notin \left(\cap_{z\in Q} K_Q(z)\right)^*$. This is a contradiction. ■

The next theorem shows that the AM F^* is the closure of a union of LAMs over $z \in \operatorname{gph} F$.

Theorem 2.5. Let F be a closed convex multivalued mapping. Then

$$F^*(y^*) = \overline{\bigcup_{z\in \operatorname{gph} F} F^*(y^*; z)}, \quad y \in F(x; y^*).$$

□ Taking into account the definition of adjoint mapping and LAM, the only formula of the theorem is obtained immediately from the preceding theorem if we set $Q = \operatorname{gph} F$:

$$F^*(y^*) = \overline{\bigcup_{z\in \operatorname{gph} F} F^*(y^*; z)}.$$

It only remains to observe that, by Theorem 2.1, $F^*(y^*; z) = \varnothing$ for $z = (x, y)$, $y \notin F(x; y^*)$. ■

It is elementary to verify that if F is a convex multivalued mapping, then αF is a convex multivalued mapping for any real $\alpha \neq 0$. Let $G(x) = \alpha F(x)$.

Lemma 2.7. $G^*(y^*;(x, \alpha y)) = \|\alpha\| F^*(y^* \operatorname{sgn} \alpha;(x, y))$.

□ Let H_F, H_G be the Hamiltonian functions for F and G, respectively. Clearly,

$$\begin{aligned} H_G(x, y^*) &= \sup_y\{\langle y, y^*\rangle : y \in G(x)\} = \sup_y\{\alpha\langle y, y^*\rangle : y \in F(x)\} \\ &= |\alpha| \sup_y\{\langle y, \operatorname{sgn} \alpha y^*\rangle : y \in F(x)\} = |\alpha| H_F(x, \operatorname{sgn} \alpha y^*). \end{aligned}$$

Now use Theorem 2.1. ■

For any multivalued mapping $F: X \to P(Y)$, its inverse is denoted by F^{-1} and is defined to be the multivalued mapping

$$F^{-1}(y) = \{x : y \in F(x)\}.$$

Obviously, $F^{-1}: Y \to P(X)$. It is easy to see that $(F^{-1})^{-1} = F$ and $F^{-1}(F(\text{dom } F)) = \text{dom } F$. Furthermore,

$$\text{gph } F^{-1} = \{(y,x) : x \in F^{-1}(y)\} = \{(y,x) : y \in F(x)\} = \{(y,x) : (x,y) \in \text{gph } F\}.$$

Theorem 2.6. Let $F: X \to P(Y)$ be a convex mapping and $F^{-1}: Y \to P(Y)$ be its inverse mapping. Then we have

$$(F^{-1})^*(x^*; (y,x)) = -(F^*)^{-1}(-x^*, (x,y)), \quad (x,y) \in \text{gph } F.$$

□ Remember that, by definition, $K_{\text{gph } F}(z) = \text{cone}(\text{gph } F - z)$. Clearly,

$$K_{\text{gph } F^{-1}}(y,x) = \{(\bar{y},\bar{x}) : (\bar{x},\bar{y}) \in K_{\text{gph } F}(x,y)\}$$

Then,

$$\begin{aligned} K^*_{\text{gph } F^{-1}}(y,x) &= \{(y^*,x^*) : \langle y^*,\bar{y}\rangle + \langle x^*,\bar{x}\rangle \geq 0, \ (\bar{x},\bar{y}) \in K_{\text{gph } F}(x,y)\} \\ &= \{(y^*,x^*) : (x^*,y^*) \in K^*_{\text{gph } F}(x,y)\}. \end{aligned}$$

Thus, by definition of LAM, we can write

$$\begin{aligned} (F^{-1})^*(x^*; (y,x)) &= \{y^* : (y^*, -x^*) \in K^*_{\text{gph } F^{-1}}\} = \{y^* : (-x^*, y^*) \in K^*_{\text{gph } F}\} \\ &= \{y^* : -x^* \in F^*(-y^*; (x,y))\} = -\{-y^* : -x^* \in F^*(-y^*; (x,y))\} \\ &= -(F^*)^{-1}(-x^*, (x,y)) \blacksquare \end{aligned}$$

Given multivalued mappings $F_i: X \to P(Y)$, $i = 1, \ldots, N$, their sum is defined to be the multivalued mapping

$$F(x) = (F_1 + \cdots + F_N)(x) \equiv F_1(x) + \cdots + F_N(x), \ \ x \in \bigcap_{i=1}^{N} \text{dom } F_i. \tag{2.16}$$

Denote by $H_i(x, y^*)$ and $F_i(x, y^*)$ the Hamiltonian function and Argmaximum set corresponding to $F_i(x)$, $i = 1, \ldots, N$ for a given $y^* \in Y^*$.

Theorem 2.7. Let F_i: $X \to P(Y)$, $i = 1, \ldots, N$ be convex and suppose that all the Hamiltonian functions $H_i(\cdot, y^*)$, $i = 1, \ldots, N$ except possibly one are continuous at a point $\bar{x} \in \cap_{i=1}^{N} \text{dom} F_i$. Then

$$F^*(y^*; (x,y)) = F_1^*(y^*; (x,y_1)) + \cdots + F_N^*(y^*; (x,y_N)),$$

$$y = y_1 + \cdots + y_N, \quad y_i \in F_i(x), \ \ i = 1, \ldots, N.$$

□ In the standard way, it can be shown that the sum of multivalued mappings defined by Eq. (2.16) is convex if F_i, $i = 1, \ldots, N$ are convex. Furthermore,

$$H(x, y^*) = \sum_{i=1}^{N} H_i(x, y^*), \quad x \in \bigcap_{i=1}^{N} \operatorname{dom} F_i \tag{2.17}$$

and so

$$F(x, y^*) = \sum_{i=1}^{N} F_i(x;\ y^*). \tag{2.18}$$

Moreover, by Theorem 2.1

$$F^*(y^*; (x, y)) = \begin{cases} \partial_x H(x, y^*), & y \in F(x; y^*), \\ \varnothing, & y \notin F(x; y^*). \end{cases} \tag{2.19}$$

Now, by hypothesis and Theorem 1.29, we obtain

$$\partial_x H(x, y^*) = \sum_{i=1}^{N} \partial_x H_i(x; y^*). \tag{2.20}$$

Now by substituting Eqs. (2.17), (2.18), and (2.20) into Eq. (2.19), we have the needed result. ■

Now, consider the intersection of $F_i : X \to P(Y)$, $i = 1, \ldots, N$:

$$F(x) = \left(\bigcap_{i=1}^{N} F_i\right)(x) \equiv \bigcap_{i=1}^{N} F_i(x), \ \operatorname{gph} F = \bigcap_{i=1}^{N} (\operatorname{gph} F_i). \tag{2.21}$$

Theorem 2.8. Suppose $F_i : X \to P(Y)$, $i = 1, \ldots, N$ are convex multivalued mappings and either

1. $\operatorname{gph} F_1 \cap \operatorname{int} \operatorname{gph} F_2 \cap \cdots \cap \operatorname{int} \operatorname{gph} \mathrm{F_N} \neq \varnothing$
   or
2. $\operatorname{ri} \operatorname{gph} F_1 \cap \operatorname{ri} \operatorname{gph} F_2 \cap \cdots \cap \operatorname{ri} \operatorname{gph} F_N \neq \varnothing$ is satisfied. Then we have

$$F^*(y^*; z) = \sum_{i=1}^{N} F_i^*(y_i^*; z), \ y^* = \sum_{i=1}^{N} y_i^*,$$

where F^* $(y^*; z)$ is the LAM to the intersection map in Eq. (2.21).
□ Obviously, $K_{\operatorname{gph} F}(z) = \cap_{i=1}^{N} K_{\operatorname{gph} F_i}(x)$, $z \in \operatorname{gph} F$ and now the hypothesis implies that either $K_{\operatorname{gph} F_1} \cap intK_{\operatorname{gph} F_2} \cap \ \ldots \cap \ intK_{\operatorname{gph} F_N} \neq \varnothing$ or ri $K_{\operatorname{gph} F_1} \cap$ ri $K_{\operatorname{gph} F_2} \cap \ldots \cap$ri $K_{\operatorname{gph} F_N} \neq \varnothing$.

Then, by applying Theorems 1.11 and 1.30, we have

$$K^*_{\text{gph } F}(z) = K^*_{\text{gph } F_1}(z) + \cdots + K^*_{\text{gph } F_N}(z),$$

Let us rewrite the latter formula in the form

$$(x^*,y^*)=(x_1^*,y_1^*)+\cdots+(x_N^*,y_N^*);\ \ (x^*,y^*)\in K^*_{\text{gph}\,F}(z),\ \ (x_i^*,y_i^*)\in K^*_{\text{gph}\,F_i}(z),\ i=1,\ldots,N,$$

where $x^* = \sum_{i=1}^N x_i^*; y^* = \sum_{i=1}^N y_i^*$, i.e., $F^*(y^*;z) = \sum_{i=1}^N F_i^*(y_i^*;\, z)$, $y^* = \sum_{i=1}^N y_i^*$. ■

Lemma 2.8. Suppose that $F_i : X \to P(Y)$, $i = 1, \ldots, N$ are convex multivalued mappings and the hypotheses of Theorem 2.8 are satisfied. Then, we have

$$F(x;y^*) = \bigcap_{i=1}^N F_i(x;y_i^*),\ \ y^* = \sum_{i=1}^N y_i^*$$

where $F(x;\, y^*)$ is the Argmaximum set corresponding to the intersection map in Eq. (2.21).

□ Let $x^* \in F^*(y^*;\, z)$ and $x_i^* \in F_i^*(y_i^*;z)$, $i = 1,\ldots,N$. By Theorem 2.8, $y^* = \sum_{i=1}^N y_i^*$. By the definition of LAM, we can write

$$\begin{aligned}&\langle x_1 - x,\ \ x^*\rangle - \langle y_1 - y,\ \ y^*\rangle \geq 0,\quad (x_1,\ \ y_1)\in \text{gph}\, F = \bigcap_{i=1}^N(\text{gph}\, F_i),\\&\langle x_i - x,\ \ x_i^*\rangle - \langle y_i - y,\ \ y_i^*\rangle \geq 0,\quad (x_i,\ \ y_i)\in \text{gph}\, F_i,\ \ i = 1,\ \ \ldots, N.\end{aligned}$$

If $x_1 = x$, $y_1 \in F(x)$, then the first relation implies that

$$\langle y, y^*\rangle \geq \langle y_1, y^*\rangle,\quad y_1 \in F(x);$$

i.e., $y \in F(x;\, y^*)$ and $\langle y,\, y^*\rangle = H(x,\, y^*)$, where H is the Hamiltonian function for F. Similarly, if $x_i = x$, $y_i \in F_i(x)$, the second inequality yields $y \in F_i(x;y_i^*)$, $(i = 1,\ldots,N)$. ■

Corollary 2.2. With the assumptions of Theorem 2.8, we have

$$F^*(y^*;z) = \begin{cases} \sum_{i=1}^N \partial_x H_i(x,y_i^*), & y \in \bigcap_{i=1}^N F_i(x,y_i^*),\\ \varnothing, & y \notin \bigcap_{i=1}^N F_i(x,y_i^*),\end{cases}$$

where $y^* = \sum_{i=1}^N y_i^*$ and H_i is the Hamiltonian function for F_i.

Now, consider the Cartesian product $F = F_1 \times F_2 \times \cdots \times F_N$ of the multivalued mappings $F_i : X \to P(Y)$, $i = 1, \ldots, N$.

Lemma 2.9. Let $F_i : X \to P(Y)$, $i = 1, \ldots, N$ be convex-valued mappings. Moreover, let H_i and H be the Hamiltonian functions for F_i and the Cartesian product $F = F_1 \times F_2 \times \cdots \times F_N$, respectively. Then we have

$$H(x, y^*) = \sum_{i=1}^{N} H_i(x, y_i^*), \quad y^* = (y_1^*, \ldots, y_N^*).$$

□ We shall confine ourselves to the case $N = 2$; the case of a greater number of Cartesian product follows from this case by induction.

$$\begin{aligned} H(x, y^*) &= \sup_y \{\langle y, y^* \rangle : y \in (F_1 \times F_2)(x)\} \\ &= \sup_{y_1, y_2} \{\langle y_1, y_1^* \rangle + \langle y_2, y_2^* \rangle : y_1 \in F_1(x), y_2 \in F_2(x)\} \\ &= \sup_{y_1} \{\langle y_1, y_1^* \rangle : y_1 \in F_1(x)\} + \sup_{y_2} \{\langle y_2, y_2^* \rangle : y_2 \in F_2(x)\} \\ &= H_1(x, y_1^*) + H_2(x, y_2^*), \quad y^* = (y_1^*, y_2^*). \blacksquare \end{aligned}$$

Theorem 2.9. Let $F_i : X \to P(Y)$, $i = 1, \ldots, N$ be convex mappings and suppose that all the Hamiltonian functions $H_i(\cdot, y^*)$, $i = 1, \ldots, N$ except possibly one are continuous at a point common to every dom F_i. Then, for F, the Cartesian product map, we have

$$F^*(y^*; z) = \sum_{i=1}^{N} F_i^*(y_i^*; (x, y_i)), \quad y = (y_1, \ldots, y_N) \in F(x; y^*)$$

□ It is easily checked that $F(x, y^*) \equiv F_1(x; y_1^*) \times \cdots \times F_N(x; y_N^*)$, $y^* = (y_1^*, \ldots, y_N^*)$. Now use Lemma 2.9 and Theorems 1.29 and 2.1. ■

Let us now consider the composition $F^T = F_{T-1} \circ F_{T-2} \circ \cdots \circ F_0$ of the convex multivalued functions $F_t : X^t \to P(X^{t+1})$, $t = 0,1, \ldots, T-1$, where $F^T : X^0 \to P(X^T)$ and X^t are finite-dimensional Euclidean spaces. For two such mappings, define as follows

$$\begin{aligned} &(F_t \circ F_{t-1})(x_{t-1}) = \cup_{x_t \in F_{t-1}(x_{t-1})} F_t(x_t), \\ &F_t \circ F_{t-1} : X^{t-1} \to P(X^{t-1}), \quad t = 1, \ldots, T-1. \end{aligned}$$

Since

$$\text{gph}\,(F_t \circ F_{t-1}) = \{(x_{t-1}, x_{t+1}) : (x_{t-1}, x_t) \in \text{gph}\, F_{t-1}, \quad (x_t, x_{t+1}) \in \text{gph}\, F_t\},$$

it is easy to see that $F_t \circ F_{t-1}$ is a convex mapping and so the composition F^T is also a convex map.

Theorem 2.10. Let $F_t : X^t \to P(X^{t+1})$, $t = 0, 1, \ldots, T-1$ be convex mappings. Let $x_t^0 \in X^t$ be a point such that either

1. $(x_t^0, x_{t+1}^0) \in \text{int gph}\, F_t$, $\quad t = 0, \ldots, T-2$, $\quad (x_{T-1}^0, x_T^0) \in \text{gph}\, F_{T-1}$
or
2. $(x_t^0, x_{t+1}^0) \in \text{ri gph}\, F_t, t = 0, \ldots, T-1$.

Then, for a point $(\tilde{x}_0, \tilde{x}_T)$, we have

$$(F^T)^*(x_T^*;\ (\tilde{x}_0, \tilde{x}_T)) \\ = \{x_0^* : x_0^* \in F_0^*(x_1^*; (\tilde{x}_0, \tilde{x}_1)), x_1^* \in F_1^*(x_2^*; (\tilde{x}_1, \tilde{x}_2)), \ldots, x_{T-1}^* \in F_{T-1}^*(x_T^*; (\tilde{x}_{T-1}, \tilde{x}_T))\}$$

or, briefly,

$$(F^T)^*(\cdot; (\tilde{x}_0, \tilde{x}_T)) = F_0^*(\cdot; (\tilde{x}_0, \tilde{x}_1)) \circ F_1^*(\cdot; (\tilde{x}_1, \tilde{x}_2)) \circ \cdots \circ F_{T-1}^*(\cdot; (\tilde{x}_{T-1}, \tilde{x}_T)),$$

where $\tilde{x}_t$, $t = 1, \ldots, T-1$ are arbitrary points satisfying $\tilde{x}_{t+1} \in F_t(\tilde{x}_t)$, $t = 0, \ldots, T-1$.

□ We shall confine ourselves to the case $T = 2$; the case of a greater number of compositions follows by induction. By the definition of LAM,

$$x_0^* \in (F_1 \circ F_0)^*(x_2^*; (\tilde{x}_1, \tilde{x}_2))$$

if and only if

$$\langle x_0 - \tilde{x}_0, x_0^* \rangle - \langle x_2 - \tilde{x}_2, x_2^* \rangle \geq 0 \tag{2.22}$$

for all $(x_0, x_2) \in \text{gph}\,(F_1 \circ F_0)$. Taking into account the expression for gph $(F_1 \circ F_0)$ and the fact that $(\tilde{x}_0, \tilde{x}_2) \in \text{gph}(F_t \circ F_{t-1})$, we determine that there is a point $\tilde{x}_1$ such that

$$(\tilde{x}_0, \tilde{x}_1) \in \text{gph}\, F_0, \quad (\tilde{x}_1, \tilde{x}_2) \in \text{gph}\, F_1.$$

Rewrite Eq. (2.22) in the equivalent form

$$\begin{gathered} \langle x_0 - \tilde{x}_0, x_0^* \rangle - \langle x_1 - \tilde{x}_1, 0 \rangle - \langle x_2 - \tilde{x}_2, x_2^* \rangle \geq 0 \\ (x_0, x_1) \in \text{gph}\, F_0, \quad (x_1, x_2) \in \text{gph}\, F_1. \end{gathered} \tag{2.23}$$

Consider the space $X^0 \times X^1 \times X^2$ with elements consisting of triplets (x_0, x_1, x_2), $x_0 \in X^0$, $x_1 \in X^1$, $x_2 \in X^2$. Put

$$Q_1 = \{(x_0, x_1, x_2) : (x_0, x_1) \in \text{gph}\, F_1\}, \quad Q_2 = \{(x_0, x_1, x_2) : (x_1, x_2) \in \text{gph}\, F_2\}$$

It is not hard to see that

$$\text{cone}(Q_1 \cap Q_2 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)) = \text{cone}(Q_1 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)) + \text{cone}(Q_2 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2))$$

On the other hand,

$$\begin{aligned} &\text{cone}(Q_1 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)) = K_{\text{gph}\, F_1}(\tilde{x}_0, \tilde{x}_1) \times X^2, \\ &\text{cone}(Q_2 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)) = X^0 \times K_{\text{gph}\, F_2}(\tilde{x}_1, \tilde{x}_2). \end{aligned}$$

Thus,

$$\begin{aligned}&[\text{cone}(Q_1 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2))]^* = K^*_{gphF_1}(\tilde{x}_0, \tilde{x}_1) \times \{0\},\\&[\text{cone}(Q_2 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2))]^* = \{0\} \times K^*_{gphF_2}(\tilde{x}_1, \tilde{x}_2).\end{aligned} \tag{2.24}$$

Then, by using the Theorems 1.11 and 1.30, we have

$$\begin{aligned}&[\text{cone}(Q_1 \cap Q_2 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2))]^*\\&\quad = [\text{cone}(Q_1 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2))]^* + [\text{cone}(Q_2 - (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2))]^*.\end{aligned}$$

Now, it is easy to see that the inequality in Eq. (2.23) means that $(x_0^*, 0, -x_2^*)$ belongs to the latter dual cone. Taking into account Eq. (2.24), we claim that $(x_0^*, 0, -x_2^*)$ has the representation

$$\begin{aligned}&x_0^* = x_0^{1*}, \quad 0 = -x_1^{1*} + x_1^{2*}, \quad -x_2^* = -x_2^{2*},\\&(x_0^{1*}, -x_1^{1*}) \in K^*_{\text{gph } F_1}(\tilde{x}_0, \tilde{x}_1), \quad (x_1^{2*}, -x_2^{2*}) \in K^*_{\text{gph } F_2}(\tilde{x}_1, \tilde{x}_2),\end{aligned}$$

which clearly, by the definition of LAM, is equivalent to the relations

$$x_0^* = x_0^{1*}, x_0^{1*} \in F_0^*(x_1^{1*}; (\tilde{x}_0, \tilde{x}_1)), \quad x_1^{1*} \in F_1^*(x_2^*; (\tilde{x}_1, \tilde{x}_2)). \blacksquare$$

Let now $F : X^1 \to P(X^2)$ be a convex multivalued function. Moreover, let A and B be linear transformations from X^0 to X^1 and from X^2 to X^3, respectively. Construct a new mapping $G : X^0 \to P(X^3)$ defined by $G(x) = BF(Ax)$. We now compute the LAM to G. For the composition mapping $F \circ A$, we calculate its Hamiltonian function $H_{F \circ A}$. We can write

$$\begin{aligned}&H_G(x, x_3^*) = \sup_{x_3} \left\{ \langle x_3, x_3^* \rangle : x_3 \in B(F \circ A)(x) \right\} = \sup_{x_2} \left\{ \langle Bx_2, x_3^* \rangle : x_2 \in (F \circ A)(x) \right\}\\&\sup_{x_2} \left\{ \langle x_2, B^* x_3^* \rangle : x_2 \in (F \circ A)(x) \right\} = H_{F \circ A}(x, B^* x_3^*).\end{aligned}$$

Thus, $G^*(x_3^*; (x, x_3)) = (F \circ A)^*(B^* x_3^*; (x, x_2))$. Assume that there exists a pair of points, $A\bar{x}_2$ and $\bar{x}_2$, such that $(A\bar{x}, \bar{x}_2) \in \text{ri (gph } F)$. Then, by Theorem 2.10,

$$\begin{aligned}(F \circ A)^*(B^* x_3^*; (x, x_2)) = A^* F^*(B^* x_3^*; (Ax, x_2)) \text{ and } G^*(x_3^*; (x, x_3))\\= A^* F^*(B^* x_3^*; (Ax, x_2))\end{aligned}$$

2.4 Locally Adjoint Mappings in Concrete Cases

First, we show that with the use of the LAM we can calculate the subdifferential of even some complicated functions.

Let $F : X \to P(Y)$ be a bounded closed convex multivalued mapping and φ be a proper convex function such that dom $\varphi = Z$. By Theorem 1.18, it is continuous on z and at any point has the subdifferential $\partial_z \varphi(z)$. Let us define a function

$$f(x) = \min_y \{\varphi(x, y) : y \in F(x)\}. \tag{2.25}$$

Since F is bounded and closed and φ is continuous, f is finite for all $x \in \operatorname{dom} F$ and is continuous on ri dom F. Denote $\tilde{Y} = Y \times \mathbb{R}$, $\tilde{Z} = X \times \tilde{Y}$ so that

$$\tilde{y} = (y, y^0), \quad \tilde{z} = (x, y, y^0).$$

Suppose $\tilde{F}$ is a multivalued mapping, the graph of which is given by

$$\operatorname{gph} \tilde{F} = \{(x, y, y^0) : (x, y) \in \operatorname{gph} F, \quad y^0 \geq \varphi(x, y)\}. \tag{2.26}$$

Since gph F is closed, so is gph $\tilde{F}$. Furthermore, φ is a continuous convex function. Thus,

$$\tilde{F}(x) = \{(y, y^0) : y \in F(x), y^0 \geq \varphi(x, y)\} \tag{2.27}$$

is a closed convex multivalued mapping. Clearly, dom $F = \operatorname{dom} \tilde{F}$ and the Hamiltonian function H_F of F is $-\infty$, if $x \notin \operatorname{dom} F$. We calculate the Hamiltonian function $H_{\tilde{F}}$ of $\tilde{F}$:

$$H_{\tilde{F}}(x, y^*, y^{0*}) = \sup_{(y^0, y)} \{\langle y, y^* \rangle + y^0 y^{0*} : y \in F(x), y^0 \geq \varphi(x, y)\},$$

where $x \in \operatorname{dom} F$. We have

$$H_{\tilde{F}}(x, y^*, y^{0*}) = \begin{cases} +\infty, & \text{if } y^{0*} > 0; \\ H_F(x, y^*), & \text{if } y^{0*} = 0; \\ \max_y \left\{\langle y, y^* \rangle + y^{0*} \varphi(x, y) : y \in F(x)\right\}, & \text{if } y^{0*} < 0. \end{cases}$$

As is obvious, $H_{\tilde{F}}(x, 0, -1) = -f(x)$ and hence, by Lemma 2.5, $-f$ is concave and f is convex. In fact, by Theorem 2.1, the calculation of the subdifferential $\partial f(x)$ is reduced to the calculation of the LAM to $\tilde{F}$. Taking $y_0^0 = \varphi(x_0, y_0)$, we calculate $K_{\operatorname{gph} \tilde{F}}(x_0, y_0, y_0^0)$: by Definition 2.8 of $K_{\operatorname{gph} F}(\tilde{z})$, $\bar{\tilde{z}} = (\bar{x}, \bar{y}, \overline{y^0}) \in K_{\operatorname{gph} \tilde{F}}(\tilde{z}_0)$, $\tilde{z}_0 = (x_0, y_0, y_0^0)$ if and only if $\tilde{z}_0 + \lambda \bar{\tilde{z}} \in \operatorname{gph} \tilde{F}$. For sufficiently small $\lambda > 0$, i.e., according to Eq. (2.26),

$$(x_0 + \lambda \bar{x},\ y_0 + \lambda \bar{y}) \in \operatorname{gph} F, \tag{2.28}$$

$$y_0^0 + \lambda \overline{y^0} \geq \varphi(x_0 + \lambda x, y_0 + \lambda \bar{y}). \tag{2.29}$$

The cone defined by Eq. (2.28) clearly is $K_a(z_0) \times \mathbb{R}$, $z_0 = (x_0, y_0)$ because here $\overline{y^0}$ can be chosen arbitrarily. Hence,

$$(K_{\text{gph } F}(z_0) \times \mathbb{R})^* = K^*_{\text{gph } F}(z_0) \times \{0\}. \tag{2.30}$$

On the other hand, the cone defined by Eq. (2.29) is cone$(\tilde{M} - \tilde{z}_0)$, where

$$\tilde{M} = \{(x, y, y^0) : \varphi(x, y) - y^0 \leq 0\}, \quad \tilde{M} \subseteq \tilde{Z}.$$

Choosing a sufficiently large number y^0, we can always arrange that $\varphi(x, y) - y^0 < 0$. Note that φ is continuous and its subdifferential is convex and compact. By Theorem 1.33, for the function $\varphi(x, y) - y^0$ at a point $(x_0, y_0, \varphi(x_0, y_0))$, we have

$$\text{cone}(\tilde{M} - \tilde{z}_0) = \{(-\lambda x_0^*, \ -\lambda y_0^*, \lambda) : (x_0^*, y_0^*) \in \partial_z \varphi(z_0), \quad \lambda \geq 0\}, \tag{2.31}$$

where

$$z_0 = (x_0, y_0), \quad \partial_{\tilde{z}}(\varphi(z_0) - y_0) = (\partial_z \varphi(z_0), \ -1).$$

Now let $(\overline{x_0}, \overline{y_0}) \in K_{\text{gph } F}(z_0)$. Choose $\overline{y_0^0}$ so that

$$\overline{y_0^0} > \varphi'(z_0, \overline{z_0}), \quad \overline{z_0} = (\overline{x_0}, \overline{y_0}),$$

where $\varphi'(z_0, \overline{z_0})$ is the directional derivative of φ at z_0 with respect to $\overline{z_0}$. Then, for sufficiently small $\lambda > 0$,

$$\overline{y_0^0} > \frac{\varphi(x_0 + \lambda \overline{x_0}, y_0 + \lambda \overline{y_0}) - \varphi(x_0, y_0)}{\lambda}. \tag{2.32}$$

On the other hand, since $y^0 = \varphi(x_0, y_0)$, the inequality in Eq. (2.29) implies that

$$\overline{y^0} \geq \frac{\varphi(x_0 + \lambda \overline{x}, y_0 + \lambda \overline{y}) - \varphi(x_0, y_0)}{\lambda}.$$

Moreover, φ is continuous and, by virtue of Eq. (2.32), the inequality in Eq. (2.29) is satisfied for all $(\overline{x}, \overline{y}, \overline{y_0})$ around $(\overline{x_0}, \overline{y_0}, \overline{y_0^0})$. This means that the point $(\overline{x_0}, \overline{y_0}, \overline{y_0^0})$ is an interior point of the cone defined by Eq. (2.29). Furthermore, note that

$$K_{\text{gph } \tilde{F}}(x_0, y_0, y_0^0) = [K_a(z_0) \times \mathbb{R}] \cap \text{cone } (\tilde{M} - \tilde{z}_0).$$

Then, by Theorem 1.11, it follows from Eqs. (2.30) and (2.31) that $(x^*, y^*, y^{0*}) \in K^*_{\text{gph } \tilde{F}}(\tilde{z}_0)$ if and only if

$$\begin{gathered} x^* = x_1^* - \lambda x_0^*, \quad y^* = y_1^* - \lambda y_0^*, \quad y^{0*} = \lambda, \\ (x_1^*, y_1^*) \in K^*_{\text{gph } F}(z_0), \quad (x_0^*, y_0^*) \in \partial_z \varphi(z_0), \quad \lambda \geq 0. \end{gathered}$$

In particular, $x^* \in \tilde{F}^*(y^*, y^{0*}; \tilde{z}_0)$; i.e., $(x^*, \ -y^*, \ -y^{0*}) \in K^*_{\text{gph } \tilde{F}}(\tilde{z}_0)$, if and only if

$$\begin{array}{ll} x^* = x_1^* - \lambda x_0^*, & -y^* = y_1^* - \lambda y_0^*, \quad -y^{0*} = \lambda, \\ x_1^* \in F^*(-y_1^*;\ z_0), & (x_0^*, y_0^*) \in \partial_z \varphi(z_0), \quad \lambda \ge 0. \end{array} \tag{2.33}$$

Now let $x_0 \in \text{dom}\ F$, $y^* = 0$, $y^{0*} = -1$. Suppose that the minimum of $\varphi(x_0, y)$ on $y \in F(x_0)$ is attained at y_0. Then, from the expression of the above calculated $H_{\tilde{F}}(x, y^*, y^{0*})$, we obtain

$$(y_0, \varphi(x_0, y_0)) \in \tilde{F}(x; 0, -1)$$

because $\langle y, y^* \rangle + y^{0*} y^0$, where $y^* = 0$, $y^{0*} = -1$, attains its maximum on $y \in F(x_0)$, $y^0 \ge \varphi(x_0, y)$ just at $(y^0, \varphi(x_0, y_0))$. Since $f(x) = -H_{\tilde{F}}(x, 0, -1)$, then $\partial f(x_0) = \partial_x(-H_{\tilde{F}}(x_0, 0, -1))$. Thus, by Theorem 2.1, we have

$$\partial f(x_0) = -\tilde{F}^*(0, -1; \tilde{z}_0), \quad \tilde{z}_0 = (x_0, y_0, \varphi(x_0, y_0)).$$

Therefore, by using Eq. (2.33), we have that $x^* \in \tilde{F} * (0, -1; \tilde{z}_0)$ if and only if

$$\begin{array}{ll} x^* = x_1^* - x_0^*, & y_1^* = y_0^*, \\ x_1^* \in F^*(-y_1^*; z_0), & (x_0^*, y_0^*) \in \partial_z \varphi(z_0). \end{array} \tag{2.34}$$

Theorem 2.11. Let $F: X \to P(Y)$ be bounded, closed, and convex, and suppose that φ is a proper convex function such that dom $\varphi = Z$. Furthermore, suppose that

$$f(x) = \min_y \{\varphi(x, y) : y \in F(x)\}. \tag{2.35}$$

Then f is a convex function and for any $x_0 \in \text{dom}\ F$,

$$\partial f(x_0) = \{x^* - F^*(-y^*; z_0) : (x^*, y^*) \in \partial_z \varphi(z_0)\}, \tag{2.36}$$

where $z_0 = (x_0, y_0)$ and $y_0 \in F(x_0)$ is any point that minimizes φ in Eq. (2.35). In particular, if φ is differentiable, then

$$\partial f(x_0) = \varphi'_x(z_0) - F^*(-\varphi'_y(z_0), z_0),$$

where $\varphi'_x(z_0)$ and $\varphi'_y(z_0)$ are the vectors of partial derivatives with respect to x and y, correspondingly.

☐ Indeed, we can rewrite Eq. (2.34) as follows:

$$-x^* = x_0^* - x_1^*, \quad x_1^* \in F^*(-y_0^*; z_0), (x_0^*, y_0^*) \in \partial_z \varphi(z_0),$$

which means that $-x^* \in x_0^* - F^*(-y_0^*;\ z_0)$, $(x_0^*,\ y_0^*) \in \partial_x \varphi(z_0)$. Consequently, taking into account $\partial f(x_0) = -\tilde{F}^*(0, -1; \tilde{z}_0)$ and denoting x_0^* and y_0^* again by x^* and y^*, we obtain Eq. (2.36). Furthermore, if φ is differentiable, then $\partial_z\varphi(z_0)$ consists of the unique vector-gradient and $x^* = \varphi'_x(z_0),\ y^* = \varphi'_y(z_0)$. ■

Let us consider the multivalued mapping defined by

$$F(x) = \{A : x \in X\},$$

where A is a closed convex set in Y. Observe that gph $F = X \times A$, so

$$K_{\text{gph } F}(z_0) = X \times \text{cone}(A - y_0).$$

Therefore, $K^*_{\text{gph } F}(z_0)$ has the form

$$K^*_a(z_0) = \{0\} \times [\text{cone}(A - y_0)]^*.$$

It follows that

$$F^*(y^*; z_0) = \begin{cases} 0, & \text{if } -y^* \in [\text{cone}(A - y_0)]^*, \\ \varnothing, & \text{if } -y^* \in [\text{cone}(A - y_0)]^*. \end{cases} \tag{2.37}$$

Theorem 2.12. Let $\varphi: Z \to \mathbb{R}$ be a convex function continuous in x on some neighborhood of x_0, and $y \in A$. Furthermore, let

$$f(x) = \min_y \{\varphi(x, y) : y \in A\} \tag{2.38}$$

and suppose that the minimum is attained at $y_0 \in A$, where $x = x_0$. Then

$$\partial f(x_0) = \{x^* : (x_0^*, y_0^*) \in \partial_z \varphi(z_0), y^* \in [\text{cone}(A - y_0)]^*\}.$$

□ Note that in the proof of Theorem 2.11, the boundedness of F was used for the existence of a point $y_0 \in F(x_0)$, where $\varphi(x_0, y)$ has a minimum. Therefore, in the formulation of the theorem, instead of the boundedness of F, we may assume that the minimum is attained at $y_0 \in A$, where $x = x_0$. Observe also that, in place of the condition dom $\varphi = Z$, it is enough to assume the continuity of φ in x in some neighborhood of x_0, and $y \in A$. Now use Eq. (2.37) and the previous theorem. ■

Example 2.1. Let A be a convex set in Y and let gph $F = X \times A$; i.e., $F(x) = A$ for all $x \in X$.

Then,

$$H(x, y^*) = \sup_y \{\langle y, y^* \rangle : y \in A\} = H_A(y^*),$$

where H_A is the support function of A. On the other hand,

$$M(x^*, y^*) = \inf_{(x,y)}\{\langle x, x^*\rangle - \langle y, y^*\rangle : x \in X, y \in A\} = \begin{cases} -\infty, & \text{if } x^* \neq 0, \\ -H_M(y^*), & \text{if } x^* = 0. \end{cases}$$

Above, we have seen that the LAM to F has the form shown in Eq. (2.37).

Example 2.2. Let $F(x) = Ax + U$, where U is a convex subset of Y and A is an operator from X to Y. Then

$$\begin{aligned} H(x, y^*) &= \sup_y\{\langle y, y^*\rangle : y \in F(x)\} \\ &= \langle Ax, y^*\rangle + \sup_y\{\langle u, y^*\rangle : u \in U\} = \langle Ax, y^*\rangle + H_U(y^*), \end{aligned}$$

and by Eq. (2.6), we have

$$M(x^*, y^*) = \inf_x\{\langle x, x^*\rangle - H(x, y^*)\} = \inf_x\{\langle x, x^* - A^*y^*\rangle - H_U(y^*)\}.$$

Thus,

$$M(x^*, y^*) = \begin{cases} -\infty & \text{if } A^*y^* \neq x^*, \\ H_U(y^*), & \text{if } A^*y^* = x^*. \end{cases}$$

If $y_0 = Ax_0 + u_0$, $u_0 \in U$, then it is easy to see that

$$\begin{aligned} F(x_0; y^*) &= \{y_0 \in Ax_0 + U : \langle y_0, y^*\rangle = H(x_0, y^*)\} = \{u_0 \in U : \langle u_0, y^*\rangle = H_U(y^*)\} \\ &= \{u_0 \in U : -y^* \in K_U^*(u_0)\}, \end{aligned}$$

where $K_U^*(u_0) = \{\bar{u} : \bar{u} = \lambda(u - u_0)\ \forall \lambda > 0, \ \ u \in U\}$.

Therefore, by Theorem 2.1, we conclude that

$$F^*(y^*; z_0) = \begin{cases} A^*y^*, & \text{if } -y^* \in [\operatorname{cone}(U - u_0)]^*, \\ \varnothing, & \text{if } -y^* \notin [\operatorname{cone}(U - u_0)]^*. \end{cases}$$

Example 2.3. Let $Y = \mathbb{R}$ and let f be any proper convex function. Let us define F by

$$\operatorname{gph} F = \operatorname{epi} f, \quad F(x) = \{y : y \geq f(x)\}.$$

Then, $y^* \in \mathbb{R}$ and

$$H(x, y^*) = \sup_y\{yy^* : y \geq f(x)\} = \begin{cases} \infty \cdot \operatorname{sgn} y^*, & \text{if } y^* \neq 0, \quad x \notin \operatorname{dom} f, \\ +\infty, & \text{if } y^* > 0, \quad x \in \operatorname{dom} f, \\ y^* f(x), & \text{if } y^* \leq 0, \quad x \in \operatorname{dom} f. \end{cases}$$

On the other hand,

$$M(x^*, y^*) = \inf_{(x,y)}\{\langle x, x^*\rangle - yy^* : y \geq f(x)\} = \begin{cases} -\infty & \text{if } y^* > 0, \\ -H_{\operatorname{dom} f}(-x^*), & \text{if } y^* = 0. \end{cases}$$

In the case where $y^* < 0$, we can write

$$M(x^*, y^*) = \inf_x \{\langle x, x^*\rangle - y^* f(x)\} = y^* \sup_x \left\{\left\langle x, \frac{x^*}{y^*}\right\rangle - f(x)\right\} = y^* f^*\left(\frac{x^*}{y^*}\right).$$

Now, let $x_0 \in \operatorname{dom} f$, $y_0 = f(x_0)$ and $y^* \leq 0$. Then, since $y \geq f(x_0)$, the maximum of yy^* attains if $y = y_0 = f(x_0)$ or $y_0 \in F(x; y^*)$. Then, by virtue of Theorem 2.1 and the above expression for $H(x, y^*)$, we obtain that

$$F^*(y^*; z_0) = \begin{cases} y^* \partial f(x_0), & \text{if } y^* < 0, \\ \partial \delta_{\operatorname{dom} f}(x_0), & \text{if } y^* = 0. \end{cases}$$
$$z_0 = (x_0, f(x_0)), \quad x_0 \in \operatorname{dom} f.$$

Example 2.4. Let $\varphi : Z \to \mathbb{R}$, $Z = X \times Y$ be a convex function. Set

$$\operatorname{gph} F = \{z : \varphi(z) \leq 0\}, \quad F(x) = \{y : \varphi(x, y) \leq 0\}.$$

Let $\varphi(z_0) = 0$ and φ be continuous at z_0. Furthermore, let z_1 be such that $\varphi(z_1) < 0$. Then, by Theorem 1.34,

$$K^*_{\operatorname{gph} F}(z_0) = [\operatorname{cone}(\operatorname{gph} F - z_0)]^* = -\operatorname{cone} \partial_z \varphi(z_0),$$

so

$$F^*(y^*; z_0) = \{-\lambda x_0^* : y^* = \lambda y_0^*, \quad (x_0^*, y_0^*) \in \partial_z \varphi(z_0), \quad \lambda \geq 0\}. \tag{2.39}$$

Thus, we can formulate the next result.

Lemma 2.10. Let $\varphi : Z \to \mathbb{R}$, $Z = X \times Y$ be a convex function and suppose that φ is continuous at z_0, $\varphi(z_0) = 0$. Furthermore, let z_1 be a point such that $\varphi(z_1) < 0$. Then, Eq. (2.39) holds. ■

Example 2.5. Let $\varphi(z) = \max_{1 \leq i \leq m} \varphi_i(z)$ and suppose that φ_i is continuous at z_0. Then, according to Theorem 1.32,

$$\partial \varphi(z) = \overline{\operatorname{conv}}\left(\bigcup_{i \in I(z)} \partial_z \varphi_i(z)\right), \quad I(z) = \{i : \varphi_i(z) = \varphi(z)\}. \tag{2.40}$$

By Lemma 1.34 and Remark 1.2, both $\partial_z\varphi_i(z)$, $i \in I(z)$ are closed and bounded. Then their union is closed and bounded too. Now by Lemma 1.8 and Theorem 1.3, Eq. (2.40) takes the form

$$\partial\varphi(z) = \operatorname{conv}\left(\bigcup_{i \in I(z)} \partial_z\varphi_i(z)\right). \tag{2.41}$$

Observe that by Lemma 1.5,

$$z^* \in \operatorname{conv}\left(\bigcup_{i \in I(z)} \partial_z\varphi_i(z)\right) \tag{2.42}$$

if and only if

$$z^* = \lambda_1 z_1^* + \cdots + \lambda_k z_k^*, \quad \sum_{i=1}^{k} \lambda_1 = 1, \quad \lambda_i \geq 0,$$

where each z_j^* belongs to one of the sets $\partial_z\varphi_i(z)$. Let us write

$$J_i = \{j : z_j^* \in \partial_z\varphi_i(z)\}, \quad \gamma_i = \sum_{j \in J_i} \lambda_j, \quad i \in I(z).$$

Then

$$z^* = \sum_{i=1}^{k} \gamma_i \left(\sum_{j \in J_i} \frac{\lambda_j}{\gamma_j} z_j^*\right). \tag{2.43}$$

Since $\partial_z\varphi_I(z)$ is convex and $\sum_{j \in J_i} \lambda_j/\gamma_i = 1$,

$$\sum_{j \in J_i} \frac{\lambda_j}{\gamma_i} z_j^* \in \partial_z\varphi_i(z). \tag{2.44}$$

By virtue of Eqs. (2.43) and (2.44), it follows that Eq. (2.42) is fulfilled if and only if

$$z^* = \sum_{i \in I(z)}^{k} \gamma_i z_i^*, \quad z_i^* \in \partial_z\varphi_i(z), \quad \gamma_i \geq 0, \quad \sum_{i \in I(z)} \gamma_i = 1.$$

Now Eq. (2.41) gives us

$$\partial_2\varphi(z) = \left\{\sum_{i \in I(z)} \gamma_i z_i^* : z_i^* \in \partial_z\varphi_i(z), \quad \gamma_i \geq 0, i \in I(z), \quad \sum_{i \in I(z)} \gamma_i = 1\right\}. \tag{2.45}$$

Finally, substituting $\partial z\varphi(z_0)$, given by Eq. (2.45), into Eq. (2.39), writing $\lambda_i = \lambda\gamma_i$, and taking into account that $\varphi(z_0) = 0$ implies $I(z_0) = I_0$, we have established the result given in Theorem 2.13.

Theorem 2.13. Suppose that φ_i, $i = 1, \dots, m$ are convex functions, continuous at z_0, where $\varphi(z_0) = 0$. Moreover, assume that there is a point z_1 such that $\varphi_i(z_1) < 0$, $i = 1, \dots, m$. Then the LAM to the multivalued mapping given by

$$F(x) = \{y : \varphi_i(x,y) \le 0, \quad i = 1, \dots, m\}$$

has the form

$$F^*(y^*; z_0) = \left\{ -\sum_{i \in I_0} \lambda_i x_i^* : y^* = \sum_{i \in I_0} \lambda_i y_i^*, \quad (x_i^*, y_i^*) \in \partial_z \varphi_i(z), \quad \lambda_i \ge 0, \quad i \in I_0 \right\}, \tag{2.46}$$

where $I_0 = \{i : \varphi_i\,(x_0, y_0) = 0\}$. ■

Example 2.6. Let $f_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$ be convex functions and

$$\text{gph}\, F = \{(x,y) : f_i(y) \le x_i, \quad i = 1, \dots, m\}$$

or

$$\text{gph}\, F = \{(x,y) : f(y) \le x\},$$

where $f(y) = (f_1(y), \dots, f_m(y))$, $x = (x^1, \dots, x^m)$ so that

$$F(x) = \{y : f(y) \le x\}.$$

From here it should be

$$H(x, y^*) = \sup_y \{\langle y, y^* \rangle : f(y) \le x\},$$

and

$$M(x^*, y^*) = \inf_{(x,\,y)} \{\langle x, x^* \rangle - \langle y, y^* \rangle : f(y) \le x\} = -\infty$$

if at least one of the components of x^* is less than zero. If $x^* \ge 0$, then

$$M(x^*, y^*) = \inf_y \{\langle f(y), x^* \rangle - \langle y, y^* \rangle\}.$$

Now, in the previous theorem, let $\varphi_i(z) = f_i(y) - x^i$. Then, if we choose $z_1 = (x_1, y_1)$, where y_1 is arbitrary and $x_1^i > f_i(y_1)$, then $\varphi_I(z_1) < 0$, $i = 1, \dots, m$. On the other hand, it is easy to see that

$$\partial_z \varphi_i(z) = (-e_i, \partial_y f_i(y)), \tag{2.47}$$

where $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ (the ith component of e_i is 1). Therefore, $(x_i^*, y_i^*) \in \partial_z \varphi_i(z_0)$ if and only if $x_i^* = -e_i, y_i^* \in \partial_y f_i(y)$. Here, note that in Eq. (2.46), $\varphi_i(z_0) = \varphi(z_0) = 0$ implies that $\lambda_i \neq 0$. Also note that $f_i(y_0) < x_0^i$, $i = 1, \dots, m$, then $K_{\text{gph } F}(z_0) = Z, K^*_{\text{gph } F}(z_0) = \{0\}$, and $x^* \in F^*(y^*; z_0)$ if and only if $x^* = 0$, $y^* = 0$. Furthermore, $\sum_{i=1}^m \lambda_i e_i = \lambda$. Consequently, taking into account Eq. (2.47) in the formula in Eq. (2.46), we have proved the following theorem.

Theorem 2.14. Let f_i: $\mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$ be convex functions and

$$\text{gph } F = \{(x, y) : f_i(y) \leq x_i, i = 1, \dots, m\}.$$

Then

$$F^*(y^*; z_0) = \left\{ \lambda \in \mathbb{R}^m : \lambda_i \geq 0, \lambda_i(f_i(y_0) - x_0^i) = 0, \quad i = 1, \dots, m, \right.$$
$$\left. y^* + \sum_{i=1}^m \lambda_i y_i^* = 0 \right\} \blacksquare$$

Now, suppose that we have a convex set $M \subseteq X$ and

$$\text{gph } \tilde{F} = \text{gph } F \cap (M \times Y) = \{(x, y) : \varphi(x, y) \leq 0, x \in M\}.$$

Then

$$K_{\text{gph } \tilde{F}}(z_0) = K_{\text{gph } F}(z_0) \cap \text{cone } ((M - x_0) \times (Y - y_0)), \quad z_0 = (x_0, y_0) \in \text{gph } F.$$

Observe that if there is a point (x_1, y_1), such that $x_1 \in M$, $\varphi(x_1, y_1) < 0$, then int gph $F \cap (M \times Y) \neq \varnothing$, and so

$$\text{int } K_{\text{gph } F}(z_0) \cap \text{cone } ((M - x_0) \times (Y - y_0)) \neq \varnothing.$$

Therefore, by Theorem 1.11

$$\begin{aligned} K^*_{\text{gph } \tilde{F}}(z_0) &= K^*_{\text{gph } F}(z_0) + [\text{cone}((M - x_0) \times (Y - y_0))]^* \\ &= K^*_{\text{gph } F}(z_0) + [\text{cone}(M - x_0)]^* \times \{0\}. \end{aligned}$$

Now, if $\varphi(z_0)=0$

$$K^*_{\mathrm{gph}\,\tilde{F}}(z_0) = -\,\mathrm{cone}\,\partial_z\varphi(z_0) + [\mathrm{cone}(M-x_0)]^* \times \{0\}$$

and $(x_1^*,\,-y_1^*) \in K^*_{\mathrm{gph}\,\tilde{F}}(z_0)$; i.e., $x^* \in \tilde{F}^*(y^*;z_0)$ if and only if

$$x^* = -\lambda x_0^* + x_1^*,\quad y^* = \lambda y_0^*,\quad (x_0^*,y_0^*) \in \partial_z\varphi(z_0),\quad x_1^* \in [\mathrm{cone}(M-x_0)]^*,\quad \lambda \geq 0.$$

What we have thus obtained may be formulated as shown in Theorem 2.15.

Theorem 2.15. Let $\varphi: Z \to \mathbb{R}$, $Z = X \times Y$ be a convex function and φ be continuous at a point z_0, $\varphi(z_0)=0$. Furthermore, let $z_1=(x_1,\,y_1)$ be a point such that $\varphi(z_1)<0$, $x_1 \in M$, where M is a convex set. Then

$$\tilde{F}*(y^*;z_0) = \{-\lambda x_0^* + x_1^* : y^* = \lambda y_0^*,\, (x_0^*,y_0^*) \in \partial_z\varphi(z_0),\, x_1^* \in [\mathrm{cone}(M-x_0)]^*,\quad \lambda \geq 0\}. \blacksquare$$

Example 2.7. Let K be a closed convex cone in $Z = X \times Y$ and gph $F = K$. Then,

$$M(x^*,y^*) = \sup_{(x,y)}\{\langle x,x^*\rangle - \langle y,y^*\rangle : (x,y) \in K\} = \begin{cases} 0, & \text{if } (x^*,\ -y^*) \in K^*, \\ -\infty, & \text{if } (x^*,\ -y^*) \notin K^*. \end{cases}$$

Further, for $\lambda > 0$,

$$H(\lambda x,y^*) = \sup_y \left\{\langle y,y^*\rangle : (\lambda x,y) \in K\right\} = \lambda \sup_y \left\{\left\langle \frac{y}{\lambda},y^*\right\rangle : \left(x,\frac{y}{\lambda}\right) \in \frac{1}{\lambda}K\right\}.$$

Observe that since K is the cone, so $\lambda^{-1}K = K$. Thus, denoting $y_1 = \lambda^{-1}y$, we have

$$H(\lambda x,y^*) = \lambda \sup_{y_1}\{\langle y_1,y^*\rangle : (x,y_1) \in K\} = \lambda H(x,y^*);$$

i.e., the Hamiltonian function $H(x,\,y^*)$ is the positively homogeneous in x.

Take a point $(x_0,\,y_0) \in K$. Since $K_{\mathrm{gph}\,F}(z_0) = \mathrm{cone}(K - z_0)$, Lemma 1.20 implies that $(x^*,\,-y^*) \in K^*_{\mathrm{gph}\,F}(z_0)$ if and only if

$$\langle x-x_0,x^*\rangle - \langle y-y_0,y^*\rangle \geq 0 \quad \forall (x,y) \in K.$$

Rewriting this inequality in the form

$$\langle x,x^*\rangle - \langle y,y^*\rangle \geq \langle x_0,x^*\rangle - \langle y_0,y^*\rangle$$

and applying Lemma 1.18, we derive that $(x^*, -y^*) \in K^*$ and the infimum of the left-hand side over $(x, y) \in K$ is zero. Therefore,

$$0 \geq \langle x_0, x^* \rangle - \langle y_0, y^* \rangle.$$

On the other hand, $(x_0, y_0) \in K$ and $(x^*, -y^*) \in K^*$. Thus,

$$0 \leq \langle x_0, x^* \rangle - \langle y_0, y^* \rangle;$$

i.e., $\langle x_0, x^* \rangle - \langle y_0, y^* \rangle$. Consequently,

$$F^*(y^*; z_0) = \{x^* : (x^*, -y^*) \in K^*, \langle x_0, x^* \rangle = \langle y_0, y^* \rangle\}.$$

Example 2.8. (Polyhedral mapping) Let $X = Y = \mathbb{R}^n$, and let A, B be $m \times n$ matrices. Moreover, let A_i, B_i be the ith row of A and B, and let d_i be the ith component of the m-dimensional column vector d. Define a polyhedral mapping, the graph of which is polyhedral set in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\text{gph } F = \{(x, y) : Ax - By \leq d\}. \tag{2.48}$$

Observe that

$$F(x) = \{y : Ax - By \leq d\}.$$

Take $z_0 = (x_0, y_0) \in \text{gph } F$ and denote the set of active indices by

$$I_0 = \{i : A_i x_0 - B_i y_0 = d_i, \ i = 1, \ldots, m\}.$$

At first, let us calculate the cone

$$K_{\text{gph } F}(z) = \{\bar{z} : z + \lambda z \in \text{gph } F \text{ for sufficiently small } \lambda > 0\}.$$

Let $i \in I_0$. Then, the inequality

$$A_i(x_0 + \lambda\bar{x}) - B_i(y_0 + \lambda\bar{y}) = d_i + \lambda(A_i\bar{x} - B_i\bar{y}) \leq d_i$$

holds if

$$A_i\bar{x} - B_i\bar{y} \leq 0, \quad i \in I_0. \tag{2.49}$$

If $i \in I_0$, then

$$A_i(x_0 + \lambda\bar{x}) - B_i(y_0 + \lambda\bar{y}) = (A_i x_0 - B_i y_0) + \lambda(A_i\bar{x} - B_i\bar{y}) < d_i$$

holds for small λ and arbitrarily $\bar{z}$. It follows that the cone $K_{\text{gph } F}(z_0)$ is defined completely by Eq. (2.49). Rewriting Eq. (2.49) as

$$\langle \bar{x} - A_i \rangle + \langle \bar{y} - B_i \rangle \geq 0, \quad i \in I_0$$

and then applying Theorem 1.13, we deduce that $(x^*, y^*) \in K^*_{\text{gph } F}(z_0)$ if and only if

$$x^* = -\sum_{i \in I_0} A_i^* \lambda_i, \quad y^* = \sum_{i \in I_0} B_i^* \lambda_i, \quad \lambda_i \geq 0, \tag{2.50}$$

where A_i^*, B_i^* are the transposed vectors (i.e., column vectors) of A_i and B_i, respectively. Taking $\lambda_i = 0$, $i \notin I_0$ and denoting the column vector by λ, with components λ_i, we can rewrite Eq. (2.50) in the form

$$K^*_{\text{gph } F}(z_0) = \{(x^*, y^*) : x^* = -A^* \lambda, \quad y^* = B^* \lambda, \quad \lambda \geq 0, \quad \langle Ax_0 - By_0 - d, \quad \lambda \rangle = 0\}.$$

From this we have the following formula for the LAM to $F^*(y^*; z_0)$:

$$F^*(y^*; z_0) = \{-A^* \lambda : y^* = -B^* \lambda, \quad \lambda \geq 0, \quad \langle Ax_0 - By_0 - d, \lambda \rangle = 0\}. \tag{2.51}$$

Since the number of sets $I(z_0) = \{i : A_i x_0 - B_i y_0 = d_i\}$ is finite, it follows that the number of different LAM $F^*(y^*; z_0)$ is finite too.

Furthermore, by Eqs. (1.27) and (1.28), it is not hard to see that the supremum of $\langle y, y^* \rangle$, $y \in F(x)$ is either $+\infty$ or a finite number that is attained. Therefore, $(x, x^0) \in \text{epi } [-H(\cdot, -y^*)]$, $x^0 \in \mathbb{R}$ if and only if $\langle y, -y^* \rangle \leq x^0$, $y \in F(x)$, for some y; i.e., (x, x^0, y) is a solution of the system of linear inequalities

$$Ax - By \leq d, \quad \langle y, -y^* \rangle \leq x^0. \tag{2.52}$$

According to Theorem 1.14, there exist vectors $(x_j, x_j^0, y_j), j = 1, \ldots, k$ and an index set $J_0 \subseteq \{1, 2, \ldots, k\}$ such that every solution of Eq. (2.52) can be represented in the form

$$\begin{gathered} x = \sum_{j=1}^{k} \gamma_j x_j, \quad x^0 = \sum_{j=1}^{k} \gamma_j x_j^0, \quad \gamma_j \geq 0, \quad \sum_{j \in J_0} \gamma_j = 1 \\ y = \sum_{j=1}^{k} \gamma_j y_j. \end{gathered} \tag{2.53}$$

Clearly, epi $[-H(\cdot, -y^*)]$ is a set of points satisfying Eq. (2.53). Hence, by Theorem 1.14, epi $[-H(\cdot, -y^*)]$ is a polyhedral set, which can be described by a system of linear inequalities and so is closed. Thus, we have proved Theorem 2.16.

Theorem 2.16. For a polyhedral mapping, Eq. (2.48), the Hamiltonian function $H(\cdot, y^*)$ is closed and LAM (Eq. (2.51)) is a step function in the argument z_0.

Theorem 2.17. A polyhedral mapping F is bounded if and only if $B^*\mathbb{R}^m_+ = \mathbb{R}^n$. In addition, $x \in \operatorname{dom} F$ if and only if $\langle u, d - Ax\rangle \geq 0 \ \forall u \in N = \{u \in \mathbb{R}^m : B^*u = 0, \ u \geq 0\}$. Here, $\mathbb{R}^m_+$ is a positive orthant of $\mathbb{R}^m$.

□ Obviously, $F(x)$ is both nonempty and bounded if and only if $H(x, y^*) = \max_y\{\langle y, y^*\rangle : y \in F(x)\}$ is bounded for all y^*. Then, the calculation of $H(x, y^*)$ is reduced to the following primary linear programming problem:

$$\text{maximize } \langle y, y^*\rangle \quad \text{subject to } Ax - By \leq d.$$

Its duality problem is

$$\text{maximize } \langle u, d - Ax\rangle \quad \text{subject to } B^*u = -y^*, \ \ u \in \mathbb{R}^m_+.$$

According to standard duality theorems (see, for example, Refs. [111,224,226]) the solvability of one of these problem implies the solvability of the other problem. Thus, it follows that $B^*u = y^*$ must be solvable for every $y^* \in \mathbb{R}^n$; i.e., $B^*\mathbb{R}^m_+ = \mathbb{R}^n$. This condition is necessary and sufficient for the boundedness of $F(x)$, if $F(x) \neq \varnothing$. Consider now the problem when $F(x) \neq \varnothing$. Since by the duality theorem of linear programming,

$$H(x, y^*) = \min_u\{\langle u, d - Ax\rangle, \quad B^*u = -y^*, \ \ u \geq 0\},$$

then the minimum in the duality problem is attained and finite for all y^* if $F(x) \neq \varnothing$. We now show that here the minimum value is finite if and only if

$$\langle u, d - Ax\rangle \geq 0 \ \forall \ u \in N = \{u \in \mathbb{R}^m : B^*u = 0, u \geq 0\}.$$

Indeed, since by Theorem 1.14, the set of solutions of $B^*u = -y^*; u \geq 0$ is represented as a sum of a closed bounded polytope M and a cone N, we can write

$$\begin{aligned} H(x, y^*) = \min_u \{\langle u, d - Ax\rangle, \quad B^*u = -y^*, \ \ u \geq 0\} &= \min_u\{\langle u, d - Ax\rangle, \ \ u \in M\} \\ &+ \min_u\{\langle u, d - Ax\rangle, \ \ u \in N\}. \end{aligned}$$

But $H(x, y^*)$ is finite and so $\langle u, d - Ax\rangle \geq 0 \ \ \forall u \in N$. The converse assertion is obvious.

Indeed, $H(x, y^*) = \min_u\{\langle u, d - Ax\rangle, \ \ u \in N\} = 0$ if $\langle u, d - Ax\rangle \geq 0, \ \ \forall u \in N$, so $H(x, y^*) = \min_u\{\langle u, d - Ax\rangle, \ \ u \in M\}$. ■

Corollary 2.3. $\operatorname{dom} F = \{x : \langle u_j, d - Ax\rangle \geq 0, j = 1, 2, \ldots, l\}$, where $u_j, j = 1, 2, \ldots, l$ are directions that generate the cone $N = \{u \in \mathbb{R}^m : B^*u = 0, \ u \geq 0\}$.

Theorem 2.18. A polyhedral mapping F is bounded and dom $F = \mathbb{R}^n$ if and only if

$$F(x) = C_+ x + F(0)$$

for some matrix C_+.

☐ By Theorem 2.17, dom $F = \mathbb{R}^n$ if and only if

$$\langle u, d\rangle \geq 0, \ A^* u = 0 \quad \forall u \in N.$$

Applying repeatedly Theorem 1.13 (Farkas), we obtain

$$\begin{aligned} &d = Bc + e, \ \ e \geq 0, \ c \in \mathbb{R}^n, \ \ e \in \mathbb{R}^n, \\ &\quad A = BC_+ + \Phi_+, \Phi_+ \geq 0, \\ &\quad -A = BC_- + \Phi_-, \Phi_- \geq 0, \end{aligned}$$

where C_+, C_- are $n \times n$ matrices and Φ_+, Φ_- are $m \times n$ matrices. Then, $0 = BC + \Phi$, where $C = C_+ + C_-$; $\Phi = \Phi_+ + \Phi_-$, $\Phi \geq 0$.

Remember that for every $y^* \in \mathbb{R}^n$ there is a $u \geq 0$ such that $B^* u = y^*$. Thus, transposing $0 = BC + \Phi$ and multiplying it by u, we have

$$0 = C^* y^* + \Phi^* u.$$

Since $\Phi^* \geq 0$, $u \geq 0$, then $C^* y^* \leq 0 \ \forall y^*$. But this implies $C = 0$. Hence, $\Phi = 0$; i. e., $\Phi_+ = -\Phi_-$. On the other hand, $\Phi_- \geq 0$ and so $\Phi_+ \leq 0$. Therefore, $\Phi_+ \leq 0$ and $\Phi_+ \geq 0$, which implies that $\Phi_+ = 0$. Thus, $A = BC_+$. Then

$$F(x) = \{y : B(C_+ x - y) \leq d\} = \{y : -C_+ x + y \in F(0)\} = C_+ x + F(0). \blacksquare$$

2.5 Duality Theorems for Convex Multivalued Mappings

In this section we will prove a result that can be considered as a duality theorem for a convex multivalued mapping. As we will see, this result implies a lot of other theorems of convex analysis and the theory of extremal problems.

Theorem 2.19. Let F be a convex mapping and suppose that its Hamiltonian function H, regarded as a function of x, is closed. Moreover, let x_1 be a point such that $H(x_1, y^*)$ is finite. Then,

$$\inf_{x^*}\{\langle x, x^*\rangle - M(x^*, y^*)\} = H(x, y^*).$$

☐ Let $H(x_0, y^*) \neq -\infty$, and denote $\gamma_0 = H(x_0, y^*)$. Then for an arbitrary $\varepsilon > 0$, there is a convex neighborhood U of x_0 such that $\Gamma_\varepsilon \cap \operatorname{graph} F = \varnothing$, where

$$\Gamma_\varepsilon = \{(x, y) : x \in U, \quad \langle y, y^*\rangle \geq \gamma_0 + \varepsilon\}$$

This is a direct consequence of the closedness of $H(\cdot, y^*)$. Furthermore, Γ_ε is clearly convex. Thus, by the separating theorems of convex sets, there exist vectors x_0^*, y_0^* simultaneously not zero, such that

$$\begin{aligned}&\langle x, x_0^*\rangle - \langle y, y_0^*\rangle \geq \langle \xi, x_0^*\rangle - \langle \eta, y_0^*\rangle,\\ &\quad (x,y)\in \operatorname{gph} F,\ (\xi,\eta)\in \Gamma_\varepsilon.\end{aligned} \tag{2.54}$$

Setting $x = \xi = x_0$, we have

$$-\langle y, y_0^*\rangle \leq -\langle \eta, y_0^*\rangle \tag{2.55}$$

for $y \in F(x_0)$, $\langle \eta, y^*\rangle \leq \gamma_0 + \varepsilon$.

We show that $y_0{}^* = \lambda y^*$ and $\lambda > 0$. Indeed, it follows from Eq. (2.55) that $\langle \overline{\eta}, y^*\rangle \geq 0$ implies $\langle \overline{\eta}, y_0^*\rangle \geq 0$. By Theorem 1.12, we have $y_0^* = \lambda y^*$. We now prove that $\lambda > 0$.

If $\lambda > 0$, then $y_0^* = \lambda y^* = 0$, and at $x = x_0$ the inequality in Eq. (2.54) has the form

$$\langle x, x_0^*\rangle \leq \langle \xi, x_0^*\rangle, \quad \xi \in U, \tag{2.56}$$

or

$$\langle \xi - x_0, x_0^*\rangle \geq 0, \quad \xi \in U.$$

Therefore, $x_0^* \equiv 0$. This contradicts the fact that x_0^* and y_0^* are nonzero vectors. Consequently, $\lambda > 0$, and we may take $\lambda = 1$, $y_0^* = y^*$. Then Eq. (2.54) takes the form

$$\langle x, x_0^*\rangle - \langle y, y^*\rangle \geq \langle \xi, x_0^*\rangle - \langle \eta, y^*\rangle,$$

or

$$M(x_0^*, y^*) \geq \langle x_0, x^*\rangle - \gamma_0 - \varepsilon.$$

Therefore, for a given $\varepsilon > 0$ there is a vector x_0^* such that for $\xi = x_0$,

$$H(x_0, y_0^*) + \varepsilon \geq \langle x_0, x^*\rangle - M(x_0^*, y^*),$$

or, by virtue of the arbitrariness of $\varepsilon > 0$,

$$H(x_0, y_0^*) = \inf_{x^*}\{\langle x_0, x^*\rangle - M(x^*, y^*)\}.$$

On the other hand, it follows from Definition 2.5 that

$$H(x_0, y_0^*) \leq \langle x_0, x^*\rangle - M(x_0^*, y^*).$$

Comparing the last two inequalities we have the required result.

Similarly, if $H(x_0,y^*) = -\infty$ (i.e., $F(x_0) = \varnothing$), repeating the aforementioned techniques, it can be shown that

$$\inf_{x^*}\{\langle x, x^*\rangle - M(x^*, y^*)\} = -\infty. \blacksquare$$

Corollary 2.4. If the Hamiltonian H regarded as a function of x is finite and upper semicontinuous at x_0, then

$$\inf_{x^*}\{\langle x_0, x^*\rangle - M(x^*, y^*)\} = H(x_0, y^*).$$

□ Indeed, in order to have $\Gamma_\varepsilon \cap \text{graph } F = \varnothing$ for a small ε, it suffices to have the semicontinuity of $H(\cdot, y^*)$ at x_0. ■

Recall that (Remark 2.1) a multivalued mapping F is quasisuperlinear if its graph is represented as gph $F = \Omega + K$, where Ω is a convex compactum and K is a closed convex cone.

The following result is important for further duality relations.

Proposition 2.1. For a convex mapping F, we have

$$\text{dom } M = \{(x^*, y^*) : M(x^*, y^*) > -\infty\} \subseteq (0^+ \text{ gph } F)^*.$$

In addition, if F is quasisuperlinear, then dom $M = K^*$.

□ Assume the contrary; suppose $(x_0^*, y_0^*) \in \text{dom } M$, but $(x_0^*, y_0^*) \notin (\text{gph } F)^*$. This means that there exists a pair $(\bar{x}_0, \bar{y}_0) \in \text{gph } F$ for which $\langle \bar{x}_0, x_0^*\rangle - \langle \bar{y}_0, y_0^*\rangle < 0$. By the definition of recession cone we can write $(x + \lambda\bar{x}_0, y + \lambda\bar{y}_0) \in \text{gph } F$ for all $(x, y) \in \text{gph } F$ and $\lambda > 0$. Then

$$\begin{aligned}&\langle x, + \lambda\bar{x}_0, x_0^*\rangle - \langle y + \lambda\bar{y}_0, y_0^*\rangle \\ &\quad = \langle x, x_0^*\rangle - \langle y, y_0^*\rangle + \lambda[\langle \bar{x}_0, x_0^*\rangle - \langle \bar{y}_0, y_0^*\rangle] \to -\infty, \text{ as } \lambda \to +\infty,\end{aligned}$$

and this contradiction proves the first statement of the lemma. Furthermore, when F is quasisuperlinear, we get

$$\text{dom } M = \text{dom } (M_\Omega + M_K) = \text{dom } M_\Omega \cap \text{dom } M_K = \text{dom } M_K = K^*,$$

where

$$M_A(x^*, y^*) = \inf_{(x,y)\in A}\{\langle x, x^*\rangle - \langle y, y^*\rangle\}. \blacksquare$$

Theorem 2.20. Let F be a quasisuperlinear mapping and $H(\cdot, y^*)$ be a closed proper convex function. Then, we have the duality relation

$$\inf_{x^*\in F^*(y^*)}\{\langle x, x^*\rangle - M(x^*, y^*)\} = \sup_{y\in F(x)} \langle y, y^*\rangle.$$

□ By the preceding proposition, dom $M = K^*$. Therefore, by Theorem 2.19, it is not hard to see that

$$\begin{aligned}\inf_{x^*} \{\langle x, x^*\rangle - M(x^*, y^*)\} &= \inf_{x^*}\{\langle x, x^*\rangle - M_{\Omega}(x^*, y^*) : x^* \in F^*(y^*)\} \\ &= \sup_{y \in F(x)} \langle y, y^*\rangle. \blacksquare\end{aligned}$$

Corollary 2.5. If gph $F = K$ is a convex cone and $H(\cdot, y^*)$ is a closed proper convex function, then $M(x^*, y^*) = 0$ for all $x^* \in F^*(y^*)$, so

$$\inf_{x^* \in F^*(y^*)} \langle x, x^*\rangle = \sup_{y \in F(x)} \langle y, y^*\rangle.$$

Corollary 2.6. If gph $F = K$ is a polyhedral cone, then

$$\inf_{x^* \in F^*(y^*)} \langle x, x^*\rangle = \sup_{y \in F(x)} \langle y, y^*\rangle.$$

□ It remains only to observe that by Theorem 2.16, for a polyhedral mapping, the Hamiltonian function $H(\cdot, y^*)$ is a closed function. ■

Remark 2.4. We have seen from Theorem 1.21 that if f is a closed proper convex function, then $f(x) = f^{**}(x)$. Now by using the duality, Theorem 2.19, we give an alternative way to prove this important formula. By Example 2.3, we have

$$\text{gph } F = \text{epi } f, \quad F(x) = \{y : y \geq f(x)\}$$

and

$$H(x, -1) = -f(x), \quad M(x^*, -1) = -f^*(-x^*).$$

Then by Theorem 2.19, we obtain

$$\inf_{x^*}\{\langle x, x^*\rangle + f^*(-x^*)\} = -f(x),$$

or, finally,

$$\sup_{x^*}\{\langle x, x^*\rangle - f^*(x^*)\} = f(x). \blacksquare$$

Let M be a closed convex subset of X and let N be a convex subset of Y. Furthermore, suppose that $f : Z \to \mathbb{R}$ is a function such that $f(\cdot, y)$ is convex and $f(x, \cdot)$ is concave.

Theorem 2.21. Suppose that $f: Z \to \mathbb{R}$, regarded as a function of x, is closed, proper, and $\operatorname{dom} f(\cdot, y) \cap M \neq \varnothing$, $y \in N$. Moreover, suppose that the function

$$g(x^*) = \inf_{y \in N} \sup_{x \in M} (\langle x, x^* \rangle - f(x, y)) \tag{2.57}$$

is finite and lower semicontinuous at $x^* = 0$. Then

$$\inf_{x \in M} \sup_{y \in N} f(x, y) = \sup_{y \in N} \inf_{x \in M} f(x, y). \tag{2.58}$$

□ Let us define

$$f_0(x, y) = \begin{cases} f(x, y), & \text{if } x \in M, \\ +\infty, & \text{if } x \notin M. \end{cases}$$

Since M is closed and $f(\cdot, y)$ is closed for fixed y, then

$$\operatorname{epi} f_0(\cdot, y) = \operatorname{epi} f(\cdot, y) \cap \{(x, x^0) : x \in M\},$$

which is the intersection of closed convex sets, and so $f_0(\cdot, y)$ is a closed convex function. Note that the concavity property of $f_0(x, \cdot)$ is preserved.

By virtue of $\operatorname{dom} f_0(\cdot, y) \cap M \neq \varnothing$, $y \in N$, the function $f_0(\cdot, y)$, $y \in N$ is proper. Observe that Eq. (2.58) is equivalent to

$$\inf_{x \in X} \sup_{y \in N} f_0(x, y) = \sup_{y \in N} \inf_{x \in X} f_0(x, y). \tag{2.59}$$

Let

$$f_0^*(x^*, y) = \sup_x \{\langle x, x^* \rangle - f_0(x, y)\}.$$

This function, as the supremum of the convex functions $\langle x, x^* \rangle - f_0(x, y)$, is convex in x^*. It is convex in y, too, because $f_0(x, \cdot)$ is concave and so $-f_0(x, \cdot)$ is convex.

Consider the convex multivalued mapping

$$F(x^*) = \{(y, y_0) : y^0 \geq f_0^*(x^*, y), y \in N\}.$$

Observe that for this function,

$$H(x^*, y^*, y^{0*}) = \sup_{(y, y^0)} \{\langle y, y^* \rangle + y^{0*} y^0 : y^0 \geq f_0^*(x^*, y), y \in N\}.$$

In particular, if $y^* = 0$, $y^{0*} = -1$,

$$H(x^*, 0, -1) = \sup_y \{-f_0^*(x^*, y), y \in N\} = -g(x^*).$$

By analogy,

$$\begin{aligned} M(x,0,-1) &= \inf_{(x^*,y,y^0)}\{\langle x,x^*\rangle + y^0 : y^0 \geq f_0^*(x^*,y), y\in N\} \\ &= \inf_{(x^*,y)}\{\langle x,x^*\rangle + f_0^*(x^*,y) : y\in N\} \\ &= \inf_{y\in N}\inf_{x^*}\{\langle x,x^*\rangle - (-f_0^*(x^*,y))\} = \inf_{y\in N}(-f_0(-x,y)). \end{aligned}$$

Since $f_0(\cdot,y)$ is closed, here we have used the fact that by Theorem 1.22 (see also Remark 2.4),

$$\inf_{x^*}\{\langle x,x^*\rangle - (-f_0^*(x^*,y))\} = -\sup_{x^*}\{\langle -x,x^*\rangle - f_0^*(x^*,y)\} = -f_0(-x,y).$$

Furthermore, g is semicontinuous at $x^*=0$ and so, according to Corollary 2.4,

$$\inf_x\{\langle x,0\rangle - M(x,0,-1)\} = H(0,0,-1),$$

or

$$\inf_x\{-\inf_{y\in N}(-f_0(-x,y))\} = -g(0).$$

Clearly, this relation can be converted as follows:

$$\sup_x\left\{\inf_{y\in N}(-f_0(x,y))\right\} = g(0). \tag{2.60}$$

On the other hand, by Eq. (2.57), we observe that

$$g(0) = \inf_{y\in N}\sup_{x\in X}(-f_0(x,y)). \tag{2.61}$$

Thus, by comparison of Eqs. (2.60) and (2.61), we prove Eq. (2.59), and so Eq. (2.58). ■

Theorem 2.22. Let N be a convex compact set in Y and suppose given a function $f(\cdot,y)$, $y\in N$ which is closed, proper, and convex. Moreover, suppose that $f(x,\cdot)$ is closed and concave for all fixed x and $g(0)\neq\pm\infty$. Then

$$\inf_x\sup_{y\in N} f(x,y) = \sup_{y\in N}\inf_x f(x,y).$$

□ In the case under consideration, $M=X$, and it is necessary only to verify the semicontinuity of the function

$$g(x^*) = \inf_{y\in N} f_0^*(x^*,y)$$

at $x^* = 0$, where

$$f_0^*(x^*, y) = \sup_x \{\langle x, x^* \rangle - f(x, y)\}.$$

Since $f(x, y)$ is closed on y, then $f_0^*(x^*, y)$ is closed (lower semicontinuous) on both x^* and y. Therefore, the infimum of $f_0^*(x^*, y)$ over $y \in N$ is attained.

Now, let $x_i^* \to 0, \quad g(x_i^*) \to \mu$. Denote the value of $y \in N$ by y_i, for which

$$f^*(x_i^*, y_i) = g(x_i^*).$$

Since $y_i \in N$ and N is compact, without loss of generality we may assume that $y_i \to y_0 \in N$. On the other hand, f_0^* is closed, and so
$f^*(0, y_0) \geq \inf \{f^*(0, y_0) : y \in N\} = g(0)$.

Thus, we have

$$\mu = \lim_{i \to \infty} g(x_i^*) = \lim_{i \to \infty} f^*(x_i^*, y_i) \geq f^*(0, y_0) \geq g(0),$$

and hence the semicontinuity of the function g at $x^* = 0$ is proved. ■

3 Mathematical Programming and Multivalued Mappings

3.1 Introduction

This chapter is devoted to the applications of the basic tools of multivalued mappings [14,51,109,124,196,226] to the study of mathematical programming with possibly nonsmooth data. Starting with problems of mathematical programming under functional and geometric constraints, we then consider various problems of constrained optimization: minimax problems and equilibrium constraints, the infimal convolution of convex functions, Lagrangians (see, for example, Refs. [74,226, 228]), and duality relations in convex programming problems. The key tools of our analysis are based on the extremal principle and its modifications together with the LAM calculus (see, for example, Mordukhovich [214] for more developments and discussions). In many areas of variational theory and its applications, geometric constraints are usually given as the intersections of sets. Based on the results for minimization of f_0 over a set A and calculus rules for the cone of tangent directions to set intersections, you can derive necessary optimality conditions for optimization problems with many geometric constraints. To furnish this in the nonconvex case, we will use convex upper approximations (CUAs) and the subdifferential calculus connected with them (see, for example, Pshenichnyi [226]).

It should be noted that in order for x_0 to be a point minimizing f over X, it is necessary and sufficient that $0 \in \partial f(x_0)$ (Theorem 3.1). In any case, the minimum set of f is a convex subset of X, closed if f is closed. It follows immediately from Corollary 1.2 that the minimum set of f is $\partial f^*(0)$. Thus, the minimum of f is attained if and only if f^* is subdifferentiable at $x^* = 0$. Certainly it cannot contain more than one point if f is strictly convex on dom f. At first we consider minimization problems with the so-called geometric constraints. Suppose that x_0 is a point minimizing a function f_0 defined on a set A. It is well known that the conditions that just characterize (somehow) the point x_0 are called the necessary conditions for an extremum. As a rule, if the minimized function and the set A are convex, then the above conditions are sufficient for an extremum. Later on, we will see that detailed information about A gives us different necessary conditions for various problems. Thus (Theorem 3.2), if $x_1 \in A$ is a point at which f is continuous, then in order for x_0 to be such a point it is necessary and sufficient that $\partial f(x_0) \cap K_A^*(x_0) \neq \varnothing$, where $K_A(x_0)$ is a cone of tangent directions at a point $x_0 \in A$ (the cone $K_A(x)$ is called the cone of tangent directions of the set A at the point $x \in A$ if from $\bar{x} \in K_A(x)$

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it follows that $\bar{x}$ is the tangent vector at $x \in A$). Moreover, this result will be extended to the case where A is the intersection of finitely many sets.

Throughout this book we will use the infimal convolution [137,228] of convex functions for the construction of the dual problems for various optimal control problems with a convex structure. You can find more discussions on these constructions in Refs. [144–147,149,151,159]. For the primary problem, which consists of the minimization of f over A, the dual problem is to find $\sup\{-f^*(x^*) - \delta_A^*(-x^*)\}$, where δ_A is the indicator function of A. Let v and v^* be values for the primary and the dual problems, respectively. If $v = v^*$ and is finite, and the set of solutions of the dual problem is nonempty, then x is an optimal solution to the primary problem if and only if there exists a point x^* such that $x^* \in \partial f(x) \cap K_A^*(x)$. This point is a solution to the dual problem (Theorem 3.17).

In a convex program (P), one is interested in minimizing a certain convex function on $\mathbb{R}^n$:

$$\text{infimum}\, f_0(x) \text{ subject to } x \in A, \quad A = \{x : f_i(x) \le 0,\ i = 1, \ldots, m,\ x \in D\},$$

where $f_i(x)$, $i = 0, \ldots, m$ are convex functions, D is a convex set, and $\operatorname{dom} f_i \supset D$. By introducing a special convex multivalued mapping and calculating the Hamiltonian H and M functions under the hypothesis of the existence of an interior point, we will prove that x_0 is an optimal solution in the convex problem (P) if and only if there exists a vector $y_0^* \in \mathbb{R}_+^m$ such that $L(x_0, y_0^*) \le L(x, y_0^*)$, $x \in D$, $y_0^{i*} f_i(x_0) = 0$, $i = 1, 2, \ldots, m$, where $L(x, y^*)$ is the Lagrangian of (P) (Theorem 3.6). Moreover, the vector $y_0^* \ge 0$ is said to be a Kuhn–Tucker vector for (P) if $\inf_x\{f_0(x) : x \in A\} = \inf_x\{L(x, y_0^*) : x \in D\}$.

Then the vector $y_0^* \ge 0$ is a Kuhn–Tucker vector for the convex programming problem (P) if and only if $-y_0^* \in \partial V(0)$ (Theorem 3.8). It should be noted that $V(0)$ is the optimal value in the convex programming problem, and in general $V(y) = \inf_{x,x^0}\{x^0 : f_0(x) \le x^0,\ f_i(x) \le y^i,\ i = 1, \ldots m,\ x \in D\}$ is the optimal value in the convex programming problem (P_y) obtained by replacing f_i by $f_i - y^i$ for $i = 1, \ldots, m$. Thinking of the vector y as representing "perturbations" of (P_y), we call y the perturbation function for (P).

In Section 3.1, we will investigate the Lagrangian and duality in convex programming problems. The results of this section are mostly based on the book by Pshenichnyi [226]. Let $V(0) \ne \pm\infty$ and suppose the function V is lower semicontinuous at $y = 0$. Then $\inf_x\{f_0(x) : x \in A = \sup_{y^*}\{\varphi(y^*) : y^* \ge 0\}$, $\varphi(y^*) = \inf_x\{L(x, y^*) : x \in D\}$, where f_i are closed proper convex functions (Theorem 3.11).

The Kuhn–Tucker vectors corresponding to mathematical programming (P) can be characterized in terms of a certain class of perturbations of the objective function for (P). For more generalized convex programs and bifunctions, see Ref. [226].

Observe that $f_0(x)$ can be interpreted as the "price" of x. Suppose that we are allowed to change (P) to any (P_y) that we please, except that we must pay for the change, the price vector being y_0^x per unit of perturbation vector y. Then for any y, the minimum cost will be $V(y) + \langle y, y_0^* \rangle$. A perturbation will be "worth buying" if and only if $V(y)+\langle y, y_0^*\rangle < V(0)$, where $V(0)$ is the optimal value in the unperturbed problem. We will prove that when $V(0)$ is finite, y^* is a Kuhn–Tucker vector for (P) if and only if, at the price vector $y^* = y_0^*$, no perturbation whatsoever would be worth buying.

By Definition 3.2, a vector $\bar{x}\in X$ is called a tangent direction of the set A at the point $x\in A$ if there exists a function $\varphi(\lambda)$ such that $x+\lambda\bar{x}+\varphi(\lambda)\in A$ for sufficiently small $\lambda \geq 0$ and $\lambda^{-1}\varphi(\lambda)\to 0$ as $\lambda\downarrow 0$. We have already seen that the cone of tangent directions involve directions for each of which there exists its own function $\varphi(\lambda)$. But in order to characterize the properties of the set A, this is not sufficient. However, the following notion of a *local tent* allows us to predetermine mapping in A for nearest tangent directions among themselves. For example, let $A=\{x: f_i(x)=0,\ i\in I\}$, where $I=\{1, \ldots, m\}$ is a finite index set and the f_i are continuously differentiable functions. In Section 3.4, it will be shown that if the gradient vectors $f_i'(x_0),\ i\in I,\ x_0\in A$ are linearly independent, then the cone of tangent directions $K_A(x_0)=\{\bar{x}: \langle\bar{x}, f_i'(x_0)\rangle = 0,\ \ i\in I\}$, is locally tent.

In the theory of extremal problems, in some neighborhood of a point minimizing our objective function, we deal with more simply and comparatively easily computable functions. Remember that a smooth function admits a linear approximation. As will be shown in Section 3.6, a convex function can be approached by positively homogeneous functions—i.e., directional derivatives. However, a nonsmooth and nonconvex function cannot be approximated in a neighborhood of some point with positively homogeneous functions. Just for such a class of functions, we will introduce the concept of CUAs [226]. Note that a CUA $h(\bar{x}, x)$ is defined nonuniquely and for obtaining the appropriate necessary conditions, as a rule, it is necessary to have a sufficiently wider family of CUAs. In Section 3.5, it will be proved that if $h_1(\bar{x}, x)$ and $h_2(\bar{x}, x)$ are CUAs of f at a point x, then $\lambda_1 h_1+\lambda_2 h_2,\ \lambda_1+\lambda_2=1,\ \lambda_1,\ \lambda_2\geq 0$ and $\max(h_1, h_2)$ also are CUAs (Proposition 3.8). Moreover, let h_1, h_2, respectively, be a CUA at the point x, for the function f_1, f_2, and suppose that $f=f_1+f_2$. Then $h(\bar{x}, x)=h_1(\bar{x}, x)+h_2(\bar{x}, x)$ is a CUA of f at x. In addition, if int dom $h_1(\cdot, x)\cap$ dom $h_2(\cdot, x)\neq\varnothing$, then $\partial f(x)=\partial f_1(x)+\partial f_2(x)$ (Theorem 3.20). In particular, if f is a continuously differentiable function at x, then $h(\bar{x}, x)=\langle\bar{x}, f'(x)\rangle$ is a CUA at x and $\partial f(x)=\{f'(x)\}$. Furthermore, if f is convex and continuous at x, then $h(\bar{x}, x)=f'(x, \bar{x})$ is a CUA and the subdifferential $\partial f(x)$ defined by Definitions 1.28 and 3.6 are the same.

In Section 3.6, we will extend the apparatus of LAM to the nonconvex case. In the next chapters, we will see how decisive a role the LAM plays for the construction of optimality conditions for different optimal control problems. In Chapter 2, we introduced the basic definitions of multivalued mappings and the main properties of convex mappings. One of the principal notions was LAM.

If F is a closed continuous convex-valued bounded multivalued mapping, then the Hamiltonian H is continuous and the Argmaximum set $F(x; y^*)$ is upper semicontinuous of x and y^* (Lemma 3.2). Moreover, if the conditions of Lemma 3.2 are satisfied and F is a Lipschitzian mapping, then the Hamiltonian function is also Lipschitzian (Lemma 3.3).

Now, suppose that at every point $z \in \text{gph } F$, there exists a convex cone of tangent directions $K_{\text{gph } F}(z)$. Recall that $K_{\text{gph } F}(z)$ is a cone of tangent directions, if it is convex and for all $\bar{z} \in K_{\text{gph } F}(z)$ there is a function $r:[0,1] \to Z$ such that $z + \lambda\bar{z} + r(\lambda) \in \text{gph } F$ for small $\lambda \geq 0$. Then the multivalued mapping $F^*: Y^* \to P(X^*)$ defined by

$$F^*(y^*, z) = \{x^* : (x^*, -y^*) \in K^*_{\text{gph } F}(z)\}$$

is called the LAM to F at point $z \in \text{gph } F$.

For the convex mapping, we take $K_{\text{gph } F}(z) = \text{cone}(\text{gph } F - z)$ and then the definition of LAM, which will be given in Eq. (3.55), coincides with the LAM in the sense of Definition 2.10. Note that for convex-valued mappings, by the convexity of $F(x)$, the directions $\gamma(0, y_1 - y)$, $y_1 \in F(x)$, $z = (x, y)$, $\gamma > 0$ are tangent because for sufficiently small $\lambda > 0$ we have $(x, y + \lambda\gamma(y_1 - y)) \in \text{gph } F$. Therefore, later on, we will assume that $K_{\text{gph } F}(z) \supseteq \{0, \text{cone}(F(x) - y)\}$, $z = (x, y) \in \text{gph } F$. If the latter condition is satisfied, then $F^*(y^*, z) \neq \varnothing$ if $y \in F(x; y^*)$. The closure of the convex hull of the union of $F^*(y^*;(x, y))$ over the $y \in F(x; y^*)$ is called the AM to F at $x \in \text{dom } F$ and is denoted by $F^*(y^*; x)$. Note that the adjoint differential inclusion (DFI) associated with nonconvex DFI is expressed in terms of the AM at $x \in \text{dom } F$. For convex mappings, $F^*(y^*; x) = F^*(y^*;(x, y)) = \partial_x H(x, y^*)$, where y is an arbitrary point of the Argmaximum set $F(x; y^*)$. If the multivalued mapping $F_z(\bar{x})$ is defined by

$$F_z(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_{\text{gph } F}(z)\}$$

then, by Definition 2.9, $F_z^*(x^*)$ is the AM, so $F^*(y^*, z) = F_z^*(x^*)$. Suppose that mapping F is convex-valued, closed, bounded, continuous, and Lipschitzian. In addition, assume that $H_{F_z}(\cdot, y^*)$ is a closed proper function, where $F_2(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_{\text{gph } F}(z)\}$. Then for every $y \in F(x; y^*)$, the function $-H_{F_z}(\bar{x}, -y^*) = \sup_{x*}\{\langle \bar{x}, -x^* \rangle : x^* \in F^*(-y^*; z)\}$ is the CUA for $-H(\cdot, -y^*)$ and $F^*(y^*; z) = \partial_x H(x, y^*)$ (Theorem 3.21). This result will be further extended to the case of the minimization of a function of two variables. There are different investigations in Refs. [225,226] establishing an important connection between the LAM and subdifferentials of functions admitting CUAs. In the field of different convex and nonconvex approximations of functions and sets, you can also consult Clarke [52,54,60], Demyanov [65,66], Frankowska [18,85,87], Mordukhovich [192,200,212], Mahmudov [148,160,171], Rockafellar [228−232], and Rubinov [236−238] for related and additional material. In this direction, you can find more information in Refs. [41,43,60,66,108,121,133,198,229,218,232,234,236].

In Section 3.7, our attention will be focused on the nonconvex optimality problems with geometric and functional constraints. Let x_0 be a point that minimizes the function f over the set A and let $h(\bar{x}, x_0)$ be a CUA of f at x_0. In addition, if int dom $h(\cdot, x_0) \cap K_A(x_0) \neq \varnothing$, then $\partial f(x_0) \cap K_A^*(x_0) \neq \varnothing$ (Theorem 3.23). This theorem will be extended to the intersection of finitely many sets. In particular, for a minimization problem with the equality and inequality constraints inf $f_0(x)$ subject to $f_i \le 0$, $i \in I_1$, $f_i = 0$, $i \in I_2$, we will formulate a necessary condition for optimality.

At the end of this section in terms of local tents, we will consider different mathematical programs with equilibrium constraints; consider, for example, a problem with a constraint given by the multivalued mapping F : inf $g(x, y)$ subject to $y \in F(x) \cap N$, $x \in M$, where $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ and M is some subset of X.

A variety of such problems will be considered.

3.2 Necessary Conditions for an Extremum in Convex Programming Problems

At first we consider minimization problem of f over X and then minimization problem with the so-called geometric constraint $x \in A \subset X$. Later on, we will see that specifying A will give us different necessary conditions for various problems.

Thus, let f: $X \to \mathbb{R}$ be a proper convex function and x_0 a point minimizing f over X. Clearly, this means that

$$f(x) \ge f(x_0) \quad \forall x \in X.$$

In all the statements below, we consider that if x_0 is a point minimizing f then $f(x_0) > -\infty$.

Let us rewrite this inequality in the following useful form:

$$f(x) - f(x_0) \ge \langle x - x_0, 0 \rangle, \quad 0 \in X^*.$$

Finally, by definition of the subgradient, we obtain that $x^* = 0$ is a subgradient of f at x_0; i.e., $0 \in \partial f(x_0)$. We can formulate this result as shown in Theorem 3.1.

Theorem 3.1. Suppose that f is a proper convex function on X. In order for x_0 to be a point minimizing f over X, it is necessary and sufficient that $0 \in \partial f(x_0)$.

In any case, the minimum set of f is a convex subset of X, and it is closed if f is closed. It follows immediately from Corollary 1.2 that the minimum set of f is $\partial f^*(0)$. Thus, the infimum of f is attained if and only if f^* is subdifferentiable at $x^* = 0$. Certainly, it cannot contain more than one point if f is strictly convex on dom f.

Corollary 3.1. Let F be a convex multivalued mapping and H be its Hamiltonian function. Then, x_0 is the solution to the minimization problem:

$$\inf_x \{\langle x, x^* \rangle - H(x, y^*)\}, \quad x^* \in X^*, \ \ y^* \in Y^*$$

if and only if

$$x^* \in F^*(y^*; z_0), \quad y_0 \in F(x_0, y^*), \quad z = (x, y).$$

☐ Since $\langle \cdot, x^* \rangle - H(\cdot, y^*)$ is a convex function, according to Theorem 3.1, in order for x_0 to be a solution to our minimization problem, it is necessary and sufficient that

$$0 \in \partial_x [\langle x, x^* \rangle - H(x, y^*)];$$

i.e., $x^* \in \partial_x H(x, y^*)$, and then by the definition of $M(x^*, y^*)$, it follows that y_0 is an element of the Argmaximum set $F(x_0, y^*)$. ■

Theorem 3.2. Let f be a proper convex function, and let A be a nonempty convex set over which f is to be minimized. Further, let $x_1 \in A$ be a point at which f is continuous. In order for x_0 to be a point minimizing f over A, it is necessary and sufficient that

$$\partial f(x_0) \cap K_A^*(x_0) \neq \varnothing, \tag{3.1}$$

where

$$K_A(x_0) = \text{cone}(A - x_0) = \{\bar{x} : x_0 + \lambda \bar{x} \in A, \quad \text{for sufficiently small } \lambda > 0\}.$$

☐ Recall that by definition of the indicator function,

$$\delta_A(x) = \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A. \end{cases}$$

Then, it is easy to see that the minimum of f over A is the same as the minimum of $f + \delta_A$ over X. By Theorem 1.29, the subdifferential of $f + \delta_A$ is the sum of the subdifferentials of f and δ_A. Thus, by Eq. (1.45),

$$\partial \delta_A(x_0) = -(\text{cone}(A - x_0))^* \equiv -K_A^*(x_0).$$

Therefore,

$$\partial [f(x_0) + \delta_A(x_0)] = \partial f(x_0) + \partial \delta_A(x_0) = \partial f(x_0) - K_A^*(x_0).$$

Now, by the preceding theorem, in order for x_0 to be a point minimizing f over X, it is necessary and sufficient that

$$0 \in \partial f(x_0) - K_A^*(x_0),$$

which is equivalent to Eq. (3.1). ■

Theorem 3.3. Let the conditions of the previous theorem be satisfied and suppose that

$$A = A_1 \cap A_2 \cap \ldots \cap A_k,$$

where A_i, $i = 1, \ldots, k$ are convex and

$$\operatorname{int} A_1 \cap \operatorname{int} A_2 \cap \cdots \cap \operatorname{int} A_{k-1} \cap A_k \neq \varnothing.$$

Then, in order for x_0 to be a point minimizing f over A, it is necessary and sufficient that there exist points $x_i^* \in K^*_{A_i(x_0)}$, $i = 1, 2, \ldots, k$; $x^* \in \partial f(x_0)$, such that

$$x^* = x_1^* + \cdots + x_k^*$$

□ It is not hard to see that

$$K_A(x_0) = \bigcap_{i=1}^{k} K_{A_i}(x_0), \tag{3.2}$$

and if $x_2 \in \operatorname{int} A_i$, $i = 1, \ldots, k-1$, $x_2 \in A_k$, then

$$\bar{x}_2 = \bar{x}_2 - x_0 \in \operatorname{int} K_{A_i}(x_0), \quad i = 1, \ldots, k-1; \quad \bar{x}_2 \in K_{A_k}(x_0).$$

Hence,

$$\operatorname{int} K_{A_1}(x_0) \cap \operatorname{int} K_{A_2}(x_0) \cap \cdots \cap \operatorname{int} K_{A_{k-1}}(x_0) \cap K_{A_k} \neq \varnothing. \tag{3.3}$$

Then it follows from Eqs. (3.2) and (3.3), and Theorem 1.11 that

$$K_A^*(x_0) = K_{A_1}^*(x_0) + \cdots + K_{A_k}^*(x_0). \tag{3.4}$$

Thus, the point $x^* \in \partial f(x_0) \cap K_A^*(x_0)$, which exists by Theorem 3.2, can be written as

$$x^* = x_1^* + \cdots + x_k^*, \quad x_i^* \in K_{A_i}(x_0), \quad i = 1, \ldots, k. \blacksquare \tag{3.5}$$

Theorem 3.4. Let f be a proper convex function and suppose that

$$A = \bigcap_{i=1}^{k} A_i,$$

where A_i are convex sets. Moreover, let $x_1 \in A$ be a point of continuity of f. Then, in order for x_0 to be a point minimizing f over A, it is necessary that there exist points $x_i^* \in K_{A_i}^*(x_0)$, $i = 1, 2, \ldots, k$; $x^* \in \partial f(x_0)$ and a number $\lambda \in \{0,1\}$ such that

$$\lambda x^* = x_1^* + \cdots + x_k^*. \tag{3.6}$$

Here, if $\lambda = 0$, then at least one of the points $x_1^*, \ldots, x_k^*$ is nonzero. If $\lambda = 1$, then these conditions are sufficient for the minimization of f.

□ According to Theorem 3.2, there is a vector $x^* \in \partial f(x_0)$ such that $x^* \in K_A^*(x_0)$. Taking into account Eq. (3.2), by Theorem 1.12, there are two possibilities: (1) the formula (3.4) is correct, then in Eq. (3.6), $\lambda = 1$, and by Theorem 3.3 the condition (3.6) is sufficient for optimality of our minimization problem; or (2) there exist vectors $x_i^* \in K_{A_i}^*(x_0)$, not all zero, such that $x_1^* + \cdots + x_k^* = 0$. Then Eq. (3.6) corresponds to $\lambda = 0$. ■

By an ordinary convex programming problem (P), we shall mean a problem of the following form:

$$\inf f_0(x) \text{ subject to } x \in A, \quad A = \{x : f_i(x) \le 0, \ i = 1, \ldots, m, \ x \in D\}, \tag{3.7}$$

where $f_i(x)$, $i = 0, \ldots, m$ are convex functions, D is a convex set and dom $f_i \supset D$, $i = 0, \ldots, m$. A vector x will be called a feasible solution to (P) if $x \in A$. The minimum of f_0 will be called the optimal value in (P).

Consider a multivalued mapping $F : Y \to P(X)$, defined as

$$F(y) = \{(x, x^0) : f_i(x) \le y^i, \ i = 1, \ldots, m, \quad f_0(x) \le x^0, \ x \in D\}, \tag{3.8}$$

where $Y = \mathbb{R}^m$, $X = \mathbb{R}^{m+1}$. Since the functions f_i and the set D are convex, F is a convex mapping.

Let us calculate the Hamiltonian function $H(y, x^*, x^{0^*})$:

$$H(y, x^*, x^{0^*}) = \sup_{x,x^0}\{\langle x, x^*\rangle + x^0 x^{0^*} : (x, x^0) \in F(y)\}.$$

Taking $x^* = 0$, $x^{0^*} = -1$ and denoting $H(y, 0, -1)$ by $-V(y)$, we can write

$$H(y, 0, -1) = \sup_{x,x^0}\{-x^0 : (x, x^0) \in F(y)\} = -\inf_{x,x^0}\{x^0 : (x, x^0) \in F(y)\},$$

and so

$$\begin{aligned} V(y) &= \inf_{x,x^0}\{x^0 : f_0(x) \le x^0, f_i(x) \le y^i, \ i = 1, \ldots, m, \ x \in D\} \\ &= \inf_x \{f_0(x) : f_i(x) \le y^i, \quad i = 1, \ldots, m, \quad x \in D\}. \end{aligned} \tag{3.9}$$

Obviously, if $y = 0$, then $V(0)$ is the minimum value of f_0 over A, defined by Eq. (3.7). Now let us compute the function $M(y^*, x^*, x^{0^*})$ at the point $x^* = 0$, $x^{0^*} = -1$:

$$M(y^*, 0, -1) = \inf_{y,x,x^0}\{\langle y, y^*\rangle + x^0 : (x, x^0) \in F(y)\}$$

$$= \begin{cases} \inf_x \left\{ f_0(x) + \sum_{i=1}^m y^{i*} f_i(x) : x \in D \right\} & \text{if } y^* \ge 0, \\ -\infty, & \text{if } y^{i*} < 0 \text{ for some } i = 1, \ldots, m. \end{cases}$$

Let us denote

$$L(x, y^*) = f_0(x) + \sum_{i=1}^{m} y^{i^*} f_i(x). \tag{3.10}$$

The function L is called the Lagrangian of (P). The variable y^{i^*} is known as the Lagrange multiplier associated with the ith constraint in (P). Denote

$$\varphi(y^*) = \inf_x \{L(x, y^*) : x \in D\}. \tag{3.11}$$

Then we have

$$M(y^*, 0, -1) = \begin{cases} \varphi(y^*), & \text{if } y^* \geq 0, \\ -\infty, & \text{if } y^* < 0 \text{ for some } i. \end{cases} \tag{3.12}$$

Suppose that $f_i(x_1) < 0$, $i = 1, \ldots, m$ for some $x_1 \in D$ and that $V(0)$ is finite. This means that the value of the problem (P) is finite. By Lemma 2.5, $-V(y) = H(y, 0, -1)$ is concave and so V is convex.

Since for a point y lying in some neighborhood of zero, we have $f_i(x_1) < y^i$, $i = 1, \ldots, m$, $x_1 \in D$, it follows that $V(y) \neq +\infty$ and so $0 \in \operatorname{int} \operatorname{dom} V$. Because $V(0)$ is finite and $0 \in \operatorname{int} \operatorname{dom} V$, it follows that $V(y) \neq -\infty$, so $V(y)$ is finite around $y = 0$. Therefore, by Theorem 1.17, the function V is continuous around $y = 0$. Then, by the duality theorems of multivalued mappings, it is easy to see that

$$-V(0) = H(0, 0, -1) = \langle 0, y_0^* \rangle - M(y_0^*, 0, -1) \leq \langle 0, y^* \rangle - M(y^*, 0, -1).$$

Thus, there is a vector y_0^* such that

$$V(0) = H(y_0^*, 0, -1) \geq H(y^*, 0, -1).$$

Taking into account Eqs. (3.9)–(3.12), we obtain

$$V(0) = \varphi(y_0^*) = \inf_x \{L(x, y_0^*) : x \in D\} \geq \varphi(y^*), \quad y^* \geq 0, \tag{3.13}$$

Moreover, since $V(0) = \varphi(y_0^*)$ is finite, it follows that $y_0^* \geq 0$.

Theorem 3.5. If there exists a point $x_1 \in D$ such that $f_i(x_1) < 0$, $i = 1, \ldots, m$, and

$$V(0) = \inf_x \{f_0(x) : x \in A\}$$

is finite, then there is a vector $y_0^* \geq 0$ satisfying

$$V(0) = \varphi(y_0^*) = \inf_x \{L(x, y_0^*) : x \in D\} \tag{3.14}$$

In particular, if x_0 is a point minimizing f_0 over A, then

$$f_0(x_0) \leq L(x, y_0^*), \quad x \in D \tag{3.15}$$

☐ It remains to prove only the latter assertion. Indeed, by the definition of $V(y)$ (see Eq. (3.9)) and x_0, we obtain $f_0(x_0) = V(0)$. ■

Theorem 3.6. Let $x_1 \in D$ be a point satisfying $f_i(x_1) < 0$, $i = 1, \ldots, m$. Then, in order for x_0 to be a point minimizing f_0 over A given by Eq. (3.7), it is necessary and sufficient that there exists a vector $y_0^* \in \mathbb{R}_+^m$ such that

$$L(x_0, y_0^*) \leq L(x, y_0^*), \quad x \in D,$$

$$y_0^{i*} f_i(x_0) = 0, \quad i = 1, 2, \ldots, m. \tag{3.16}$$

☐ *Necessity*. According to the preceding theorem, there is a vector $y_0^* \in \mathbb{R}_+^m$ satisfying Eq. (3.15). Since $x_0 \in A$, then $x_0 \in D$ and $f_i(x_0) \leq 0$. Thus,

$$L(x_0, y_0^*) = f_0(x_0) + \sum_{i=1}^{m} y_0^{i*} f_i(x_0) \leq f(x_0) \leq L(x_0, y_0^*).$$

It follows that

$$f(x_0) = L(x_0, y_0^*) \tag{3.17}$$

and

$$\sum_{i=1}^{m} y_0^{i*} f_i(x_0) = 0.$$

On the other hand, since $y_0^{i*} \geq 0, f_i(x_0) \leq 0$, the latter equality implies that

$$y_0^{i*} f_i(x_0) = 0, \quad i = 1, 2, \ldots, m. \tag{3.18}$$

Therefore, from Eqs. (3.15)–(3.18), we have the desired conditions for Eq. (3.16).

Sufficiency. Let $x \in A$ and suppose that Eq. (3.16) holds. Then

$$f_0(x) \geq f_0(x) + \sum_{i=1}^{m} y_0^{i*} f_i(x) = L(x, y_0^*) \geq L(x_0, y_0^*) = f_0(x_0);$$

i.e., $f_0(x) \geq f_0(x_0)$ for all $x \in A$, so x_0 is the optimal solution. ■

Suppose now that f_i, $i = 1, \ldots, m$ are convex and continuous functions. Now, the first of conditions of Eq. (3.16) means that x_0 is the optimal solution to the Lagrangian $L(\cdot\ ,y_0^*)$ over the convex set D. Since $y_0^* \geq 0$, the Lagrangian $L(\cdot, y_0^*)$ is convex and continuous. According to Theorem 3.2, the infimum of $L(\cdot, y_0^*)$ over D is attained at a point x_0 if and only if

$$\partial_x L(x_0, y_0^*) \cap K_D^*(x_0) \neq \varnothing.$$

Then, by Theorem 1.29, we can write

$$\partial_x L(x_0, y_0^*) = \partial f_0(x_0) + \sum_{i=1}^{m} y_0^{i^*} \partial f_i(x_0).$$

Hence, there is a vector $x^* \in K_D^*(x_0)$ such that

$$x^* \in \partial f_0(x_0) + \sum_{i=1}^{m} y_0^{i^*} \partial f_i(x_0). \tag{3.19}$$

Obviously, the inclusion in Eq. (3.19) is necessary and sufficient that x_0 minimize $L(\cdot, y_0^*)$ over D. Thus, we have proved Theorem 3.7.

Theorem 3.7. Let the conditions of the preceding theorem be satisfied and let f_i, $i = 0, 1, \ldots, m$ be continuous convex functions. Then, x_0 is an optimal solution to the convex programming problem (P) if and only if there exist vectors $y_0^* \in \mathbb{R}_+^m$ and $x^* \in K_D^*(x_0)$ such that

$$y_0^{i^*} f_i(x_0) = 0, \quad i = 1, 2, \ldots, m.$$

Definition 3.1. The vector $y_0^* \geq 0$ is said to be a Kuhn–Tucker vector for (P) if

$$\inf_x \{f_0(x) : x \in A\} = \inf_x \{L(x, y_0^*) : x \in D\}.$$

Theorem 3.8. The vector $y_0^* \geq 0$ is a Kuhn–Tucker vector for the convex programming problem (P) if and only if $-y_0^* \in \partial V(0)$.

☐ By definition, $-y_0^* \in \partial V(0)$ if and only if

$$V(y) \geq V(0) + \langle y, -y_0^* \rangle,$$

i.e., if

$$\inf_y \{V(y) + \langle y, y_0^* \rangle\} = V(0).$$

Taking into account Eq. (3.9) for $V(y)$, we have

$$\inf_y \inf_x \{f_0(x) + \langle y, y_0^* \rangle : f_i(x) \le y^i, \quad i = 1, 2, \ldots, m, \;\; x \in D\} = V(0). \tag{3.20}$$

It is easy to see that if $y^{i^*} < 0$ for some $i = 1, \ldots, m$, then the infimum in the left-hand side of Eq. (3.20) is $-\infty$. Since $V(0)$ is finite, the latter is excluded, and $y_0^* > 0$. Then Eq. (3.20) implies that

$$\inf_x \{f_0(x) + \sum_{i=1}^m f_i(x) y_0^{i^*} : x \in D\} = \inf_x \{L(x, y_0^*) : x \in D\}.$$

Consequently, $-y_0^* \in \partial V(0)$ if and only if

$$\inf_x \{L(x, y_0^*) : x \in D\} = V(0);$$

i.e., y_0^* is a vector of Kuhn–Tucker for the convex programming problem (P). ■

Note that in the previous statements, we have assumed that there exists a point $x_1 \in D$ such that $f_i(x_1) < 0$, $i = 1, 2, \ldots, m$. Now, instead of this, we assume that the functions f_i are continuous. We define the following sets:

$$Ai = \{x : f_i(x) \le 0\}, \quad i = 0, 1, \ldots, m.$$

Let $x_0 \in A_i$. By Theorem 1.34, there are two possible cases:

1. No such point x, for which $f_i(x) < 0$, exists. Then, according to Theorem 3.1, $0 \in \partial f_i(x_0)$ and $0 = f_i(x_0) \le f_i(x)$; i.e., x_0 is a point belonging to the set of minima of f_i.
2. There exists a point x for which $f_i(x) < 0$. Then

$$K_{A_i}^*(x_0) = \begin{cases} 0, & \text{if } f_i(x_0) < 0, \\ -\text{cone } \partial f_i(x_0), & \text{if } f_i(x_0) = 0. \end{cases} \tag{3.21}$$

Let x_0 be a point minimizing f_0 over A defined by Eq. (3.7). Since

$$A = \left(\bigcap_{i=1}^m A_i \right) \cap D,$$

then, by Theorem 3.4, there are simultaneously the points, not all zero,

$$x_i^* \in K_{A_i}^*(x_0), \;\; i = 1, 2, \ldots, m; \quad x^* \in K_D^*(x_0), \;\; x_0^* \in \partial f_0(x_0)$$

and $\lambda_0 \ge 0$ such that

$$\lambda_0 x_0 = \sum_{i=1}^m x_i^* + x^*. \tag{3.22}$$

If $f_i(x) < 0$ for some x and every i, then Eq. (3.21) is valid. Take

$$x_i^* = -\lambda_i x_{i0}^*, \quad \lambda_i \geq 0, \quad x_{i0}^* \in \partial f_i(x_0), \quad x_{00}^* = x_0^*, \tag{3.23}$$

where $\lambda_i > 0$, if $f_i(x_0) = 0$ and $\lambda_i = 0$ if $f_i(x_0) < 0$; i.e.,

$$\lambda_i f_i(x_0) = 0, \quad i = 1, 2, \ldots, m.$$

By substituting Eq. (3.23) into Eq. (3.22), we have

$$\lambda_0 x_{00}^* + \lambda_1 x_{10}^* + \cdots + \lambda_m x_{m0}^* = x^*,$$

$$\lambda_i > 0, \quad \lambda_i f_i(x_0) = 0, \quad x_{i0}^* \in \partial f_i(x_0), \quad i = 0, 1, \ldots, m; \quad x^* \in K_D^*(x_0). \tag{3.24}$$

where among the λ_i there are nonzero numbers because not all λ, x_i^*, x^* are zero.

If for some i_0 there is no point satisfying $f_{i_0}(x) < 0$, then according to the discussion above, $0 \in \partial f_i(x_0)$. Setting $x_{i_0 0}^* = 0$, $\lambda_{i_0} = 1$, $\lambda_i = 0$, $i \neq i_0$; $x^* = 0$, we deduce that the relations in Eq. (3.24) are valid.

Hence, we have proved Theorem 3.9.

Theorem 3.9. Let the functions $f_i(x)$, $i = 0, \ldots, m$ be convex and continuous. Then, in order for x_0 to be an optimal solution to the convex programming problem (P), it is necessary that there exist numbers λ_i, $i = 1, \ldots, m$ not all zero such that for some $x_i^* \in \partial f_i(x_0)$,

$$\sum_{i=1}^{m} \lambda_i x_i^* \in K_D^*(x_0),$$

where $\lambda_i \geq 0$, $\lambda_i f_i(x_i) = 0$, $i = 1, 2, \ldots, m$.

Theorem 3.10. Let f_i, $i = 0, \ldots, m$ be convex proper functions, where

$$f_i(x) = \langle x, x_i^* \rangle - \alpha_i, \quad i = k + 1, \ldots, m.$$

Moreover, let D be a convex set, ri dom $f_i \supseteq D$, $i = 1, \ldots, m$, and $x_1 \in$ ri D be a point such that $f_i(x_1) < 0$, $i = 1, \ldots, k$. Then, in order that x_0 be an optimal solution to the convex programming problem (P), it is necessary and sufficient that there exists a vector $y_0^* \in \mathbb{R}_+^m$ such that

$$L(x_0, y_0^*) \leq L(x, y_0^*), \quad x \in D,$$

$$y_0^{i*} f_i(x_0) = 0, \quad i = 1, 2, \ldots, m.$$

☐ Let us denote

$$\tilde{y}^* \in R^k, \quad \tilde{y}^* = (y_1^*, \ldots, y_k^*)$$

$$\tilde{L}(x, \tilde{y}^*) = f_0(x) + \sum_{i=1}^{m} y_i^* f_i(x),$$

$$D_0 = \{x : f_i(x) \le 0, \quad i = k+1, \ldots, m, \ x \in D\}$$

Then by Theorem 3.6, there is a vector $\tilde{y}_0^* \in \mathbb{R}_+^m$ such that

$$\tilde{L}(x_0, \tilde{y}_0^*) \le \tilde{L}(x, \tilde{y}_0^*), \quad x \in D,$$

$$y_0^{i*} f_i(x_0) = 0, \quad i = 1, \ldots, k.$$

Since

$$D \subseteq \bigcap_{i=0}^{m} \text{ri dom } f_i,$$

the function $\tilde{L}(\cdot, \tilde{y}_0^*)$ is continuous on D relative shifted subspace containing $\cap_{i=0}^{m}$ri dom f_i. Then, for this subspace, on the basis of Theorem 3.2, we conclude that there exists a vector $x^* \in \partial_x \tilde{L}(x_0, \tilde{y}_0^*)$ so that $x^* \in K_{D_0}^*(x_0)$. On the other hand, it is obvious that

$$K_{D_0}(x_0) = K_D(x_0) \cap K_{D_1}(x_0),$$

where

$$D_1 = \{x : f_i(x) = \langle x, x_i^* \rangle - \alpha_i \le 0, \quad i = k+1, \ldots, m\}.$$

and $K_{D_1}(x_0)$ is a polyhedral cone. It is easy to compute that

$$K_{D_1}^*(x_0) = \left\{ x^* : x^* = -\sum_{i=k+1}^{m} \lambda_i x_i^*, \ \lambda_i \ge 0, \quad \lambda_i f_i(x_0) = 0 \right\}.$$

Since x_1 is feasible, $x_1 \in D_1$, and so ri $K_D(x_0) \cap K_{D_1}(x_0) \neq \emptyset$. Hence,

$$K_{D_0}^*(x_0) = K_D^*(x_0) + K_{D_1}^*(x_0).$$

Thus, a vector $x^* \in \partial_x \tilde{L}(x_0, \tilde{y}_0^*)$ can be represented:

$$x^* = x_1^* - \sum_{i=k+1}^{m} \lambda_1 x_i^*, \ \ \lambda_i \ge 0, \quad \lambda_i f_i(x_0) = 0, \ \ i = 1, \ldots, m, \ \ x_1^* \in K_D(x_0).$$

Setting $y_0^{i*} = \lambda_i$, $i = k+1, \ldots, m$, we have

$$L(x, y_0^*) = \tilde{L}(x_0, \tilde{y}_0^*) + \sum_{i=k+1}^{m} \lambda_i f_i(x),$$

and $x_1^* = x^* + \sum_{i=k+1}^{m} \lambda_i x_i^* \in \partial_x L(x_0, y_0^*)$. Therefore, $\partial_x L(x_0, y_0^*) \cap K_D^*(x_0) \neq \varnothing$, and by Theorem 3.2 it can be concluded that x_0 belongs to the minimum set of $L(\cdot, y_0^*)$, on a set D. Taking into account the relations $\lambda_i \geq 0$, $\lambda_i f_i(x_0) = 0$, $i = k+1, \ldots, m$, this ends the proof of theorem. ■

3.3 Lagrangian and Duality in Convex Programming Problems

In this section, we consider a convex programming problem $(\overline{P})$, consisting of the minimizing of f_0 over A:

$$A = \{x : f_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \ \ x \in D\},$$

$$\text{dom } f_i \supseteq D, \quad i = 1, \ldots, m,$$

where f_i are closed proper convex functions and D is a convex set. As before, let us denote

$$L(x, y^*) = f_0(x) + \sum_{i=1}^{m} y^{i*} f_i(x),$$

$$\varphi(y^*) = \inf_x \{L(x, y^*) : x \in D\}, \quad y^* \geq 0,$$

$$V(y) = \inf_x \{f_0(x) : f_i(x) \leq y^i, \quad i = 1, \ldots, m, \ \ x \in D\}.$$

Let us introduce

$$F(y) = \{(x, x^0) \in \mathbb{R}^{n+1} : f_i(x) \leq y^i, \quad i = 1, \ldots, m, \ \ f_0(x) \leq x^0, \ \ x \in D\}.$$

As was shown in the previous section

$$H(y, 0, -1) = -V(y) \tag{3.25}$$

$$M(y^*, 0, -1) \begin{cases} \varphi(y^*), & \text{if } y^* \geq 0, \\ -\infty, & \text{if } y^{i*} \leq 0 \text{ for some } i. \end{cases} \tag{3.26}$$

Maximizing φ over $\mathbb{R}_+^m$ is called the dual problem to the aforementioned formulated primary problem $(\overline{P})$.

Theorem 3.11. Let $V(0) \neq \pm\infty$ and the function V be lower semicontinuous at $y=0$. Then

$$\inf_x \{f_0(x) : x \in A\} = \sup_{y^*} \{\varphi(y^*) : y^* \geq 0\}; \tag{3.27}$$

i.e., the value of the primary problem $(\overline{P})$ is equal to the supremum of φ over $\mathbb{R}_+^m$.
☐ By the duality results for multivalued mappings (Corollary 2.4), it is easy to see that

$$\inf_{y^*} \{\langle 0, y^* \rangle - M(y^*, 0, -1)\} = H(0, 0, -1). \tag{3.28}$$

By hypothesis, $H(y,0,-1) = -V(y)$ is upper semicontinuous at $y=0$, so the conditions of Corollary 2.4 are fulfilled. Hence, by using Eqs. (3.25) and (3.26), from Eq. (3.28), we obtain

$$\inf_{y^*} \{-M(y^*, 0, -1)\} = -V(0)$$

or

$$\inf_{y^*} \{\varphi(y^*) : y^*\} = -V(0).$$

Thus,

$$\sup_{y^*} \{\varphi(y^*) : y^* \geq 0\} = V(0) = \inf_x \{f_0(x) : x \in A\}. \blacksquare$$

Note that if $x \in A$, $y^* \geq 0$, then

$$\varphi(y^*) \leq L(x, y^*) = f_0(x) + \sum_{i=1}^m y^{i^*} f_i(x) \leq f_0(x).$$

Thus, for all feasible solutions of the primary and dual problems, $x \in A$, $y^* \geq 0$,

$$\varphi(y^*) \leq f_0(x).$$

On the other hand, it is easy to see that

$$\psi(x) = \sup_{y^*} \{L(x, y^*) : y^* \geq 0\} = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ +\infty & \text{if } f_i(x) > 0 \text{ for some } i. \end{cases}$$

Thus, Eq. (3.27) is equivalent to

$$\inf_{x\in D}\sup_{y^*\geq 0} L(x,y^*) = \sup_{y^*\geq 0}\inf_{x\in D} L(x,y^*), \tag{3.29}$$

i.e., the duality relation in Eq. (3.27) is valid if and only if the inequality in Eq. (3.29) is fulfilled.

Theorem 3.12. Let D be a closed convex set and suppose the minimum set D_* of f_0 over A is bounded. Then, the duality relation in Eq. (3.27), or equivalently, Eq. (3.29), is valid.

☐ Define the multivalued mapping

$$\overline{F}(y,y^0) = \{x : f_i(x) \leq y^i, \quad i=0,\ldots,m, \;\; x\in D\}.$$

By virtue of the closure of f_i and D, the mapping $\overline{F}$ is closed, convex, and $D_* = \overline{F}(0,V(0))$. Then by hypothesis, $\overline{F}(0,V(0))$ is bounded. Therefore, it follows that by Lemma 2.1, the mapping $\overline{F}$ is bounded and so $\overline{F}(y,y^0)$ is a closed bounded set, and hence compact.

Now, let $y_j\to 0$, $\lim_{j\to\infty}V(y_j)=\mu$. Since the case $\mu=+\infty$ is trivial, we may assume that $\mu<+\infty$. Hence,

$$V(y_j) = \inf_x\{f_0(x) : x\in\overline{F}(y_j,\mu+\varepsilon) \tag{3.30}$$

for large j and arbitrary $\varepsilon>0$. Since $\overline{F}(y_j,\mu+\varepsilon)$ is compact and f_0 is closed proper, it follows that the infimum in Eq. (3.30) is attained at some point $x_j\in\overline{F}(y_j,\mu+\varepsilon)$. Besides, $\overline{F}$ is bounded, so the sequence x_j is bounded. Thus, we can choose a convergent subsequence. Without loss of generality, assume that $x_j\to x_0$. Since $\overline{F}$ is closed, $x_0\in\overline{F}(0,\mu+\varepsilon)$. Therefore, by definition of $\overline{F}$, it follows that $x_0\in A$ and $f_0(x_0)\leq\mu+\varepsilon$. But in this case, $V(0)\leq\mu+\varepsilon$, and since $\varepsilon>0$ is arbitrary, $\mu=\lim_{j\to\infty}V(y_j)\geq V(0)$; i.e., V is lower semicontinuous at $y=0$. Thus, applying Theorem 3.11, we have the desired result. ■

Theorem 3.13. Let y_0^* be a Kuhn–Tucker vector for the convex programming problem $(\overline{P})$. Then the duality relations in Eqs. (3.27) and (3.29) are valid and

$$\varphi(y_0^*) = \sup_{y^*}\{\varphi(y^*) : y^*\geq 0\}.$$

☐ If y_0^* is a Kuhn–Tucker vector for the problem $(\overline{P})$, then by Theorem 3.7, $-y^*\in\partial V(0)$. Therefore,

$$V(y)\geq V(0)-\langle y,y_0^*\rangle$$

for all y. In particular, if $y \to 0$, then $\lim V(y) \geq V(0)$; i.e., the function V is lower semicontinuous at $y = 0$. By Theorem 3.11, Eqs. (3.27) and (3.29) then follow that the relations in Eqs. (3.27) and (3.29) are valid. Furthermore, by Definition 3.1,

$$\varphi(y_0^*) = V(0) = \inf_x \{f_0(x) : x \in A\}.$$

On the other hand, as was shown above, $\varphi(y^*) \leq f_0(x)$ for every $y^* \geq 0$ and $x \in A$. Hence,

$$\varphi(y_0^*) = \inf_x \{f_0(x) : x \in A\} \geq \varphi(y^*), \quad y^* \geq 0. \blacksquare$$

Note that if in the convex programming problem $f_i(x) = \langle x, x_i^* \rangle - \alpha_i$, $i = 0, \ldots, m$, $D = X$, then we have a linear programming problem. Obviously, for this problem, F is polyhedral, so by Theorem 2.16, the function $H(y,0,-1) = -V(y)$ is closed.

Thus, we have proved the following theorem.

Theorem 3.14. If in the convex programming problem $f_i(x) = \langle x, x_i^* \rangle - \alpha_i$, $i = 0, \ldots, m$, $D = X$, then for the obtained linear programming problem, the duality relations in Eqs. (3.27) and (3.29) are true.

Remember that $V(0)$ is the optimal value in the convex programming problem, and in general $V(y)$ is the optimal value in the convex programming (P_y) obtained by replacing f_i by $f_i - y_i$ for $i = 1, \ldots, m$ (see Eq. (3.9)). Thinking of the vectors y as representing "perturbations" of (P), we call y the perturbation function for (P). Let us assume that $f_0(x)$ can be interpreted as the "price" of x. Then for any y, the minimum cost will be $V(y) + \langle y, y_0^* \rangle$. A perturbation will be "worth buying" if and only if

$$V(y) = \langle y, y_0^* \rangle < V(0),$$

where $V(0)$ is the optimal value in the unperturbed problem. We prove that when $V(0)$ is finite, y^* is a Kuhn–Tucker vector for (P) if and only if, at the price vector $y^* = y_0^*$, no perturbation whatsoever would be worth buying. In fact

$$\inf_y \{V(y) + \langle y_0^*, y \rangle\} = \inf_y \inf_x \{f_0(x) + \langle y, y_0^* \rangle ; f_i(x) \leq y^i,$$
$$i = 1, 2, \ldots, m, \ \ x \in D\}.$$

But the right-hand side of this equality is

$$\inf_x \{f_0(x) + \sum_{i=1}^m f_i(x) y_0^{i*} : x \in D\}, \quad y_0^* \geq 0.$$

Thus, if $V(0)$ is finite and $y^* = y_0^*$, the inequality

$$V(y) + \langle y, y^* \rangle \geq V(0)$$

holds for all y if and only if $y^* \geq 0$ and

$$\inf_x \{f_0(x) + \sum_{i=1}^m f_i(x) y^{i^*} : x \in D\} = V(0).$$

The latter condition means that y^* is a Kuhn–Tucker vector for (P).

Let us return to the problem stated at the beginning of Section 3.2:

$$\inf_{x \in A} f(x). \quad (G)$$

Suppose that f is a closed and proper convex function and that A is a closed convex set. In order to investigate this problem, to construct its dual problem, and to establish the duality relations, we need the following supplementary results. First, we use the theorem of duality of operations of addition and infimal convolution of convex functions.

Theorem 3.15. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ be functions not identically equal to $+\infty$. Then

$$(f_1 \oplus f_2 \oplus \cdots \oplus f_m)^* = f_1^* + f_2^* + \cdots + f_m^*,$$

$$(f_1 + f_2 + \cdots + f_m)^* \leq f_1^* \oplus f_2^* \oplus \cdots \oplus f_m^*.$$

In addition, if all functions are closed proper, and convex, and if all functions except possibly one are continuous at x_0 and these functions are finite at x_0, then

$$(f_1 + f_2 + \cdots + f_m)^* = f_1^* \oplus f_2^* \oplus \cdots \oplus f_m^*,$$

where for each $x^* = x_1^* + \cdots + x_m^*$ the inf is attained; i.e.,

$$(f_1 + f_2 + \cdots f_m)^*(x^*) = f_1^*(x_1^*) + f_2^*(x_2^*) + \cdots + f_m^*(x_m^*).$$

□ We shall confine ourselves to the case $m = 2$; the case of a greater number of summands is proved by induction. By definition of infimal convolution and conjugate of functions, we establish at once the first equality of the theorem:

$$\begin{aligned}(f_1 \oplus f_2)^*(x^*) = {} & \sup_x \{\langle x, x^* \rangle - \inf_{x_1 + x_2 = x} (f_1(x_1) + f_2(x_2))\} \\ & = \sup_{x_1, x_2} \{\langle x_1, x^* \rangle + \langle x_2, x^* \rangle = f_1(x_1) - f_2(x_2)\} = f_1^*(x_1^*) + f_2^*(x_2^*).\end{aligned}$$

Furthermore, by Young–Fenchel inequality, for all x_1^*, x_2^*, and x we can write

$$f_1^*(x_1^*) + f_2^*(x_2^*) \geq \langle x_1^* + x_2^*, x \rangle - f_1(x) - f_2(x),$$

and hence,

$$f_1^*(x_1^*) + f_2^*(x_2^*) \geq (f_1 + f_2)^*(x_1^* + x_2^*).$$

In particular, this inequality holds for all x_1^*, x_2^* such that $x_1^* + x_2^* = x^*$. Thus,

$$f_1^* \oplus f_2^* \geq (f_1 + f_2)^*,$$

and the first part of the theorem is proved.

By Theorem 6.5.7 of Ref. [138], $f_1^* \oplus f_2^*$ is a closed and proper convex function and for all x^* the inf is attained. Applying the already proved first formula of the theorem for f_1^*, f_2^*, and Theorem 1.21, we have

$$(f_1^* \oplus f_2^*)^* = f_1^{**} + f_2^{**} = f_1 + f_2.$$

Since $f_1^* \oplus f_2^*$ is a closed and proper convex function, then again by Theorem 1.21, it follows from the latter formula that

$$f_1^* \oplus f_2^* = (f_1^* \oplus f_2^*)^{**} = (f_1 + f_2)^*. \blacksquare$$

Proposition 3.1. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$ be functions not identically equal to $+\infty$. Then

$$\partial f_1(x_1) \cap \partial f_2(x_2) \subset \partial(f_1 \oplus f_2)(x_1 + x_2).$$

Moreover, if $\partial f_1(x_1) \cap \partial f_2(x_2) \neq \varnothing$, then for all $x = x_1 + x_2$, the infimum in the definition of $f_1 \oplus f_2$ is attained.

□ Let us denote $f = f_1 \oplus f_2$. By the preceding theorem, $f^* = f_1^* + f_2^*$. Let $x^* \in \partial f_1(x_1) \cap \partial f_2(x_2)$. Then, by Theorem 1.27, we obtain

$$f_1(x_1) + f_1^*(x^*) = \langle x_1, x^* \rangle,$$

$$f_2(x_2) + f_2^*(x^*) = \langle x_2, x^* \rangle.$$

By summing these inequalities, we have

$$f_1(x_1) + f_1^*(x^*) + f_2(x_2) + f_2^*(x^*) = \langle x, x^* \rangle,$$

where $x = x_1 + x_2$. Since by definition,

$$\begin{aligned} &f(x) \leq f_1(x_1) + f_2(x_2), \\ &f^*(x^*) = f_1^*(x^*) + f_2^*(x^*) \end{aligned} \tag{3.31}$$

it follows that

$$f(x)+f^*(x^*)\le\langle x,x^*\rangle, \tag{3.32}$$

whence $x^*\in\partial f(x)$. Actually, the inequality in Eq. (3.32) and so the first inequality of Eq. (3.31) are satisfied as equalities. Thus, we have seen that if $\partial f_1(x_1)\cap\partial f_2(x_2)\neq\varnothing$, then for any $x=x_1+x_2$, the infimum in the definition of $f_1\oplus f_2$ is attained. ■

Proposition 3.2. If $f_i:\mathbb{R}^n\to\mathbb{R}$, $i=1, 2$ are functions not identically equal to $+\infty$ and if for all $x=x_1+x_2$ the infimum in the definition of $f_1\oplus f_2$ is attained, then

$$\partial(f_1\oplus f_2)(x)=\partial f_1(x_1)\cap\partial f_2(x_2).$$

□ According to Proposition 3.1, it is sufficient to show that

$$\partial(f_1\oplus f_2)(x)=\partial f_1(x_1)\cap\partial f_2(x_2).$$

Let us denote $f=f_1\oplus f_2$. Let $x^*\in\partial f(x)$, i.e., $f(x)+f^*(x^*)=\langle x,x^*\rangle$. By Theorem 3.15, $f^*=f_1^*+f_2^*$. On the other hand, by hypothesis, $f(x)=f_1(x)+f_2(x)$, whence

$$f_1(x_1)+f_1^*(x^*)+f_2(x_2)+f_2^*(x^*)=\langle x_1,x^*\rangle+\langle x_2,x^*\rangle.$$

Since always

$$f_1(x_1)+f_1^*(x^*)\ge\langle x_1,x^*\rangle,$$

$$f_2(x_2)+f_2^*(x^*)\ge\langle x_2,x^*\rangle,$$

we can deduce that simultaneously

$$f_1(x_1)+f_1^*(x^*)=\langle x_1,x^*\rangle,$$

$$f_2(x_2)+f_2^*(x^*)=\langle x_2,x^*\rangle,$$

or $x^*\in\partial f_1(x_1)$, $x^*\in\partial f_2(x_2)$ are true. ■

Proposition 3.3. If $f_i:\mathbb{R}^n\to\mathbb{R}$, $i=1, 2$ are functions not identically equal to $+\infty$, and if $\partial f_1(x_1)\cap\partial f_2(x_2)\neq\varnothing$, then

1. $\partial f(x)=\partial f_1(x_1)\cap\partial f_2(x_2)$, where $f=f_1\oplus f_2$, $x=x_1+x_2$;
2. the set $\partial f_1(\tilde{x}_1)\cap\partial f_2(\tilde{x}_2)$, $\tilde{x}_1+\tilde{x}_2=x$ is either empty or coincides with $\partial f(x)$;
3. the set $\partial f_1(\tilde{x}_1)\cap\partial f_2(\tilde{x}_2)$ is nonempty if and only if for every $x=\tilde{x}_1+\tilde{x}_2$ the infimum in the definition of $f_1\oplus f_2$ is attained.

1. □ Since $\partial f_1(x_1) \cap \partial f_2(x_2) \neq \varnothing$, by virtue of Proposition 3.1 for any $x = x_1 + x_2$, the infimum in the definition of $f_1 \oplus f_2$ is attained. From this, Proposition 3.2 implies the validity of (1).
2. This assertion follows immediately from (1).
3. The necessity condition was already proved in Proposition 3.1. Furthermore, if for every $x = \tilde{x}_1 + \tilde{x}_2$ the infimum in the definition of $f_1 \oplus f_2$ is attained, then by Proposition 3.2, $\partial f(x) = \partial f_1(\tilde{x}_1) \cap \partial f_2(\tilde{x}_2)$. But since $\partial f(x) = \partial f_1(x_1) \cap \partial f_2(x_2) \neq \varnothing$, we conclude that $\partial f_1(\tilde{x}_1) \cap \partial f_2(\tilde{x}_2) \neq \varnothing$. ■

Now recall that the problem (G) can be transformed as follows:

$$\begin{aligned}\inf_{x \in A} f(x) &= \inf\{f(x) + \delta_A(x)\} = -\sup\{-f(x) - \delta_A(x)\} \\ &= -\sup\{\langle x, 0\rangle - [f(x) + \delta_A(x)]\} = -(f + \delta_A)^*(0) \geq -(f^* \oplus \delta_A^*)(0) \\ &= -\inf\{f^*(x^*) + \delta_A^*(-x^*)\} = \sup\{-f^*(x^*) - \delta_A^*(-x^*)\} \end{aligned} \tag{3.33}$$

Here, setting $f_1 = f, f_2 = \delta_A$, we have used Theorem 3.15.
The problem

$$\sup\{-f^*(x^*) - \delta_A^*(-x^*)\} \quad (G^*)$$

labeled (G^*), we call the dual problem to the primary problem (G).

Let us denote $\tilde{\delta}_A(x) = \delta_A(-x)$. Clearly, $\tilde{\delta}_A^*(x^*) = \delta_A^*(-x^*)$ for all x^*. Also, the values of the problems (G) and (G^*) are denoted by v and v^*, respectively.

Proposition 3.4. For all feasible solutions x and x^* of problems (G) and (G^*), the inequality $v \geq v^*$ holds. Furthermore, the following assertions are equivalent:

1. $v = v^*$.
2. $(f^* \oplus \delta_A^*)^{**}(0) = (f^* \oplus \delta_A^*)(0)$.
3. $(f \oplus \tilde{\delta}_A)^{**}(0) = (f \oplus \tilde{\delta}_A)(0)$.

□ The inequality $v \geq v^*$ follows directly from Eq. (3.33). Moreover, by Theorem 3.15,

$$f^{**} + \delta_A^{**} = (f^* \oplus \delta_A^*)^*,$$

so

$$f + \delta_A = (f^* \oplus \delta_A^*)^*.$$

Thus, $(f + \delta_A)^* = (f^* \oplus \delta_A^*)^{**}$. Then taking into account that $v = -(f + \delta_A)^*(0)$ (see Eq. (3.33)) we can write

$$\begin{aligned} v &= \inf[f(x) + \delta_A(x)] = \inf[f(x) + \tilde{\delta}_A(-x)] \\ &= (f \oplus \tilde{\delta}_A)(0) = -(f + \delta_A)^*(0) = -(f^* \oplus \delta_A^*)^{**}(0). \end{aligned}$$

Similarly,

$$v^* = (f \oplus \tilde{\delta}_A)^{**}(0) = -(f^* \oplus \delta_A^*)(0).$$

Then comparing these two equalities, we have the desired result. ■

The set of solutions of problems (G) and (G^*) are denoted by P and Q, respectively:

$$P = \{x : f(x) + \delta_A(x) = v\}$$
$$Q = \{x : \ -f^*(x^*) - \delta_A^*(-x^*) = v^*\}.$$

Note that if v and v^* are finite, then by Theorems 3.15,

$$P = \partial(f + \delta_A)^*(0) = \partial(f^* \oplus \delta_A^*)^{**}(0),$$
$$Q = \partial(f^* + \tilde{\delta}_A^*)^*(0) = \delta(f \oplus \tilde{\delta}_A)^{**}(0).$$

On the other hand, the set of solutions P of (G) consists of the points x for which the infimum in $f \oplus \tilde{\delta}_A$ is attained at $0 = x + (-x)$. Similarly, the set of solutions Q of (G^*) consists of the points x^* for which the infimum in $f^* \oplus \ \delta_A^*$ is attained at $0 = x^* + (-x^*)$.

Theorem 3.16. If a pair (x, x^*) of points x and x^* is such that $x^* \in \partial f(x) \cap K_A^*(x)$, then x and x^* are solutions of the problems (G) and (G^*), respectively.
□ It is clear that

$$0 \in \partial f(x) - K_A^*(x) = \partial f(x) = \partial \delta_A(x) \subset \partial(f + \delta_A)(x);$$

i.e.,

$$0 \in \partial(f + \delta_A)(x).$$

According to Theorem 3.1, the latter inclusion means that x is a solution to the problem (G). By hypothesis, $x^* \in \partial f(x)$ and $x^* \in K_A^*(x) = -\partial\delta_A(x)$. Since f and δ_M are closed convex proper functions, then by Corollary 1.2, $x \in \partial f^*(x^*)$ and $x \in \delta_A^*(-x^*) = \tilde{\delta}_A^*(x^*)$. Therefore,

$$0 \in \partial f^*(x^*) + \partial\tilde{\delta}_A^*(x^*) \subset \partial(f^* + \tilde{\delta}_A^*)(x^*),$$

$$0 \in \partial(f^* + \tilde{\delta}_A^*)(x^*),$$

and hence x^* is a solution to the problem (G^*). ■

Proposition 3.5. If $v = v^*$ and is finite and the set Q is nonempty, then

$$P = \partial(f^* \oplus \delta_A^*)(0), \quad Q = \partial(f \oplus \tilde{\delta}_A)(0).$$

☐ By Proposition 3.4, if $v = v^*$, then $(f^* \oplus \delta_A^*)^{**}(0) = (f^* \oplus \delta_A^*)(0)$. Therefore, $\partial(f^* \oplus \delta_A^*)^{**}(0) = \partial(f^* \oplus \delta_A^*)(0)$, and by virtue of the finiteness of v, we obtain that $P = \partial(f^* \oplus \delta_A^*)(0)$. The assertion for Q is proved by analogy. ■

Proposition 3.6. If $v = v^*$ and both are finite and the set Q is nonempty, then

$$P = \partial f^*(x^*) \cap \partial \delta_A^*(-x^*),$$

where x^* is an arbitrary element of Q.
☐ The nonempty set Q is the set of points x^* for which the infimum at $0 = x^* + (-x^*)$ is attained. Applying Proposition 3.2 to the case f^* and δ_A^*, we have

$$\partial(f^* \oplus \delta_A^*)(0) = \partial f^*(x^*) \cap \partial \delta_A^*(-x^*).$$

Since $v = v^*$ is finite, by Proposition 3.5, $P = \partial(f^* \oplus \delta_A^*)(0)$. If $\partial f^*(\tilde{x}^*) \cap \partial \delta_A^*(-\tilde{x}^*)$ is nonempty, then by Proposition 3.3

$$\partial(f^* \oplus \delta_A^*)(0) = \partial f^*(\tilde{x}^*) \cap \partial \delta_A^*(-\tilde{x}^*) \neq \varnothing,$$

where the infimum is attained at $0 + \tilde{x}^* + (-\tilde{x}^*)$. Consequently, $\tilde{x} \in Q$. ■

Theorem 3.17. If $v = v^*$ and both are finite and the set Q is nonempty, then a point x is the solution to the primary problem (G) if and only if there exists a point x^* such that $x^* \in \partial f(x) \cap K_A^*(x)$. This point is then a solution to the dual problem (G^*).
☐ According to the previous proposition, $x \in P$ if and only if $x \in \partial f^*(x^*) \cap \partial \delta_A^*(-x^*)$ for a point x^*—i.e., if and only if there is a point x^* such that $x^* \in \partial f(x)$ and $x^* \in K_A^*(x) = -\partial \delta_A(x)$. This point then belongs to the set of solutions of Q. ■

In the next theorem, a sufficient condition ensuring the equality $v = v^*$ and the existence of a solution to the dual problem (G^*) is formulated.

Theorem 3.18. If v is finite and there exists a point $x_1 \in A$ at which f is continuous, then $v = v^*$ and the set of solutions Q of dual problem (G^*) is nonempty.
☐ According to Theorem 6.5.7 of Ref. [138], the function $f^* \oplus \delta_A^*$ is closed, proper, and convex, so

$$(f^* \oplus \delta_A^*)^{**}(0) = (f^* \oplus \delta_A^*)(0).$$

By Proposition 3.4, this implies that $v = v^*$. Moreover, for each x^*, the infimum in the infimal convolution is attained; i.e., there are points x_1^* and x_2^* such that $x^* = x_1^* + x_2^*$ and

$$(f^* \oplus \delta_A^*)(x^*) = f^*(x_1^*) + \delta_A^*(x_2^*).$$

Since Q is the set of points x_1^* for which the infimal convolution is attained at $0 = x_1^* + (-x_1^*)$, then Q is nonempty. ■

Theorem 3.19. If there exists either (1) a point x^* such that $\partial f^*(x^*) \cap \partial \delta_A^*(-x^*) \neq \varnothing$ or (2) a point x such that $\partial f(x) \cap K_A^*(x) \neq \varnothing$, then both the sets of solutions P and Q are nonempty. Moreover, $v = v^*$, where v is finite, and

$$P = \partial f^*(x^*) \cap \partial \delta_A^*(-x^*), \quad x^* \in Q,$$

$$Q = \partial f(x) \cap K_A^*(x), \quad x \in P.$$

□ For simplicity, consider case (1). By Proposition 3.3, it is easy to see that

$$\partial(f^* \oplus \delta_A^*)(0) = \partial f^*(x^*) \cap \partial \delta_A^*(-x^*) \neq \varnothing,$$

and the set Q consisting of those x^* such that
$(f^* \oplus \delta_A^*)(0) = f^*(x^*) + \delta_A^*(-x^*)$ is nonempty. Since $f(x^*)$ and $\delta_A^*(-x^*)$ are finite, $-v^*$ is finite. Furthermore, $(f^* \oplus \delta_A^*)^{**}(0) = (f^* \oplus \delta_A^*)(0)$, so $v = v^*$. Furthermore,

$$\partial(f^* \oplus \delta_A^*)^{**}(0) = \partial(f^* \oplus \delta_A^*)(0) = P$$

and $P \neq \varnothing$. ■

3.4 Cone of Tangent Directions and Locally Tents

Definition 3.2. Let A be an arbitrary subset of X. A vector $\bar{x} \in X$ is called a tangent direction of the set A at a point $x \in A$, if there exists a function $\varphi(\lambda)$ such that

$$x + \lambda \bar{x} + \varphi(\lambda) \in A$$

for sufficiently small $\lambda \geq 0$ and $\lambda^{-1}\varphi(\lambda) \to 0$ as $\lambda \downarrow 0$. Since $\alpha \bar{x}, \alpha \geq 0$ is a vector of tangent direction, if $\bar{x}$ is the same, then it is clear that such vectors form a cone.

Definition 3.3. The cone $K_A(x)$ is called a cone of tangent directions of the set A at a point $x \in A$ if from $\bar{x} \in K_A(x)$ it follows that $\bar{x}$ is the tangent vector at $x \in A$.

It should be pointed out that the cone $K_A(x)$ is not uniquely defined. In any case we shall see that the wider a cone of tangent directions we have the essentially necessary condition for a minimum. If A is convex, then

$$K_A(x) = \operatorname{cone}(A - x) = \{\bar{x} : \bar{x} = \lambda(x_1 - x), \quad x_1 \in A, \ \lambda > 0\} \tag{3.34}$$

is the cone of tangent directions to A. Therefore in Definition 3.2, it is sufficient to take $\varphi(\lambda) \equiv 0$. By convention, for a convex set A, we define the cone of tangent directions to A by Eq. (3.34). The following examples were considered by Pshenichnyi [226].

Example 3.1. Let A be defined as follows:

$$A = \{x : f_i(x) = 0, \quad i \in I\}, \tag{3.35}$$

where $I = \{1, \ldots, m\}$ is a finite index set and f_i are continuously differentiable functions.

Proposition 3.7. Let $x_0 \in A$; i.e., for x_0, suppose that the system in Eq. (3.35) is satisfied and the gradient vectors $f_i'(x_0)$, $i \in I$ are linearly independent. Then

$$K_A(x_0) = \{\bar{x} : \langle \bar{x}, f_i'(x_0) \rangle = 0, \quad i \in I\} \tag{3.36}$$

is a cone of tangent directions.
☐ Let $f'(x_0)$ be a matrix with rows $f_i'(x_0)$, $i \in I$. Clearly, $f'(x_0)$ is an $m \times n$ ($X = \mathbb{R}^n$) matrix. Then the condition $\bar{x} \in K_M(x_0)$ can be written as $f'(x_0)\bar{x} = 0$. Let $(f'(x_0))^*$ be the matrix transpose of $f'(x_0)$ and $B = f'(x_0)(f'(x_0))^*$. It is easy to show that a square matrix B of the size $m \times m$ is nonsingular. Indeed, suppose that B is singular. Then there is a nonzero vector $y \in \mathbb{R}^m$ such that $By = 0$. Thus,

$$\langle y, By \rangle = \|f'(x_0)^* y\|^2 = \Big\| \sum_{j \in I} f_j'(x_0) y^j \Big\|^2 = 0;$$

i.e., $\sum_{j \in I} f_j'(x_0) y^j = 0$ under the hypothesis that not all y^j are equal to zero. The latter equality implies that the set of vectors $f_i'(x_0)$, $i \in I$, is linearly dependent. This contradiction proves that B is a nonsingular matrix.

Consider the system of equations

$$g_i(\lambda, y) \equiv f_i(x_0 + \lambda \bar{x} + (f'(x_0))^* y) = 0, \quad i \in I \tag{3.37}$$

where y is an unknown and λ is a parameter. It is not hard to compute that

$$\frac{\partial g_i(0,0)}{\partial \lambda} = \langle \bar{x}, f_i'(x_0) \rangle = 0, \quad i \in I.$$

On the other hand, the matrix with the elements $\partial g_i(0,0)/(\partial y^j)$, $i, j = 1, \ldots, m$, is equal to B, so is nonsingular. By the familiar implicit function theorem, the system in Eq. (3.37) for a sufficiently small $\lambda > 0$ has a solution $y(\lambda)$ such that $\lambda^{-1} y(\lambda) \to 0$ as $\lambda \to 0$. If we set $\varphi(\lambda) = [f'(x_0)^* y(\lambda)]$, then $\lambda^{-1}\varphi(\lambda) \to 0$ as $\lambda \to 0$, and according to Eq. (3.37), $x_0 + \lambda \bar{x} + \varphi(\lambda) \in A$ for sufficiently small $\lambda > 0$; i.e., $\bar{x}$ is a tangent vector.

We compute the dual cone of the cone given by Eq. (3.36). Since the system of equations $\langle \bar{x}, f_i'(x_0)\rangle = 0,\ \ i \in I$ is equivalent to the system of inequalities,

$$\langle \bar{x}, f_i'(x_0)\rangle \geq 0, \quad i \in I,$$

$$\langle \bar{x}, -f_i'(x_0)\rangle \geq 0, \quad i \in I,$$

then by Theorem 1.13, $K_A^*(x_0)$ consists of the elements of the form

$$x^* = \sum_{i \in I} \alpha_i f_i'(x_0) - \sum_{i \in I} \beta_i f_i'(x_0), \quad \alpha_i \geq 0, \ \ \beta_i \geq 0.$$

Denoting $\lambda_i = \alpha_i - \beta_i$, we have

$$K_A^*(x_0) = \left\{ x^* : x^* = \sum_{i \in I} \lambda_i f_i'(x_0), \quad \lambda_i \in \mathbb{R}, \ \ i \in I \right\}.$$

Example 3.2. Let the set A be defined as follows:

$$A = \{x : f_i(x) \leq 0, \quad i \in I_1, \ \ f_i(x) = 0, \ \ i \in I_2\}$$

where I_1 and I_2 are finite index sets and f_{i} are continuously differentiable functions.

Let $x_0 \in A$ and

$$I_1(x_0) = \{i \in I_1 : f_i(x_0) = 0\}.$$

We show that if the vectors $f_i'(x_0)$, $i \in I_2$ are linearly independent, then a direction $\bar{x}$ satisfying the relations

$$\langle \bar{x}, f_i'(x_0)\rangle < 0, \quad i \in I_1(x_0),$$

$$\langle \bar{x}, f_i'(x_0)\rangle = 0, \quad i \in I_2 \tag{3.38}$$

is a tangent vector. Indeed, according to the previous example, there is a function $\varphi(\lambda)$, $\lambda^{-1}\varphi(\lambda) \to 0$ as $\lambda \to 0$ such that

$$f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) = 0, \quad i \in I_2.$$

On the other hand, for $i \in I_1$, we can write

$$f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) = f_i(x_0) + \lambda\langle \bar{x} + \lambda^{-1}\varphi(\lambda), f_i'(\xi_i)\rangle, \tag{3.39}$$

where ξ_i is some point of the segment with end points x_0 and $f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) < 0$ for sufficiently small $\lambda > 0$. On the other hand if $i \in I_1(x_0)$, then Eq. (3.39) implies that

$$f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) = \lambda\langle\bar{x}, f_i'(x_0)\rangle + \lambda\langle\bar{x}, f_i'(\xi_i) - f_i'(x_0)\rangle + \langle\varphi(\lambda), f_i'(\xi_i)\rangle.$$

Here, since $\xi_i \to x_0$ as $\lambda \to 0$, then the latter two terms are higher-order infinite small with respect to λ. Hence, by virtue of Eq. (3.38),

$$f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) < 0, \quad i \in I_1(x_0)$$

for small $\lambda > 0$. Therefore, for a small $\lambda > 0$, the relations

$$\begin{aligned} &f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) < 0, \quad i \in I_1, \\ &f_i(x_0 + \lambda\bar{x} + \varphi(\lambda)) = 0, \quad i \in I_2 \end{aligned}$$

are satisfied. Consequently, $x_0 + \lambda\bar{x} + \varphi(\lambda) \in A$. The obtained inclusion means that $\bar{x}$ is a tangent direction of the set A at the point $x_0 \in A$. Hence,

$$K_A(x_0) = \{\bar{x} : \langle\bar{x}, f_i'(x_0)\rangle < 0, \;\; i \in I_1(x_0), \quad \langle\bar{x}, f_1'(x_0)\rangle \;\; = 0, \;\; i \in I_2\}.$$

Thus, applying Theorem 1.13, it is easy to see that

$$K_A^*(x_0) = \left\{ x^* : x^* = -\sum_{i \in I_1(x_0)} \lambda_i f_i'(x_0) - \sum_{i \in I_2} \lambda_i f_i'(x_0), \quad \lambda_i \geq 0, \;\; i \in I_1(x_0) \right\}.$$

By Definition 3.2, we have already seen that the cone of tangent directions involve directions for each of which there exists a function $\varphi(\lambda)$. But in order to predetermine properties of the set A, this is not sufficient. Nevertheless, the following notion of a local tent allows us to predetermine mapping in A for the nearest tangent directions among themselves.

Definition 3.4. The cone $K_A(x_0)$ of tangent directions of the set A at a point $x_0 \in A$ is called a local tent, if for each $\bar{x}_0 \in \operatorname{ri} K_A(x_0)$, there exists a convex cone $K \subseteq K_A(x_0)$ and a continuous mapping ψ defined in a neighborhood of the origin such that

1. $\bar{x}_0 \in \operatorname{ri} K, \quad \operatorname{Lin} K = \operatorname{Lin} K_A(x_0)$;
2. $\psi(\bar{x}) = \bar{x} + r(\bar{x}), \|\bar{x}\|^{-1} r(\bar{x}) \to 0$ as $\bar{x} \to 0$;
3. $x_0 + \psi(\bar{x}) \in A, \;\; \bar{x} \in K \cap (\varepsilon B)$ for some $\varepsilon > 0$, where B is the unit Euclidean ball.

If ψ is a continuously differentiable function, then a local tent is said to be a smooth local tent.

Example 3.3. Let the set A be defined as in Example 3.1 and assume that at x_0 the conditions of Proposition 3.7 are satisfied. We show that the cone $K_A(x_0)$ defined by Eq. (3.36) is a local tent.

Let us form the system of equations

$$g_i(\overline{x}, y) \equiv f_i(x_0 + \overline{x} + (f'(x_0))^* y) - \langle \overline{x}, -f_i'(x_0) \rangle = 0, \quad i \in I \tag{3.40}$$

where y is unknown and $\overline{x}$ is a parameter. It is not hard to see that

$$g_i(0,0) = 0, \ \ i \in I, \quad g_{i\overline{x}}'(0,0) = 0, \ \ i \in I$$

and the matrix $\partial g_i(0,0)/(\partial y^j)$, $i, j \in I$, coincides with the matrix $B = f'(x_0)[f'(x_0)]^*$ and so is nonsingular. Then by Theorem 1.1 of Chapter 5 of Ref. [225], for sufficiently small $\overline{x}$, there is a smooth solution $y(\overline{x})$ such that

$$\frac{y(\overline{x})}{\|\overline{x}\|} \to 0 \tag{3.41}$$

Denote $K = K_A(x_0)$ and set

$$\psi(\overline{x}) = \overline{x} + (f'(x_0))^* y(\overline{x}) \tag{3.42}$$

Take $\overline{x} \in K \cap (\varepsilon B)$, $\varepsilon > 0$ such that $y(\overline{x})$ is defined in the ball εB. By Eq. (3.36), Eq. (3.40) can be rewritten as:

$$f_i(x_0 + \psi(\overline{x})) = 0, \quad i \in I;$$

i.e., $x_0 + \psi(\overline{x}) \in A$. From this and the formulas in Eqs. (3.41) and (3.42), we obtain that $K_A(x_0)$ is a local tent.

Example 3.4. Now, let A be defined as in Example 3.2. We show that the cone $K_A(x_0)$ defined in this example is a local tent.

Obviously, ri $K_A(x_0) = K_A(x_0)$. Let $\overline{x}_0 \in K_A(x_0)$. Denote

$$\delta = \frac{1}{\|\overline{x}_0\|} \max_i \left\{ \langle \overline{x}_0, f_i'(x_0) \rangle : i \in I_1(x_0) \right\} < 0$$

and

$$K = \{\overline{x} : \langle \overline{x}_0, f_i'(x_0) \rangle \le \delta \|\overline{x}\|, \ \ i \in I_1(x_0), \quad \langle \overline{x}_0, f_i'(x_0) \rangle = 0, \ \ i \in I_2\}. \tag{3.43}$$

Then $\overline{x}_0 \in \text{ri } K$ and $K \subseteq K_A(x_0)$. Choose $\psi(\overline{x})$ as in Example 3.3. By the mean value theorem, it is not hard to see that

$$f_i(x_0 + \psi(\overline{x})) = f_i(x_0) + \langle \overline{x}_0, f_i'(x_0) \rangle + 0(\|x\|), \quad i \in I_1. \tag{3.44}$$

Here, if $i \in I_1 \backslash I_1(x_0)$, then $f_i(x_0) < 0$. Hence, it follows from (3.44) that for small $\bar{x}$,

$$f_i(x_0 + \psi(\bar{x})) < 0, \quad i \in I_1 \backslash I_1(x_0) \tag{3.45}$$

and if $i \in I_1(x_0)$, then for small $\bar{x} \in K$,

$$f_i(x_0 + \psi(\bar{x})) \le \delta \|\bar{x}\| + 0(\|x\|) < 0, \quad i \in I_1(x_0). \tag{3.46}$$

On the other hand, by construction of $\psi(\bar{x})$,

$$f_i(x_0 + \psi(\bar{x})) = 0, \quad i \in I_2 \tag{3.47}$$

Thus, the formulas in Eqs. (3.45) and (3.47) imply that $x_0 + \psi(\bar{x}) \in A$ for small $\bar{x} \in K$; i.e., $K_A(x_0)$ is a local tent.

It can be shown that if A is a convex set, then $K_A(x_0) = \text{cone}(A - x_0)$, $x_0 \in A$, is a local tent. Furthermore, an important property of a local tent is that under more general assumptions, the intersection of local tents is a local tent. We will not consider such investigations here; for a detailed consideration, refer to Pshenichnyi [226].

3.5 CUA of Functions

Remember that a smooth function admits a linear approximation. As will be shown in Section 3.6, a convex function can be approached by positively homogeneous functions—i.e., directional derivatives. However, a nonsmooth and nonconvex function cannot be approximated in a neighborhood of some point with positively homogeneous functions. Just for such a class of functions, we will introduce the concept of CUAs. Such a CUA approach was strongly developed and applied to new classes of extremal problems by Pshenichnyi [226]. Note that a CUA $h(\bar{x}, x)$ is defined nonuniquely, and for obtaining the appropriate necessary conditions, as a rule, it is necessary to have a sufficiently wider family of CUA. In Section 3.5, it will be proved that if $h_1(\bar{x}, x)$ and $h_2(\bar{x}, x)$ are CUAs of f at a point x, then $\lambda_1 h_1 + \lambda_2 h_2$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \ge 0$ and $\max(h_1, h_2)$ also are CUAs (Proposition 3.8). Moreover, let h_1, h_2, respectively, be a CUA at the point x, for the function f_1, f_2, and suppose $f = f_1 + f_2$. Then $h(\bar{x}, x) = h_1(\bar{x}, x) + h_2(\bar{x}, x)$ is a CUA of f at x. In addition, if $\text{int dom } h_1(\cdot, x) \cap \text{dom } h_2(\cdot, x) \ne \varnothing$, then $\partial f(x) = \partial f_1(x) + \partial f_2(x)$ (Theorem 3.20). In particular, if f is a continuously differentiable function at x, then $h(\bar{x}, x) = \langle \bar{x}, f'(x) \rangle$ is a CUA at x and $\partial f(x) = \{f'(x)\}$. Furthermore, if f is convex and continuous at x, then $h(\bar{x}, x) = f'(x, \bar{x})$ is a CUA and the subdifferential $\partial f(x)$ defined by Definitions 1.29 and 3.6 are the same.

Let $f: X \to \mathbb{R} \cup \{\pm\infty\}$ be an arbitrary function. Let us denote dom $f = \{x : |f(x)| < +\infty\}$.

Take a point $x \in \operatorname{dom} f$, $\bar{x} \in X$, $\bar{x} \neq 0$, and define

$$\Omega(\bar{x}, x) = \sup_{r(\cdot)} \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda\bar{x} + r(\lambda)) - f(x)}{\lambda}$$

where the exterior supremum is taken on all $r(\cdot)$, such that $\lambda^{-1} r(\lambda) \to 0$ as $\lambda \downarrow 0$. Obviously, $\Omega(\bar{x}, x)$, finite or not, always exists. Also, it is easy to see that $\Omega(\cdot, x)$ is a positively homogeneous function; i.e., for any $\lambda > 0$,

$$\Omega(\lambda\bar{x}, x) = \lambda\Omega(\bar{x}, x).$$

If f is Lipschitzian in some neighborhood of x, then

$$|f(x + \lambda\bar{x} + r(\lambda)) - f(x + \lambda\bar{x})| \leq L\|r(\lambda)\|.$$

Therefore,

$$\frac{f(x + \lambda\bar{x} + r(\lambda)) - f(x + \lambda\bar{x})}{\lambda} \to 0 \text{ as } \lambda \downarrow 0.$$

Moreover,

$$\begin{aligned}\Omega(\bar{x}, x) &= \sup_{r(\cdot)} \limsup_{\lambda \downarrow 0} \left[\frac{f(x + \lambda\bar{x} + r(\lambda)) - f(x)}{\lambda} + \frac{f(x + \lambda\bar{x} + r(\lambda)) - f(x + \lambda\bar{x})}{\lambda}\right] \\ &= \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda\bar{x}) - f(x)}{\lambda}\end{aligned}$$

i.e., for Lipschitzian functions, we have

$$\Omega(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda\bar{x}) - f(x)}{\lambda}.$$

Definition 3.5. The function $h(\cdot, x)$ is called a CUA of a function $f: X \to \mathbb{R} \cup \{\pm\infty\}$ at every fixed point $x \in \operatorname{dom} f = \{x : |f(x)| < +\infty\}$, if

1. $h(\bar{x}, x) \geq \Omega(\bar{x}, x)$ for all $\bar{x} \neq 0$;
2. $h(\cdot, x)$ is a closed (lower semicontinuous) positively homogeneous convex function.

It is clear that CUA is not unique; there are a lot of CUAs.

Proposition 3.8. If $h_1(\bar{x}, x)$ and $h_2(\bar{x}, x)$ are CUAs of f at a point x, then $\lambda_1 h_1 + \lambda_2 h_2$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$ and $\max(h_1, h_2)$ is also a CUA. ■

Definition 3.6. If $h(\bar{x}, x)$ is a CUA of f at a point x, then the set

$$\partial h(0, x) = \{x^* \in X^* : h(\bar{x}, x) \geq \langle \bar{x}, x^* \rangle, \ \ \bar{x} \in X\}$$

is called a subdifferential of f at a point x and is denoted by $\partial f(x)$.

Obviously, $\partial(\lambda f) = \lambda \partial f$, $\lambda \geq 0$.

According to Definition 3.5 and Theorems 1.24 and 1.31, a subdifferential $\partial h(0, x)$ of a closed positively homogeneous convex function $h(\cdot, x)$ always exists and

$$\partial h(0, x) = \text{dom } h^*(\cdot, x) \tag{3.48}$$

where

$$h^*(\bar{x}^*, x) = \sup_{\bar{x}} \{\langle \bar{x}, \bar{x}^* \rangle - h(\bar{x}, x)\}.$$

On the other hand, by Theorem 1.25, it follows from Eq. (3.48) that

$$h(\bar{x}, x) = \sup_{x^*} \{\langle \bar{x}, \bar{x}^* \rangle : x^* \in \partial h(0, x)\} = \sup_{x^*} \{\langle \bar{x}, \bar{x}^* \rangle : x^* \in \partial f(x)\} \tag{3.49}$$

In addition, the subdifferential $\partial f(x)$ in Eq. (3.49) is defined using the function $h(\bar{x}, x)$. It is also clear that $h(\bar{x}, x)$ is the support function of $\partial f(x)$. Thus, $h(\bar{x}, x)$ and $\partial f(x)$ determine each other one to one. The function $h(\cdot, x)$ defined by Eq. (3.49) must be a CUA of f at x. The convexity and closedness of the subdifferential immediately follows from Definition 3.6 and Lemma 1.34. As with the CUA, the subdifferential of f is not unique.

Proposition 3.9. If h_1 and h_2 are CUAs for the function f at a point x and $h_1 \geq h_2$, then $\partial_1 f(x) \supseteq \partial_2 f(x)$, where $\partial_1 f(x)$ and $\partial_2 f(x)$ are the subdifferentials defined by h_1 and h_2, respectively.

□ Indeed, according to Definition 3.6, $x^* \in \partial_2 f(x)$ if and only if $\langle \bar{x}, x^* \rangle \leq h_2(\bar{x}, x)$. Then, since $h_1 \geq h_2$, it follows that $\langle \bar{x}, x^* \rangle \leq h_2(\bar{x}, x) \leq h_1(\bar{x}, x)$; i.e., $x^* \in \partial_1 f(x)$. ■

Theorem 3.20. Let h_1 and h_2 be CUAs for the functions f_1 and f_2 at x, respectively, and $f = f_1 + f_2$. Then $h(\bar{x}, x) = h_1(\bar{x}, x) + h_2(\bar{x}, x)$ is a CUA for f at x. In addition, if int dom $h_1(\cdot, x) \cap$ dom $h_2(\cdot, x) \neq \varnothing$, then

$$\partial f(x) = \partial f_1(x) + \partial f_2(x).$$

☐ By definition,

$$\Omega(\bar{x},x) = \sup_{r(\cdot)} \limsup_{\lambda\downarrow 0} \left[\frac{f_1(x+\lambda\bar{x}+r(\lambda))}{\lambda} + \frac{f_2(x+\lambda\bar{x}+r(\lambda)) - f_1(x) - f_2(x)}{\lambda}\right]$$
$$\leq \sup_{r(\cdot)} \limsup_{\lambda\downarrow 0} \frac{f_1(x+\lambda\bar{x}+r_1(\lambda)) - f_1(x)}{\lambda} + \sup_{r(\cdot)} \limsup_{\lambda\downarrow 0} \frac{f_2(x+\lambda\bar{x}+r_2(\lambda)) - f_2(x)}{\lambda}$$
$$= \Omega_1(\bar{x},x) + \Omega_2(\bar{x},x).$$

Since $h_1 \geq \Omega_1$, $h_2 \geq \Omega_2$, $h \geq \Omega$. Besides, as a sum of closed positively homogeneous convex functions, h is the same. It follows that h is a CUA for f at x. Furthermore, by virtue of Theorems 1.18 and 1.29, we have,

$$\partial h(0,x) = \partial h_1(0,x) + \partial h_2(0,x),$$

i.e., $\partial f(x) = \partial f_1(x) + \partial f_2(x)$. ■

Example 3.5. Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ k, & \text{if } x \geq 0, \end{cases}$$

where k is a constant. It is easy to see that

$$\Omega(\bar{x},0) = \begin{cases} -\infty, & \text{if } \bar{x} < 0, \\ 0, & \text{if } \bar{x} > 0. \end{cases}$$

Hence any function of the form $h(\bar{x},0) = \lambda\bar{x}$, $\lambda \geq 0$ is a CUA for f at $x = 0$. Then $\partial f(0) = \{\lambda\}$, $\lambda \geq 0$. Thus, if $k > 0$, then f has a set of subdifferentials, each of which consists of a unique point $\lambda \geq 0$. If $k < 0$, then

$$\Omega(\bar{x},0) = \begin{cases} +\infty, & \text{if } \bar{x} < 0, \\ 0, & \text{if } \bar{x} > 0. \end{cases}$$

Let

$$h_0(\bar{x},0) = \begin{cases} +\infty, & \text{if } \bar{x} < 0, \\ 0, & \text{if } \bar{x} \geq 0. \end{cases}$$

Since $h_0(\bar{x},0)$ is a closed positively homogeneous convex function, h_0 is a CUA for $f(k < 0)$ at $x = 0$. It can easily be calculated that

$$\partial f(0) = \partial h_0(0,0) = (-\infty, 0].$$

Since $h_0(\bar{x}, 0) = \Omega(\bar{x}, 0), \bar{x} \neq 0$, for every other CUA, the inequality $h \geq h_0$ will be satisfied. By Proposition 3.9, it follows that $\partial f(0) \supseteq (-\infty, 0]$ for an arbitrary subdifferential $\partial f(0)$. This example is instructive in that there exists a subdifferential of a discontinuous function.

Theorem 3.21. Suppose that for a given function f, the function $\Omega(\cdot, x)$ is convex and closed, i.e., $\Omega(0, x) = 0$. Then $\partial F(0, x)$ is a subdifferential of f at x and for any other subdifferential $\partial f(x)$ satisfied the inclusion $\partial f(x) \supseteq \partial\Omega(0, x)$.
□ By hypothesis, $\Omega(\bar{x}, x)$ is a CUA, and for any other CUA the inequality $h \geq \Omega$ holds. ■

Corollary 3.2. If f is a continuously differentiable function at a point x, then $h(\bar{x}, x) = \langle \bar{x}, f'(x) \rangle$ is a CUA at x and $\partial f(x) = \{f'(x)\}$.
□ In fact, for a continuously differentiable function, $\Omega(\bar{x}, x) = \langle \bar{x}, f'(x) \rangle$. ■

Corollary 3.3. If f is convex and continuous at x, then

$$h(\bar{x}, x) = f'(x, \bar{x}) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda\bar{x}) - f(x)}{\lambda}$$

is a CUA and the subdifferential $\partial f(x)$ defined by Definitions 1.29 and 3.6 is the same.
□ Since by Theorem 1.17 a continuous convex function satisfies a Lipschitzian condition, then it is easy to verify that $\Omega(\bar{x}, x) = f'(x, \bar{x})$. Moreover, by Lemma 1.34 and Remark 1.2, $\partial f(x_0)$ is a convex compact set and

$$f'(x, \bar{x}) = \max_{x^*} \{\langle \bar{x}, x^* \rangle : x^* \in \partial f(x)\}.$$

Clearly, this formula implies that $f'(x, \cdot)$ is a closed function. Now, applying Theorem 1.31 shows that

$$\partial\Omega(0, x) = \partial f'(x, 0) = \partial f(x).$$

Thus, $\Omega(\cdot, x)$ is a closed positively homogeneous convex function satisfying the conditions of Theorem 3.21, and its subdifferential coincides with the usual subdifferential of the convex function f. ■

In order to calculate CUA and the corresponding subdifferentials for some of the operations, such as pointwise operations, pointwise supremum, and the Euclidean distance function, refer to Ref. [225].

We emphasize again that for a given function at a point x, there exist a lot of CUAs. However, if h_1 and h_2 are two CUAs at x and $h_1 \geq h_2$, then h_1 is a worse approximation than h_2 in some neighborhood of x.

Definition 3.7. Let h be a CUA of f at a point x. Then h is called the principal CUA, if there is no another h_1 such that

$$h(\bar{x}, x) \geq h_1(\bar{x}, x), \quad \forall \bar{x}.$$

The subdifferential corresponding to h is called the principal subdifferential.

Lemma 3.1. If $\Omega(\bar{x}, x)$ defined at $\bar{x} = 0$ as $\Omega(0, x) = 0$ is a closed convex function, then there is the unique principal subdifferential. In particular, if f is a continuous convex function, then its usual subdifferential is the single principal subdifferential.
□ This is obvious from the preceding definition. ■

3.6 LAM in the Nonconvex Case

In Chapter 2, the basic definitions of multivalued mappings and the main properties of convex mappings were introduced. One of the principal notions was that of a LAM. In this section, we will extend the apparatus of LAM to the nonconvex case. Of course, in the next chapters, we will see how decisive a role the LAM plays for the construction of optimality conditions for different optimal control problems.

Suppose now that a multivalued mapping F is convex valued; i.e., $F(x)$ is convex in Y. For such a definition to be meaningful, we introduce the Hamiltonian function and Argmaximum set defined as in the convex case:

$$H(x, y^*) = \sup_y \{\langle y, y^* \rangle : y \in F(x)\}, \quad y^* \in Y^* = \mathbb{R}^n$$

$$F(x; y^*) = \{y \in F(x) : \langle y, y^* \rangle = H(x, y^*)\}.$$

Lemma 3.2. Let F be a closed, continuous, convex-valued, and bounded multivalued mapping. Then its Hamiltonian function H is continuous and the Argmaximum set $F(x; y^*)$ is upper semicontinuous in x and y^*.
□ Let $x_0 \in \operatorname{dom} F$. Since F is continuous at x_0, then for every open ball $\varepsilon B \subseteq Y$ with centered origin and radius $\varepsilon > 0$ there is a neighborhood V of x_0 such that

$$F(X) \subseteq F(x_0) + \varepsilon B, \quad F(x_0) \subseteq F(x) + \varepsilon B \tag{3.50}$$

for all $x \in \varepsilon B$. Let y_0^* be fixed. Then it follows from the first inclusions of Eq. (3.50) that

$$\|F(x)\| = \sup_y \{\|y\| : y \in F(x_0)\} \leq \sup_{y,u} \{\|y + u\| : y \in F(x_0), \ u \in \varepsilon B\} \leq \|F(x_0)\| + \varepsilon.$$

Then we have

$$\begin{aligned}H(x,y^*) &\geq \sup_{y,u}\{\langle y+u,y^*\rangle : y\in F(x_0),\quad \|u\| <\varepsilon\}\\&= \sup_{y,u}\{\langle y,y_0^*\rangle + \langle u,y^*\rangle + \langle y,y^*-y_0^*\rangle : y\in F(x_0),\ \ \|u\| <\varepsilon\}\\&\leq H(x_0,y_0^*) + \sup_u\{\langle u,y^*\rangle :\ \|u\|<\varepsilon\} + \sup_y\{\langle y,y^*-y_0^*\rangle : y\in F(x_0)\}\\&\leq H(x_0,y_0^*) + \varepsilon\|y^*\| + \|F(x_0)\|\,\|y^*-y_0^*\|.\end{aligned}$$

Similarly, from the second inclusion of Eq. (3.50), we derive that

$$H(x_0,y_0^*)\leq H(x,y^*) + \varepsilon\|y_0^*\| + \|F(x)\|\,\|y^*-y_0^*\|$$

Thus,

$$|H(x,y^*)-H(x_0,y_0^*)|\leq \varepsilon\max\left(\|y^*\|,\|y^*\|\right) + (\|F(x_0)\|+\varepsilon)\|y^*-y_0^*\| \tag{3.51}$$

and so $H(x, y^*)$ is continuous.

To prove the second assertion of the lemma, we must show that for any neighborhood U of zero in Y, there exists an $\varepsilon>0$ such that

$$F(x,y^*)\subseteq F(x_0,y_0^*) + U, \tag{3.52}$$

as soon as $\|x-x_0\|<\varepsilon,\ \ \|y^*-y_0^*\|<\varepsilon$. Suppose that the inclusion in Eq. (3.52) does not hold. Observe that since F is closed and bounded, $F(x, y^*)$ is also closed and bounded. Now, let $x_i\to x_0$, $y_i\to y_0$, but that for every i there is a point $y_i\in F(x_i;y_i^*)$ such that

$$y_i\notin F(x_0;y_0^*) + U \tag{3.53}$$

Since F is continuous, then for x sufficiently close to x_0, all $F(x)$ are bounded. On the other hand, since $F(x;y^*)\subseteq F(x)$, all the $F(x_i;y_i^*)$ are bounded sets and y_i is a bounded sequence. There is no loss of generality if we assume $y_i\to y_0$. By the definition of the Argmaximum set $F(x; y^*)$, we have

$$\langle y_i,y_i^*\rangle = H(x_i,y_i^*).$$

By passing to the limit and using the continuity of H, we obtain

$$\langle y_0,y_0^*\rangle = H(x_0,y_0^*) \tag{3.54}$$

By virtue of the closure of F, the inclusion $y_i\in F(x_i)$ implies that $y_0\in F(x_0)$. Taking into account Eq. (3.54), this means that $y_0\in F(x_0;y_0^*)$. But this contradicts Eq. (3.53). ■

Lemma 3.3. Let the conditions of Lemma 3.2 be satisfied. Moreover, let F be a Lipschitzian mapping. Then the Hamiltonian function is also Lipschitzian.
□ Since F is Lipschitzian, it follows from Eq. (3.50) that the radius ε of the ball εB satisfies the inequality $\varepsilon \leq L\|x - x_0\|$. Therefore, Eq. (3.51) implies that

$$|H(x,y^*) - H(x_0,y_0^*)| \leq L\|x - x_0\| \max(\|y^*\|,\|y^*\|) + (\|F(x_0)\| + L\|x - x_0\|)\|y^* - y_0^*\|$$

which in turn means that the Hamiltonian function is Lipschitzian. ■

Suppose now that at every point $z \in \operatorname{gph} F$, there exists a convex cone of tangent directions $K_{\operatorname{gph} F}(z)$. Recall that $K_{\operatorname{gph} F}(z)$ is a cone of tangent directions if it is convex, and for all $\bar{z} \in K_{\operatorname{gph} F}(z)$ there is a function $r{:}[0,1] \to Z$ such that $z + \lambda\bar{z} + r(\lambda) \in \operatorname{gph} F$ for small $\lambda \geq 0$.

Definition 3.8. The multivalued mapping $F^*{:}Y^* \to P(X^*)$ defined by

$$F^*(y^*, z) = \{x^* : (x^*, -y^*) \in K^*_{\operatorname{gph} F}(z)\} \tag{3.55}$$

is called the LAM to F at the point $z \in \operatorname{gph} F$.

If for the convex mapping we take $K_{\operatorname{gph} F}(z) = \operatorname{cone}(\operatorname{gph} F - z)$ and by definition the LAM in Eq. (3.55) coincides with the LAM of Definition 2.10. Note that for convex-valued mappings, by the convexity of $F(x)$, the directions $\gamma(0, y_1 - y)$, $y_1 \in F(x)$, $z = (x, y)$, $\gamma > 0$ are tangent directions, because for sufficiently small $\lambda > 0$ we have

$$(x, y + \lambda\gamma(y_1 - y)) \in \operatorname{gph} F.$$

Therefore, later we assume that

$$K_{\operatorname{gph} F}(z) \supseteq (0, \operatorname{cone}(F(x) - y)), \quad z = (x, y) \in \operatorname{gph} F. \tag{3.56}$$

Lemma 3.4. If the conditions in Eq. (3.56) are satisfied, then $F^*(y^*, z) \neq \varnothing$ if $y \in F(x; y^*)$.
□ If $x^* \in F^*(y^*; z)$, then by definition

$$\langle \bar{x}, x^* \rangle - \langle \bar{y}, y^* \rangle \geq 0, \quad (\bar{x}, \bar{y}) \in k_{\operatorname{gph} F}(z)$$

In particular, if Eq. (3.56) is satisfied, then

$$\langle y_1 - y, -y^* \rangle \geq 0, \ \ y_1 \in F(x) \quad \text{or} \quad \langle y_1 - y, y^* \rangle \leq 0, \ \ y_1 \in F(x)$$

i.e.,

$$H(x, y^*) \leq \langle y, y^* \rangle.$$

But $y \in F(x)$, so

$$H(x, y^*) \geq \langle y, y^* \rangle.$$

From the last two inequalities we conclude that $H(x, y^*) = \langle y, y^* \rangle$; i.e., $y \in F(x; y^*)$. ■

Definition 3.9. The mapping

$$F^*(y^*, x) = \overline{\text{conv}}\left(\bigcup_{y \in F(x;\, y^*)} F^*(y^*; (x, y)) \right)$$

is called the AM to F at $x \in \text{dom } F$.

Lemma 3.5. If F is a convex mapping, then

$$F^*(y^*; x) = F^*(y^*; (x, y)) = \partial_x H(x, y^*)$$

where y is an arbitrary point of the Argmaximum set $F(x; y^*)$.
☐ The proof of this lemma follows immediately from Theorem 2.1. ■

If a multivalued mapping $F_z(\bar{x})$ is defined by

$$F_z(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_{\text{gph } F}(z)\}$$

then by Definition 2.9, $F_z^*(x^*)$ is the AM at $x \in \text{dom } F$, so

$$F^*(y^*, z) = F_z^*(x^*)$$

In Theorem 3.21, $H_F(\cdot\,, y^*)$ denotes the Hamiltonian functions of F_z.

Theorem 3.22. Suppose that F is a convex-valued, closed, bounded, continuous, and Lipschitzian mapping. In addition, suppose that $H_{F_z}(\cdot, y^*)$ is a closed proper function, where $F_z(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_{\text{gph } F}(z)\}$. Then for every $y \in F(x, y^*)$, $z \in \text{gph } F$, the function

$$-H_{F_z}(\bar{x}, -y^*) = \sup_{x^*}\{\langle \bar{x}, -x^* \rangle : x^* \in F^*(-y^*; z)$$

is a CUA for $-H(\cdot\,, y^*)$ and

$$F^*(y^*, z) = \partial_x H(x, y^*).$$

☐ If $\bar{z} = (\bar{x}, \bar{y}) \in K_{\text{gph } F}(z)$, $z = (x, y)$, $y \in F(x)$, then by definition of a cone of tangent directions there is a function $r(\lambda)$, $\lambda^{-1} r(\lambda) \to 0$, $\lambda \downarrow 0$ $(r(\lambda) \in Z)$ such that $z + \lambda \bar{z} + r(\lambda) \in \text{gph } F$ for small $\lambda \geq 0$. This means that

$$y + \lambda \bar{y} + r_y(\lambda) \in F(x + \lambda \bar{x} + r_x(\lambda)), \quad r = (r_x, r_y), \quad r_x(\lambda) \in X, \quad r_y(\lambda) \in Y.$$

Then by Lemma 3.3, $-H$ is a Lipschitzian function. As was shown earlier for such functions, we can write

$$\Omega(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}[-H_F(x + \lambda\bar{x}, -y^*) + H_F(x, -y^*)]$$

Similarly, it can be shown that regardless of the choice of $r(\lambda)$, $\lambda^{-1}r(\lambda) \to 0$, $\lambda \downarrow 0$

$$\Omega(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}[-H(x + \lambda\bar{x} + r_x(\lambda), -y^*) + H(x, -y^*)].$$

Moreover, by the definition of H_F and the condition $y \in F(x; y^*)$, it follows that

$$\begin{aligned}\frac{1}{\lambda}[-H_F(x+\lambda\bar{x}+r_x(\lambda), -y^*)+H_F(x, -y^*)] &\leq \frac{1}{\lambda}[-\langle y+\lambda\bar{y}+r_x(\lambda), -y^*\rangle + \langle y, -y^*\rangle] \\ &= \langle \bar{y}, y^*\rangle + \langle \lambda^{-1}r(\lambda), y^*\rangle.\end{aligned}$$

Then

$$\begin{aligned}\Omega(\bar{x}, x) &= \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}[-H_F(x + \lambda\bar{x} + r_x(\lambda), -y^*) + H_F(x, -y^*)] \\ &\leq \limsup_{\lambda \downarrow 0}[\langle \bar{y}, y^*\rangle + \langle \lambda^{-1}r(\lambda), y^*\rangle] = \langle \bar{y}, y^*\rangle.\end{aligned}$$

Thus,

$$\Omega(\bar{x}, x) \leq \inf_{\bar{y}}\{\langle \bar{y}, y^*\rangle : \bar{y} \in F_z(\bar{x})\}.$$

The right-hand side of this inequality is $-H_{F_z}(\bar{x}, -y^*)$. Here, if $\bar{x} \notin \operatorname{dom} F_z$, then by convention $H_{F_z}(\bar{x}, -y^*) = -\infty$ and this equality holds for every $\bar{x}$. Now, since F_z is quasisuperlinear, by Corollary 2.5, we can write

$$-H_{F_z}(\bar{x}, -y^*) = \sup_{x*}\{\langle \bar{x}, -x^*\rangle : x^* \in F_z^*(-y^*)\}.$$

On the other hand, $F^*(y^*, z) = F_z^*(x^*)$ and $-H_{F_z}(\bar{x}, -y^*) = \sup_{x*}\{\langle \bar{x}, -x^*\rangle : x^* \in F^*(-y^*, z)\}$ is closed, positively homogeneous, and convex in $\bar{x}$. Consequently, $-H_{F_z}$ is a CUA for $-H(\cdot, -y^*)$. Thus, using Theorem 1.31, we obtain

$$\partial[-H(x, -y^*)] = \partial[-H_{F_z}(0, -y^*)] = -F^*(-y^*; z)$$

or

$$\partial_x H(x, y^*) = F^*(y^*, z). \blacksquare$$

This theorem can be extended to the case of the minimization of a function of two variables:

$$W(x) = \inf_y \{\varphi(x, y) : y \in F(x)\}, \quad F(x, \varphi) = \{y \in F(x) : \varphi(x, y) = W(x)\} \tag{3.57}$$

Theorem 3.23. Let φ be continuously differentiable and suppose that $H_{F_z}(\cdot, y^*)$ is closed. Moreover, let W be Lipschitzian. Then for all $y \in F(x, \varphi)$, $z = (x, y)$, the function h defined by

$$h(\bar{x}, x) = \langle \bar{x}, \varphi'_x(z) \rangle - \inf_{x^*} \{\langle x, x^* \rangle : x^* \in F^*(-\varphi'_y(z); z)\}$$

is a CUA of W at x and

$$\partial W(x) = \varphi'_x(z) - F^*(-\varphi'_y(z); z).$$

☐ In the same way as in the proof of Theorem 3.22, it can be shown that

$$\Omega(\bar{x}, x) = \limsup_{\lambda \downarrow 0} \frac{W(x + \lambda \bar{x} + r_x(\lambda)) - W(x)}{\lambda}$$

regardless of the choice of $r(\lambda) = (r_x(\lambda), r_y(\lambda))$, $\lambda^{-1} r(\lambda) \to 0$, $\lambda \downarrow 0$. By Eq. (3.57), obviously

$$\frac{W(x + \lambda \bar{x} + r_x(\lambda)) - W(x)}{\lambda} \le \frac{\varphi(x + \lambda \bar{x} + r(\lambda)) - \varphi(x)}{\lambda}.$$

Thus,

$$\Omega(\bar{x}, x) \le \limsup_{y \downarrow 0} \frac{\varphi(x + \lambda \bar{x} + r(\lambda)) - \varphi(x)}{\lambda} = \langle \bar{x}, \varphi'_x(z) \rangle + \langle \bar{y}, \varphi'_y(z) \rangle.$$

Since for a fixed $\bar{x}$ the point $\bar{y}$ can be chosen arbitrarily from the set

$$F_z(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_{\mathrm{gph}\, F}(z)\}$$

then

$$\Omega(\bar{x}, x) \le \langle \bar{x}, \varphi'_x(z) \rangle + \inf_{\bar{y}} \{\langle \bar{y}, \varphi'_y(z) \rangle : \bar{y} \in F_z(\bar{x})\},$$

or

$$\Omega(\bar{x}, x) \le \langle \bar{x}, \varphi'_x(z) \rangle - H_{F_z}(\bar{x}, -\varphi'_y(z)).$$

If $F_z(\bar{x}) = \varnothing$, then by convention $H_{F_z}(\bar{x}, -y^*) = -\infty$ and this equality holds trivially for every $\bar{x}$. Observe that F_z is quasisuperlinear and $F^*(y^*, z) = F_z^*(x^*)$. Then by Corollary 2.5, we have

$$\sup_{\bar{y}}\{\langle\bar{y}, -\varphi_y'(z)\rangle : \bar{y} \in F_z(\bar{x})\} = \inf_{x*}\{\langle x, x^*\rangle : x^* \in F^*(-\varphi_y'(z); z)\}.$$

Thus, if we denote

$$h(\bar{x}, x) = \langle\bar{x}, \varphi_x'(z)\rangle - \inf_{x^*}\{\langle\bar{x}, x^*\rangle : x^* \in F^*(-\varphi_y'(z); z)\}$$

then $\Omega(\bar{x}, x) \le h(\bar{x}, x)$, where h is a closed positively homogeneous function. This means that $h(\bar{x}, x)$ is a CUA of W at x. Then, by Theorem 1.30, we have

$$\partial W(x) = \partial h(0, x) = \varphi_x'(z) - F^*(-\varphi_y'(z); z). \blacksquare$$

Remark 3.1. It is clear that the result of this theorem coincides with the result of Theorem 2.11 given for the convex case.

3.7 Necessary Conditions for an Extremum in Nonconvex Problems

Next our attention will be focused on nonconvex optimality problems with geometric and functional constraints.

Theorem 3.24. Let x_0 be a point that minimizes the function f over the set A and let $h(\bar{x}, x_0)$ be a CUA of f at x_0. If, in addition, int dom $h(\cdot, x_0) \cap K_A(x_0) \neq \varnothing$, then

$$\partial f(x_0) \cap K^*{}_A(x_0) \neq \varnothing.$$

$\square$ It is easy to see that since x_0 is a point that minimizes the function f, then $h(\bar{x}, x_0) \ge 0$ for all $\bar{x} \in K_A(x_0)$. Indeed, suppose this is not so, i.e., $\bar{x} \in K_A(x_0)$ implies that $h(\bar{x}, x_0) < 0$. By definition, there is a function $r(\lambda)$, $\lambda^{-1} r(\lambda) \to 0$, as $\lambda \downarrow 0$, such that $x_0 + \lambda\bar{x} + r(\lambda) \in A$. Thus,

$$\limsup_{\lambda \downarrow 0} \frac{f(x_0 + \lambda\bar{x} r(\lambda)) - f(x_0)}{\lambda} \le \Omega(\bar{x}, x_0) \le h(\bar{x}, x_0) < 0;$$

i.e., for sufficiently small $\lambda > 0$ we obtain $f(x_0 + \lambda\bar{x} r(\lambda)) < f(x_0)$. This contradiction proves that $h(\bar{x}, x_0) \ge 0$. Therefore, a convex function $h(\cdot, x_0)$ over the convex set $K_A(x_0)$ is minimized at $\bar{x} = 0$. Because of Theorem 3.2, it follows that $\partial f(x_0) \cap K_A^*(x_0) \neq \varnothing$. $\blacksquare$

Corollary 3.4. Let the function f admit a CUA $h(\cdot, x_0)$ at x_0. Then, in order for x_0 to be a point that minimizes the function f over all of X, it is necessary that $0 \in \partial f(x_0)$.

□ Clearly, if $A = X$, then $K_A(x_0) = X$ and $K_A^*(x_0) = \{0\}$. ■

For illustration, consider the following example.

Example 3.6. Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ k, & \text{if } x \geq 0. \end{cases}$$

We emphasize that the condition of Corollary 3.4 must be satisfied for an arbitrary subdifferential. In Example 3.5, we have seen that any nonnegative number is a subdifferential, if $k > 0$. Then the condition $0 \in \partial f(0)$ does not hold. On the other hand, for any subdifferential $\partial f(0) \supseteq (-\infty, 0]$, if $k < 0$. Consequently, $0 \in \partial f(0)$. This example shows that the obtained necessary condition is useful even if the minimized objective function is discontinuous.

Theorem 3.25. Let x_0 be a point minimizing a function f over the set $A = \cap_{i=1}^m A_i$ and suppose that the cones $K_{A_i}(x_0)$ are local tents. If, in addition, f admits a CUA $h(\bar{x}, x_0)$ at x_0 and

$$\text{int dom}\, h(\cdot, x_0) \cap \left(\bigcap_{i=1}^{m} K_{A_i}(x_0) \right) \neq \varnothing,$$

then there exists a number $\lambda \geq 0$ and vectors $x_i^* \in K_{A_i}^*(x_0)$, not all equal to zero, such that

$$\lambda x_0^* = \sum_{i=1}^{m} x_i^*, \quad x_0^* \in \partial f(x_0).$$

□ If the cones $K_i \equiv K_{A_i}(x_0)$ are separable, then there exist vectors x_i^*, $i = 1, \ldots, m$ such that $\sum_{i=1}^m x_i^* = 0$, $x_i^* \in K_i^*$ and the required result of theorem is true in the case $\lambda = 0$. Assume that these cones K_i, $i = 1, \ldots, m$ are nonseparable. Then for every vector $\bar{x}_0 \in \text{ri } K$, $K = \cap_{i=1}^m K_i$ there exists a cone Q and a function ψ such that

$$\bar{x}_0 \in \text{ri } Q, \quad \text{Lin } Q = \text{Lin } K, \quad Q \subseteq K,$$

$$\psi(\bar{x}) = \bar{x} + r(\bar{x}), \quad \|\bar{x}\|^{-1} r(\bar{x}) \to 0, \quad \text{whenever } \bar{x} \to 0$$

and besides $x_0 + \psi(\bar{x}) \in A$ for all $\bar{x} \in Q \cap (\varepsilon B)$, $\varepsilon > 0$. Hence, for sufficiently small $\lambda > 0$

$$x_0 + \psi(\lambda \bar{x}_0) = x_0 + \lambda \bar{x}_0 + r(\lambda \bar{x}_0) \in A, \quad \lambda^{-1} r(\lambda \bar{x}_0) \to 0 \text{ as } \lambda \downarrow 0.$$

It follows that $\bar{x}_0$ is a tangent direction. Thus, ri K is a cone of tangent directions to M at x_0. By hypothesis, int dom $h(\cdot, x_0) \cap K \neq \varnothing$, so we suppose that $\bar{x}_1 \in$ int dom $h(\cdot, x_0)$ and $\bar{x}_1 \in K$. Since $\bar{x}_1$ is an interior point of the convex set dom $h(\cdot, x_0)$, there can be found a vector $\bar{x}_2 \in$ ri K belonging to int dom $h(\cdot, x_0)$. Therefore, ri K, $x_0 \in A$, is the cone of tangent directions for which the hypothesis of Theorem 3.24 is satisfied. Thus, $\partial f(x_0) \cap (ri\ K)^* \neq \varnothing$. According to Lemma 1.7 $\overline{K} = \overline{(\text{ri}\ \ K)}$ and so $(\text{ri } K)^* = K^*$. Consequently, $\partial f(x_0) \cap K^* \neq \varnothing$. By assumption, the cones K_i, $i = 1, \ldots, m$ are nonseparable. Hence, by Theorem 1.11, we can write $K^* = \sum_{i=1}^m K_i^*$. This means that there exist vectors $x_0^* \in \partial f(x_0)$ and $x_i^* \in K_i^*$ such that $x_0^* = \sum_{i=1}^m x_i^*$. ■

Corollary 3.5. Let x_0 be a solution to the minimization problem with the equality constraints

$$\inf f_0(x) \text{ subject to } f_i(x) = 0, \quad i = 1, \ldots, m$$

where $f_i(x)$, $i = 0, \ldots, m$ are continuously differentiable functions. Then, there exist numbers y^{i^*}, $i = 0, \ldots, m$ not all equal to zero, such that

$$\sum_{i=1}^m y^{i^*} f_i'(x_0) = 0, \quad y^{0^*} \geq 0$$

□ Let us denote $A = \{x : f_i(x) = 0,\ i = 1, \ldots, m$. There are two possibilities: the gradients $f_i'(x_0)$, $i = 1, \ldots, m$ are either linearly independent or linearly dependent. In the first case, we put $y^{0^*} = 0$ and take y^{i^*} $i = 1, \ldots, m$, satisfying

$$\sum_{i=1}^m y^{i^*} f_i'(x_0) = 0.$$

In the second case, by Example 3.1,

$$K_A(x_0) = \{\bar{x} : \langle \bar{x}, f_i'(x_0) \rangle = 0, \quad i \in I\},$$

$$K_A^*(x_0) = \left\{ x^* : x^* = \sum_{i \in I} \lambda_i f_i'(x_0), \quad \lambda_i \in \mathbb{R},\ \ i \in I \right\}.$$

Since f_0 is a smooth function, $\partial f(x_0) = \{f_0'(x_0)\}$. Then, by Theorem 3.24,

$$f_0'(x_0) = \sum_{i=1}^m \lambda_i f_i'(x_0),$$

whence setting $y^{0^*} = 1$, $y^{i^*} = -\lambda_i$, $i = 1, \ldots, m$, we have the required result. ■

Corollary 3.6. Let x_0 be a solution to the minimization problem with equality and inequality constraints

$$\inf\ f_0(x)\ \text{ subject to } f_i \leq 0, \quad i \in I_1,\ \ f_i = 0,\ \ i \in I_2$$

where $f_i(x)$, $i \in \{0\} \cup I_1 \cup I_2$ are continuously differentiable functions. Then there exist numbers $y^{i^*} \geq 0$, $i \in \{0\} \cup I_1 \cup I_2$, not all equal to zero, such that

$$\sum_{i \in \{0\} \cup I_1 \cup I_2} y^{i^*} f_i'(x_0) = 0, \quad y^{i^*} \geq 0, \quad i \in \{0\} \cup I_1 \cup I_2,$$

$$y^{i^*} f_i(x_0) = 0, \quad i \in I_1$$

☐ Let us denote

$$A_i = \begin{cases} \{x : f_i(x) \leq 0\}, & i \in I_1, \quad f_i(x_0) = 0, \\ X, & i \in I_1, \quad f_i(x_0) < 0, \\ \{x : f_i(x) = 0\}, & i \in I_2. \end{cases}$$

According to Examples 3.3 and 3.4, the cones

$$K_i = \begin{cases} \{\bar{x} : \langle \bar{x}, f_i'(x_0) \rangle < 0\}, & i \in I_1, \quad f_i(x_0) = 0, \\ X, & i \in I_1, \quad f_i(x_0) < 0, \\ \{\bar{x} : \langle \bar{x}, f_i'(x_0) \rangle = 0\}, & i \in I_2 \end{cases}$$

are local tents to A_i at x_0 if $f_i'(x_0) \neq 0$, $i \in I_1$, $f_i(x_0) = 0$ or $i \in I_2$. Then, using the dual cones

$$K_i^* = \begin{cases} \{-\lambda_i f_i'(x_0) : \lambda_i \geq 0\}, & i \in I_1, \quad f_i(x_0) = 0, \\ \{0\}, & i \in I_1, \quad f_i(x_0) < 0, \\ \{-\lambda_i f_i'(x_0) : \lambda_i \in \mathbb{R}\}, & i \in I_2, \end{cases}$$

the conditions of Theorem 3.25 can be rewritten

$$\lambda x_0^* = -\sum_{i \in (0) \cup I_1 \cup I_2} \lambda_i f_i'(x_0) = 0, \ \ \lambda_i \geq 0, \ \ i \in I_1, \quad \lambda_i f_i(x_0) = 0, \ \ i \in I_1$$

Setting $y^{0^*} = \lambda$, $y^{i^*} = \lambda_i$, we have the desired result. ■

Corollary 3.7. Let x_0 be a solution to the minimization problem with equality and inequality constraints

$$\inf\ f_0(x)\ \text{ subject to } f_i \leq 0,\ i \in I_1,\ f_i = 0,\ i \in I_2,\ \ x \in A,$$

where $f_i(x)$, $i \in \{0\} \cup I_1 \cup I_2$ are continuously differentiable functions and A is a convex set. Then, there exist numbers $y^{i^*} \geq 0$, $i \in \{0\} \cup I_1 \cup I_2$, not all equal to zero, such that

$$\sum_{i \in (0) \cup I_1 \cup I_2} y^{i^*} f_i'(x_0) \in (\text{cone}(M - x_0))^*; \quad y^{i^*} \geq 0, \;\; i \in \{0\} \cup I_1 \cup I_2,$$

$$y^{i^*} f_i(x_0) = 0, \quad i \in I_1$$

□ Taking into account that in the present case cone$(A - x_0)$ is a local tent, the proof is similar to the proof of Corollary 3.6. Indeed, there exist numbers $\lambda_i \geq 0$, $i \in I_1$, such that

$$\lambda f_0'(x_0) = \sum_{i \in I_1 \cup I_2} \lambda_i f_i'(x_0) + x^*, \quad x^* \in (\text{cone}(M - x_0))^*$$

and $\lambda_i = 0$, if $f_i(x_0) < 0$. Now, setting $y^{0^*} = \lambda$, $y^{i^*} = \lambda_i$, from here we deduce the needed result. ■

At the end of this section, we consider different mathematical programs with equilibrium constraints; consider a problem with the constraint given by the multi-valued mapping $F: X \to P(Y)$:

$$\inf \; f_0(y) \text{ subject to } y \in F(x) \cap N, \quad x \in M. \tag{3.58}$$

Theorem 3.26. Let y_0 be an optimal solution to problem (3.58). Assume that f_0 admits a CUA $h(\bar{y}, y_0)$ continuous in $\bar{y}$, and suppose that $K_{\text{gph } F}(x_0, y_0)$, $K_N(y_0)$, $K_M(x_0)$, $y_0 \in F(x_0) \cap N$ are local tents to gph F, N, and M, respectively. Then, there exist a number $\lambda \geq 0$ and vectors $x^* \in K_M^*(x_0)$, $y^* \in K_N^*(y_0)$, not all equal to zero, such that

$$-x^* \in F^*(y^* - \lambda y_0^*; z_0), \quad y_0^* \in \partial f_0(y_0), \quad z_0 = (x_0, y_0).$$

□ It is easy to see that the problem in Eq. (3.58) is equivalent to the following minimization problem in the space $Z = X \times Y$:

$$\text{infimum} \;\; \varphi(z) \;\; \text{subject to} \;\; z \in \text{gph } F \cap (X \times N) \cap (M \times Y) \tag{3.59}$$

where $\varphi(z) = f_0(y)$, $z = (x, y)$. Obviously,

$$\limsup_{\lambda \downarrow 0} \frac{\varphi(z_0 + \lambda \bar{z} + r(\lambda)) - \varphi(z_0)}{\lambda} \leq \Omega(\bar{y}, y_0)$$

so that $h(\bar{y}, y_0)$ is a CUA for φ at $z_0 = (x_0, y_0)$. Since $h(\bar{y}, y_0)$ does not depend on $\bar{x}$, the subdifferential $\partial\varphi(z_0)$ corresponding to this CUA has the form

$$\partial\varphi(z_0) = \{0\} \times \partial f_0(y_0), \tag{3.60}$$

i.e., $\partial\varphi(z_0) = \{(x^*, y^*): x^* = 0,\ y^* \in \partial f_0(y_0)\}$. By hypothesis, the cones $K_{\text{gph } F}(z_0)$, $X \times K_N(y_0)$, $K_M(x_0) \times Y$ are local tents to gph F, $X \times N$, $M \times Y$ at z_0, respectively. Observe that

$$\begin{aligned}(X \times K_N(y_0))^* &= \{0\} \times K_N^*(y_0),\\ (K_M(x_0) \times Y)^* &= K_M^*(x_0) \times \{0\}.\end{aligned} \tag{3.61}$$

Now, taking into account Eqs. (3.60) and (3.61), by Theorem 3.25, we can claim that there exist a number $\lambda \geq 0$ and vectors x^*, y^*, (x_1^*, y_1^*) not all equal to zero, such that

$$\begin{gathered}\lambda(0, y_0^*) = (x_1^*, y_1^*) + (0, y^*) + (x^*, 0),\\ x^* \in K_M^*(x_0), \quad y^* \in K_N^*(y_0),\\ y_0^* \in \partial f_0(y_0), \quad (x_1^*, y_1^*) \in K_{\text{gph } F}^*(z_0).\end{gathered} \tag{3.62}$$

We can rewrite Eq. (3.62) as $\lambda y_0^* = y_1^* + y^*$, $x_1^* + x^* = 0$. Then $x_1^* = -x^*$ and $(-x^*, -(y^* - \lambda y_0^*)) \in K_{\text{gph } F}^*(z_0)$. By definition of LAM, it follows that

$$-x^* \in F^*(y^* - \lambda y_0^*; z_0). \blacksquare$$

Consider now the problem with constraint given by a multivalued mapping F:

$$\inf \quad f(x) \text{ subject to } y \in F(x) \cap N \neq \varnothing, \quad x \in M, \tag{3.63}$$

where $f: X \to \mathbb{R} \cup \{\pm\infty\}$, $M \subseteq X$.

Corollary 3.8. Let x_0 be a solution to the problem in Eq. (3.63). Assume that the inverse mapping F^{-1} exists and that f admits a CUA $h(\bar{x}, x_0)$ continuous in $\bar{x}$, and that $K_{\text{gph } F}(x_0, y_0)$, $K_N(y_0)$, $K_M(x_0)$, $y_0 \in F(x_0) \cap N$ are local tents to gph F, N, and M, respectively. Then, there exist a number $\lambda \geq 0$ and vectors $x^* \in K_M^*(x_0)$, $y^* \in K_W^*(y_0)$, not all equal to zero, such that

$$x^* - \lambda x_0^* \in F^*(-y^*; z_0), \quad x_0^* \in \partial f(x_0), \quad z_0 = (x_0, y_0).$$

□ Obviously, the problem in Eq. (3.63) can be converted to a problem of the form shown in Eq. (3.58), as follows:

$$\inf \quad f(x) \text{ subject to } x \in F^{-1}(y) \cap M, \quad y \in N. \tag{3.64}$$

Then, applying Theorem 3.26 to the problem in Eq. (3.64), we claim that there exist a number $\lambda \geq 0$ and vectors $x^* \in K_M^*(x_0)$, $y^* \in K_N^*(y_0)$ not all equal to zero, such that

$$-y^* \in (F^{-1})^*(x^* - \lambda x_0^*; (y_0, x_0)), \quad x_0^* \in \partial f(x_0).$$

Thus, applying Theorem 2.6 to the latter inclusion, we can express it by the LAM $(F^*)^{-1}$

$$-y^* \in -(F^*)^{-1}(\lambda x_0^* - x^*; (x_0, y_0)),$$

which by the definition of inverse LAM means that $\lambda x_0^* - x^* \in F^*(y^*; z_0)$, $z_0 = (x_0, y_0)$. ■

Set up a minimization problem

$$\inf \ g(x, y) \ \text{ subject to } \ y \in F(x) \cap N, \quad x \in M, \tag{3.65}$$

where $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ and M is some subset of X.

Corollary 3.9. Let $z_0 = (x_0, y_0)$ be the solution to problem in Eq. (3.65). Assume that g admits a CUA $h(\bar{z}, z_0)$ continuous in $\bar{z} = (\bar{x}, \bar{y})$ and assume that $K_{\text{gph } F}(x_0, y_0)$, $K_N(y_0)$, $K_M(x_0)$, $y_0 \in F(x_0) \cap N$ are local tents to gph F, N, and M, respectively. Then, there exist a number $\lambda \geq 0$ and vectors $x^* \in K_M^*(x_0)$, $y^* \in K_N^*(y_0)$ not all equal to zero, such that

$$\lambda x_0^* - x^* \in F^*(y^* - \lambda y_0^*; z_0), \quad (x_0^*, y_0^*) \in \partial g(x_0, y_0), \quad z_0 = (x_0, y_0).$$

□ In this case, it is not hard to see that as in the proof of Theorem 3.26, we have

$$\lambda(x_0^*, y_0^*) = (x_1^*, y_1^*) + (0, y^*) + (x^*, 0), \tag{3.66}$$

where $x^* \in K_M^*(x_0)$, $y^* \in K_N^*(y_0)$, $(x_0^*, y_0^*) \in \partial g(x_0, y_0)$, $(x_1^*, y_1^*) \in K_{\text{gph } F}^*(z_0)$. Rewriting Eq. (3.66) as $y_1^* = \lambda y_0^* - y^*$, $x_1^* = \lambda x_0^* - x^*$, we derive $(\lambda x_0^* - x^*, -(y^* - \lambda y_0^*)) \in K_{\text{gph } F}^*(z_0)$. Now, by definition of LAM, it follows that $\lambda x_0^* - x^* \in F^*(y^* - \lambda y_0^*; z_0)$. ■

Corollary 3.10. Suppose that $z_0 = (x_0, y_0)$ is a solution to the problem in Eq. (3.65), where $g(x, y) = g_1(x) + g_2(y)$, and that g_1, g_2 admit continuous CUAs $\bar{h}_1(\bar{x}, x_0), \bar{h}_2(\bar{y}, y_0)$ satisfying int dom $\bar{h}_1(\cdot, x_0) \times$ dom $\bar{h}_2(\cdot, y_0) \neq \varnothing$. Then there exist a number $\lambda \geq 0$ and vectors $x^* \in K_M^*(x_0)$, $\tilde{y}^* \in K_N^*(y_0)$, not all equal to zero, such that

$$-x^* \in F^*(y^*; z_0) - \lambda \partial g_1(x_0),$$

$$\tilde{y}^* - y^* \in \lambda \partial g_2(y_0), \quad z_0 = (x_0, y_0)$$

□ Let us denote $\varphi_1(z) = g_1(x)$, $\varphi_2(z) = g_2(y)$ and let $h_1(\bar{z}, z_0)$, $h_2(\bar{z}, z_0)$ be CUAs of φ_1 and φ_2 at $z_0 = (x_0, y_0)$, respectively. Then by Theorem 3.20, $h(\bar{z}, z_0) = h_1(\bar{z}, z_0) + h_2(\bar{z}, z_0)$, $\bar{z} = (\bar{x}, \bar{y})$ is a CUA for a function g at z_0, and if int dom $h_1(\cdot, z_0) \cap \text{dom } h_2(\cdot, z_0) \neq \emptyset$, then $\partial g(z_0) = \partial\varphi_1(z_0) + \partial\varphi_2(z_0)$. On the other hand, since dom $h_1(\cdot, z_0) = \text{dom } \overline{h}_1(\cdot, x_0) \times Y$ and dom $h_2(\cdot, z_0) = X \times \text{dom } \overline{h}_2(\cdot, y_0)$, it follows that

$$\begin{aligned}\text{int dom } h_1(\cdot, z_0) \cap \text{dom } h_2(\cdot, z_0) &= [\text{int dom } \overline{h}_1(\cdot, x_0) \times Y] \cap [X \times \text{dom } \overline{h}_2(\cdot, y_0)]\\ &= [\text{int dom } \overline{h}_1(\cdot, x_0) \cap X] \times [Y \cap \text{dom } \overline{h}_2(\cdot, y_0)]\\ &= \text{int dom } \overline{h}_1(\cdot, x_0) \times \text{dom } \overline{h}_2(\cdot, y_0)\end{aligned}$$

which by hypothesis is nonempty, so $\partial g(z_0) = \partial\varphi_1(z_0) + \partial\varphi_2(z_0)$. Here,

$$\partial\varphi_1(z_0) = \partial g_1(x_0) \times \{0\}; \quad \partial\varphi_2(z_0) = \{0\} \times \partial g_1(x_0),$$

which implies that $\partial g(z_0) = \partial g_1(x_0) \times \partial g_2(y_0)$. Then, denoting $\tilde{y}^* = y^* - \lambda y_0^*$, we have $y^* - \tilde{y}^* = \lambda y_0^*$. Moreover, from the condition of Corollary 3.9, we can derive that $-x^* \in F^*(\tilde{y}^*; z_0) - \lambda x_0^*$. Thus, taking into account the formula obtained for $\partial g(z_0)$, we have the desired result. ■

Corollary 3.11. Let $\psi : X \to Y$ be a single-valued smooth vector function and f_0: $Y \to \mathbb{R}$ be a proper convex function. Moreover, let y_0 be a point minimizing f_0 with constraints

$$y = \psi(x), \quad y \in N, \ x \in M$$

Then, there exist a number $\lambda \geq 0$ and vectors $x^* \in K_M^*(x_0)$, $y^* \in K_N^*(y_0)$, $y_0 = \psi(x_0)$, and $\tilde{y}^*$, not all equal to zero, such that

$$x^* = -\psi'^*(x_0)\tilde{y}^*, \quad y^* - \tilde{y}^* \in \lambda\partial f_0(y_0)$$

□ By definition, ψ' is a linear operator such that

$$\lim_{\|\bar{x}\| \to 0} \frac{\psi(x + \bar{x}) - \psi(x) - \psi'(x)\bar{x}}{\|\bar{x}\|},$$

or, in other words,

$$\psi(x + \bar{x}) = \psi(x) + \psi'(x)\bar{x} + r(\bar{x}), \quad \|\bar{x}\|^{-1} r(\bar{x}) \to 0 \ \text{ as } \bar{x} \to 0. \tag{3.67}$$

Since X and Y are finite-dimensional Euclidean spaces, say $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $\psi'(x)$ is an $m \times n$ matrix with the entries $\partial\psi^i/\partial x^j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, where $\psi^i(x)$ is the ith component of the vector $\psi(x)$. Obviously, if $z_0 = (x_0, y_0) \in \text{gph } \psi$, i.e., $y_0 = \psi(x_0)$, then

$$K_{\text{gph } F}(x_0, y_0) = \{(\bar{x}, \bar{y}) : \bar{x} \in X, \ \bar{y} = \psi'(x_0)\bar{x}\} \tag{3.68}$$

is a local tent to gph ψ. Indeed, taking $\varphi(\bar{x},\bar{y}) = (\bar{x}, \psi(x_0+\bar{x}) - \psi(x_0))$, it follows from Eqs. (3.67) and (3.68) that

$$(x_0, y_0) + \varphi(\bar{x},\bar{y}) = (x_0+\bar{x}, \psi(x_0+\bar{x})) \in \text{gph } \psi$$

$$\varphi(\bar{x},\bar{y}) = (\bar{x}, \psi'(x_0)\bar{x} + r(\bar{x})) = (\bar{x},\bar{y}) + (0, r(\bar{x})).$$

Furthermore, using Eq. (3.68), we have

$$\begin{aligned} K^*_{\text{gph } F}(x_0,y_0) &= \{(x^*,y^*) : \langle \bar{x}, x^*\rangle + \langle \bar{y}, y^*\rangle \geq 0, \ \ \bar{x}\in X, \ \ \bar{y} = \psi'(x_0)\bar{x}\} \\ &= \{(x^*,y^*) : \langle \bar{x}, x^*\rangle + \langle \psi'(x_0)\bar{x}, y^*\rangle \geq 0, \ \ \bar{x}\in X\} \\ &= \{(x^*,y^*) : \langle \bar{x}, x^* + (\psi'(x_0)) * y^*\rangle \geq 0, \ \ \bar{x}\in X\}, \end{aligned} \tag{3.69}$$

where $(\psi'(x_0))^*$ is the transpose matrix. It follows immediately from Eq. (3.69) that $x^* = -[\psi'(x_0)]^* y^*$.Therefore,

$$K^*_{\text{gph } F}(x_0,y_0) = \{(-(\psi'(x_0))^* y^*, y^*) : y^* \in Y^*\}$$

by the definition of LAM $F^*(-y^*; z_0) = \{-\psi'^*(x_0)y^*\}$ or, what is the same,

$$F^*(y^*; z_0) = \{\psi'^*(x_0)y^*\}. \tag{3.70}$$

Thus, taking into account Eq. (3.70) and denoting $y^* - \lambda y_0^* = \tilde{y}^*$, we obtain from Theorem 3.26 the desired result. ■

4 Optimization of Ordinary Discrete and Differential Inclusions and t_1-Transversality Conditions

4.1 Introduction

In this section, a model of economic dynamics described by discrete inclusions with delay is considered; assume that all possible amount of resource (x_t, x_{t-h}), $t = 0, \ldots, T$ is connected by the relation $x_{t+1} \in F_t(x_t, x_{t-h})$, $t = 0, \ldots, T$, $x_t \in \Phi_t$, $x_T \in M$, where F_t is a convex multivalued mapping, $x_t = \xi_t$, $t = -h, -h+1, \ldots, 0$ are the vectors of initial resources, and T, h are positive integers. It is required to find a solution that minimizes $\sum_{t=1}^{T} g(x_t, t)$, where $g(\ ,\ t)$ are proper convex functions. Usually this sum can be interpreted as the total expenditure. Thus, suppose that functioning of some economic dynamics take place at the discrete times $t = 0$, $1, \ldots, T$, with the use of the apparatus of LAM necessary and sufficient conditions for optimality formulated. The following problem with no delay effect:

(P_T) infimum $g(x_T)$ subject to $x_{t+1} \in F_t(x_t)$, $t = 0, \ldots, T-1$; $x_0 \in N$, $x_T \in M$, N, $M \subseteq \mathbb{R}^n$ is equivalent to the following problem infimum $g(x_T)$ subject to $x_T \in F^T(x_0)$, $x_0 \in N$, $x_T \in M$, where $F^T \equiv F_{T-1} \circ F_{T-2} \circ \cdots \circ F_0$ is the composition of mappings F_t, $t = 0, \ldots, T-1$. By calculation, the LAM to composition, it is shown that if $\{\tilde{x}_t\}_{t=0}^{T}$ is the optimal trajectory to problem (P_T), then there exist the number $\lambda \geq 0$ and vectors $x_e^*,\ x_t^*,\ \ t = 0, \ldots, T$, not all equal to zero, such that

$$x_t^* \in F^*(x_{t+1}^*; (\tilde{x}, \tilde{x}_{t+1})), \quad t = 0, \ldots, T-1, \quad x_e^* \in K_M^*(\tilde{x}_T),$$
$$-x_0^* \in K_N^*(\tilde{x}_0), \quad x_e^* - x_T^* \in \lambda \partial g(\tilde{x}_T), \quad \lambda \geq 0.$$

Section 4.1 is devoted to optimal control problems for polyhedral discrete and differential inclusions generated by polyhedral set-valued mapping $F(x) = \{y: Ax - By \leq d\}$, where A, B are $m \times n$ matrices and $d - m$ is the dimensional vector column. We will study in detail the main properties of polyhedral mappings and derive necessary and sufficient conditions for such problems. Note that we shall describe a switching point of discontinuity of the optimal trajectory. We assume that the following condition for the generality of position is satisfied: let the set of vectors $w_{ki}, C_k w_{ki}, \ldots, C_k^{n-1} w_{ki}$ be linearly independent for each vertex of number k

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and rib w_{ki}, $i \in J_k$. Here, J_k is an index set consisting of exactly n indexes for each k. Thus, under the condition for the generality of position imposed on the coefficients of polyhedral map and on the disposition of the given polyhedron with the use of Theorem 4.10, we prove the main result, which can be called the theorem on the finiteness of the number of switchings (Theorem 4.12).

Everywhere in the considered Bolza problems for differential (nonautonomous) inclusion (DFI), $\dot{x}(t) \in F(x(t), t)$ a.e. $t \in [0, 1]$ a feasible solution to differential inclusion (DFI) is understood as absolutely continuous on function [0, 1]. (Note that a.e. denotes that this relation holds almost everywhere). Often the optimality conditions are formulated in the Euler−Lagrange and Hamiltonian forms. Clearly, DFI is a generalization of the usual differential equation, if F is single-valued mapping containing a unique element of control systems described by differential equations with control parameters, $\dot{x} = f(x, u, t)$, $u \in U$, where $F(x, t) = f(x, U, t)$. In most cases, the above DFI admits a parametric representation of such type and a solution of the DFI is also a solution of the latter differential equation for a some control function $u(t) \in U$. Moreover, the majority of mathematical and physical motivations and social and biological sciences should provide many instances of DFI. Besides, differential equations with discontinuous right-hand side and differential variational inequalities form a special class of DFI. Also the basic works of Aubin and Cellina [15] and Tolstonogov [247] studied the existence (local or global) of solutions to a DFI and the topological properties of the set of solutions, as well as the nature of its dependence on the initial state. The different qualitative problems for differential inclusions are considered in Refs. [47,50,59,67−70,80,84,86,89,125,130,135, 155,158,161,181−184,187,192,193,195,201,206,215,219−222,224,239,240,247, 253,261−264,267].

Note that there are different approaches and results in studying optimization problems for different ordinary DFI using one or another tool in nonsmooth analysis (consult Clarke [50−60], Frankowska [82−90], Mahmudov [142−173], Mordukhovich [184−214], Pshenichnyi [224−226], Rockafellar [231,232], and their reference lists). In order to construct a useful duality theory, we use the optimality conditions for convex DFI and DSI. Although, under more general assumptions, such as the closedness of N_0, M_1, the existence of locally tents, the Lipschitzian of objective functions, and the upper semicontinuity and uniform boundedness of the LAM, it can be formulated necessary condition for nonconvex DFI in terms of LAM to F at a point $x \in \text{dom}\, F$.

Lastly, by applying the method described in Sections 4.3 and 4.4, we investigate the Mayer problem for the optimization of linear optimal control ($\dot{x} = Ax + Bu$) on the linear manifold $L = \{x{:}Cx - d = 0\}$, where rank $C = k$; i.e., dim $L = n - k$, d is column vector of corresponding dimension. We shall regard as permissible any piecewise-continuous controls with values in the control domain U. If the corresponding condition for generality of position is fulfilled, then the theorems on the finiteness of the number of switchings are proved. In the case of $L = \mathbb{R}^n$ (for more detailed information, see Ref. [168]), this condition implies the well-known condition for generality for the classical linear theory of optimal control.

For nonautonomous polyhedral DFI $F(x,t)=\{y\in\mathbb{R}^n: A(t)x-B(t)y\leq \mathrm{d}(t)\}$ and $A(t)$, $B(t)$ are $(n-1)$ order continuously differentiable $m\times n$ matrices, $d(t)$ is a continuous column vector. It should be pointed out that in this case the condition for the generality of position has another specific form altogether.

In Section 4.5, a Bolza problem of optimal control theory, whose dynamic constraint is given by some class of nonconvex differential inclusion, is considered. A jump condition connected with state constraints and t_1-transversality conditions are formulated. We say that for a feasible solution $\tilde{x}(t)$ of the considered problem, the t_1-transversality condition on the terminal set M_1 is satisfied if the inequality $-\langle x^*(t),\tilde{x}(t)\rangle > W_{M_1\cap\Phi(t)}(-x^*(t))$, $t_0\leq t\leq t_1$ holds strictly. In other words, t_1-transversality condition guarantees that the point $\tilde{x}(t)$ belongs to the set M_1 only at the instant $t=t_1$. An adjoint trajectory $x^*(t)$, $t\in[t_0,t_1]$ has jumps, which are typical for control systems with state constraints, and among sufficient conditions there appears a condition of jumps (see also Ref. [221]), where the number of jump points may be countable.

Then under the t_1-transversality condition, jump conditions and monotone increasing of $J[x(\cdot),t]$ are proved to be sufficient conditions for optimality (Theorem 4.16) and the result is demonstrated in one familiar time optimal control problem,

$$\begin{cases}\dot{x}^1=x^2,\\ \dot{x}^2=u,\end{cases}$$

where u is the control parameter $|u|\leq 1$ and $|x^2|\leq 1$, $t_0=0$, $x(0)=x_0$, $x(t_1)=0$, and $\Phi(t)\equiv\{x=(x^1,x^2): -1\leq x^2\leq 1\}$. Here, x_0 is an arbitrary initial point.

In Section 4.6, we consider a Bolza problem of optimal control theory with a varying time interval given by convex and nonconvex hereditary DFI. Our main goal is to derive sufficient optimality conditions for neutral functional-differential inclusions, which contain time delays in both state and velocity variables. Both discrete and discrete-approximation inclusions are included in the same auxiliary problem. LAM and especially proved equivalence theorems represent the basic concept of obtaining optimality conditions.

Functional-differential inclusions are well known as DFI with memory expressing the fact that the velocity of the system depends not only on the state of the system at a given instant but also on the history of the trajectory until that instant. The class of functional-differential inclusions contains a large variety of DFI and control systems. It can be seen that such problems contain time delays not only in state variables but also in velocity variables. This makes the neutral-type problems essentially more complicated than delay-differential inclusions. In particular, it is known that an analog of the Pontryagin maximum principle does not generally hold for neutral systems without convexity assumptions [95], and such problems require special approaches.

Thus, optimal control problems with ordinary discrete inclusions (DSI) and DFI are one of the areas in mathematical theory of optimal processes being intensively developed (see Refs. [1,3,8,10,12,15,16,20,23,26,35,36,51,53,55,79,83,88,117,124, 133,135,140,142,143,153,154]). More specifically, we deal with similar problems with both hereditary and state constraints of the Bolza type. Observe that such problems arise frequently not only in mechanics, aerospace engineering, management science, and economies, but also in problems of automatic control, autovibration, burning in rocket motors, and biophysics [55−57,94,106,111,178,188,223,226, 260]. Moreover, neutral systems have some similarities with the so-called hybrid and differential-algebraic equations that are important in engineering control application.

As mentioned above, the presence of state constraints on an optimal control trajectory is sufficiently manifested by producing discontinuities in the corresponding adjoint arc. Thus, we show that one of the adjoint variables has jumps. Most of the appropriate results are obtained for the Mayer problem and fixed time interval (see Refs. [129,133,143,192] and their reference lists). So our aim is to establish well-verifiable sufficient conditions for optimality for functional and delay-differential inclusions with state constraints and free time. These conditions are more precise than any previously published ones because they involve useful forms of the Weierstrass−Pontryagin condition and the Euler−Lagrange-type adjoint inclusions [35,50−53,226,253,255]. As suggested by us previously [141,154,167,171,173], we expect all of these improvements to serve for the future developments of optimal control theory with hereditary differential inclusions.

Moreover, under the hypothesis (H1) involving more general assumptions, we prove the necessary and sufficient conditions for defined neutral-type discrete and discrete-approximation inclusions. Observe that these results are based on the new apparatus of locally adjoint multifunction (LAM). Another definition of the LAM was introduced by Pshenichnyi [224] and applied in articles by Mahmudov [142,146−148,150−156]. Besides, the similar notion is given by Mordukhovich [190] and is called the coderivative of multifunctions at a point. The use of LAM and convex upper approximation (CUA) for nonconvex functions and locally tents [38,224] are very suitable to get the optimality conditions for posed problems. For problems with DSI and variable delay-differential inclusions, there are other adjoint inclusions formulated in terms of LAM and therefore more subtle optimality conditions. Note that transition to the corresponding discrete-approximation problem requires special equivalence theorems. We emphasize that the key to our success is the formulation of the equivalence theorems.

The proposed finite difference methods and the results obtained can be used for numerical solutions of infinite-dimensional problems. But our main interest here is to use finite difference approximations and construct sufficient optimality conditions for neutral-type problems. In this way, for convex neutral-type problems with fixed time intervals, the sufficient conditions are proved. So, the second part of this work is devoted to the limiting procedure in discrete-approximation problems that results in sufficient conditions for functional-differential inclusions.

Thus, for furthest investigations the basic idea is to pass to the formal limit in the necessary conditions for approximate DSI problems to establish conditions sufficient for optimality of the original nonconvex neutral-type continuous time problem. Therefore, as a result of the monotonicity of the Bolza cost functional on t and t_1-transversality conditions sufficient for optimality are derived.

In the reviewed results, the arising adjoint inclusions are stated in the Euler−Lagrange form, culminating in Theorems 4.20 and 4.21. For problems connected with neutral-type functional and delay-differential inclusions, see Mahmudov [144,146,155,158], Mordukhovich [196,197,199,202,204,205,210], Mordukhovich and Trubnik [210], Mordukhovich and Wang [207], Mordukhovich and Shvartsman [211], and Medhin [178]. A great many developments on the optimization of delay-differential inclusions exist (see, for example, Refs. [57,90,91,98, 99,112,115,116,118,185,208,213,216,257,258,356].

In particular, Mordukhovich [192] establishes the value convergence of discrete-approximations as well as the strong convergences of optimal arcs in the classical Sobolev space $W^{1,2}$.

Finally, we consider the optimal control problem with the functional-differential inclusions linear in velocities involving a neutral-type operator given in the Hale form [99,201,204]: $d/(dt)[x(t) - Ax(t-h)] \in F(x(t), x(t-h), t)$. Note that because the trajectories $x(\cdot)$ are continuous (except maybe the point t_0), differential inclusions of Hale form are generally discontinuous on $[t_0 - h,\ t_1]$, while their linear combinations as a result of this Hale form behave nicely on $[t_0, t_1]$. In the nondelayed case with $A \neq 0$, a problem corresponding to implicit DFI is also of substantial interest.

Thus, the investigated examples show that for the concrete problems, the conditions of the proved theorems and well-known results of classical optimal control theory coincide. For more detailed information, consult Mordukhovich [188] and Gabasov and Kirillova [95].

In Section 4.7, under the t_1-transversality condition for higher-order DFI with sth-order differential expression:

$$L_s x = p_0 \frac{d^s x}{dt^s} + p_1 \frac{d^{s-1} x}{dt^{s-1}} + \cdots + p_{s-1} \frac{dx}{dt} + p_s x,$$

where p_i, $i = 0, 1, \ldots, s$ are some real constants, a sufficient condition is formulated. Here a feasible solution $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[t_0, t_1]$ together with the higher order derivatives until $s-1$, for which $x^{(s)}(.) \in L_1^n$. Observe that a class of functions $W_{1,s}^n([t_0, t_1])$ is a Banach space, endowed with the different equivalent norms. The adjoint differential inclusion is constructed in terms of the expression

$$L_s^* x^* = (-1)^s \frac{d^s(p_0 x^*)}{dt^s} + (-1)^{-1} \frac{d^{s-1}(p_1 x^*)}{dt^{s-1}} + \cdots + p_s x^*.$$

In particular, the time optimal control problem is studied. LAMs are used for both convex and nonconvex cases. Furthermore, the application of these results is demonstrated by solving one example.

Note that on the whole the literature investigates problems with second-order DFI. Lupulescu [138] proved the existence of viable solutions for an autonomous second-order functional-differential inclusions in the case when the multifunction that defines the inclusion is upper semicontinuous, compact valued, and contained in the subdifferential of a proper lower semicontinuous convex function. The first viability result for second differential inclusions were given by Haddad and Yarou [100] in the case in which the multifunction is upper semicontinuous and has convex compact values. The nonconvex case has been studied by Lupulescu [138] and Cernea [49] in the finite-dimensional case. The nonconvex case in Hilbert spaces has been studied by Ibrahim and Alkulaibi [107].

Mahmudov [177] analyzed the existence of Lyapunov functions for second-order differential inclusions by using the methods of the viability theory. A necessary assumption on the initial states and sufficient conditions for the existence of local and global Lyapunov functions are obtained. An application is also provided.

Haddad and Yarou [100] considered the Cauchy problem for the infinite-dimensional case and second-order differential inclusions. Second-order differential inclusions have been studied by many authors, mainly when the multifunction is convex valued. Several existence results may be found in Ref. [253].

Coverstone-Carrol et al. [61] applies the new modeling technique of higher-order differential inclusion (HODI) to the modeling and optimization of spacecraft trajectories. The spacecraft equations of motion are mathematically manipulated into differential constraints that remove explicit appearance of the control variables from the problem statement. These constraints are transformed into a nonlinear programming problem by using higher-order approximations of the derivatives of the states. In this work, the new method is first applied to a simple example to illustrate the technique and then to a three-dimensional propellant-minimizing, low-Earth-orbit to geosynchronous-Earth-orbit spacecraft transfer problem. Comparisons are made with results obtained using an established modeling technique. Agarwal and O'Regan [1] presents new fixed-point theorems for weakly sequentially upper semicontinuous maps. These results are then used to establish existence principles for second-order differential equations and inclusions. Auslender and Mechler [23] give necessary and sufficient conditions to ensure the existence of solutions to the second-order differential inclusions with state constraints. Furthermore, second-order interior tangent sets are introduced and studied to obtain such conditions. Haddad and Yarou [100], Halkin [101], Ibrahim and Alkulaibi [107], Lupulescu [137,138], Marco and Murillo [177] investigate the existence of solutions for initial and boundary value problems for second-order impulsive functional-differential inclusions in Banach spaces. Here, a fixed-point theorem for contraction multivalued maps is used. The study of impulsive functional-differential equations and inclusions are linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. That is why the perturbations are considered to take place "instantaneously," in the form of impulses. The theory of impulsive differential equations has seen considerable development (see the monograph by Lakshmikantham et al. [125]).

Furthermore, the application of the results obtained is demonstrated by solving the following well-known time optimal control problem:

$$\text{infimum } t_1 = \int_0^{t_1} 1\, dt \quad \text{subject to} \quad \frac{d^2x}{dt^2} = u, u \in [-1, 1],$$

$$x(0) = x_0, x'(0) = x_1, \quad x(t_1) = x'(t_1) = 0.$$

4.2 Optimization of Ordinary Discrete Inclusions

In this section, a model of economic dynamics described by discrete inclusions with delay is considered and, with the use of the LAM, necessary and sufficient conditions for optimality are formulated. Suppose that the functioning of some economic system takes place at the discrete times $t = 0, 1, \ldots, T$ and that at time t we have a resource vector $(x, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$, which can be transformed at time $t + 1$ to one of the vectors $y \in F_t(x, x_1)$. Here, F_t is a convex multivalued mapping, $F_t(x, x_1) \subseteq \mathbb{R}^n$. At once, note that the convexity of F is assumed for the sake of simplicity and that all results can be generalized to the nonconvex case. Moreover, assume that all possible amounts of resources (x_t, x_{t-h}), $t = 0, \ldots, T$ are connected by $x_{t+1} \in F_t(x_t, x_{t-h})$, $t = 0, \ldots, T$, $x_t \in \Phi_t$, $x_T \in M$, where T, h are fixed natural numbers and Φ_t, M are convex sets. Let $x_t = \xi_t$, $t = -h, -h + 1, \ldots, 0$ be the vectors of initial resources. Thus, we have the following problem:

$$\text{infimum} \quad \sum_{t=1}^{T} g(x_t, t) \tag{4.1}$$

subject to

$$\begin{aligned} &x_{t+1} \in F_t(x_t, x_{t-h}), \quad t = 0, \ldots, T-1, \\ &x_t = \xi_t, \ \ t = -h, -h+1, \ldots 0; \ \ x_t \in \Phi_t, \ \ x_T \in M \end{aligned} \tag{4.2}$$

where $g(\ , t)$ are proper convex functions. A set of points $\{x_{-h}, x_{-h+1}, \ldots, x_T)$ satisfying Eq. (4.2) is called a feasible solution of the problem in Eqs. (4.1) and (4.2) and is denoted by $\{x_t\}_{t=-h}^{T}$. Usually the sum in Eq. (4.1) can be interpreted as the total expenditure. In order to solve this problem, we reduce it to a convex minimization problem and then apply Theorem 3.4. Let us introduce the following sets:

$$\begin{aligned} &\tilde{M}_t = \{w : (x_t, x_{t-h}, x_{t+1}) \in gph\, F_t\}, \quad t = 0, \ldots, T-1, \\ &N = \{w : x_t = \xi_t, t = -h, -h+1, \ldots, 0\}, \\ &\tilde{\Phi}_t = \{w : x_t \in \Phi_t, t = 1, \ldots, T\}, \\ &\tilde{M}_T = \{w : x_T \in M\}, \end{aligned}$$

where $w = (x_{-h}, x_{-h+1}, \ldots, x_T) \in \mathbb{R}^{n(h+1+T)}$.

Then the posed problem is equivalent to the minimization of the convex function $\varphi(w) = \sum_{t=1}^{T} g(x_t, t)$ over the set

$$G = N \cap \left(\bigcap_{t=0}^{T} \tilde{M}_t \right) \cap \left(\bigcap_{t=1}^{T} \tilde{\Phi}_t \right).$$

It is not hard to see that

$$\tilde{K}^*_{\tilde{M}_t}(w) = \{w^* = (x^*_t, x^*_{t-h}, x^*_{t+1}) \in K^*_{gph\ F_t}(x_t, x_{t-h}, x_{t+1}), x^*_k = 0, k \neq t, t-h, t+1\}, \tag{4.3}$$

$$\begin{aligned} \tilde{K}^*_{\tilde{M}_T} &= \{w^* : x^*_t \in K^*_M(x_T),\ x^*_t = 0, t < T\}, \\ K^*_{\tilde{\Phi}_t} &= \{w^* : x^*_t \in K^*_{\Phi_t}(x_T),\ x^*_k = 0, k \neq t\}, \quad t = 1, \ldots T \\ K^*_N &= \{w^* : x^*_t = 0, t \neq -h, -h+1, \ldots, 0\}, \\ w^* &= (x^*_{-h}, x^*{}_{-h+1}, \ldots, x^*_T) \in \mathbb{R}^{n(h+1+T)} \end{aligned} \tag{4.4}$$

Let $\{x_t\}_{t=-h}^{T}$ be a feasible trajectory such that $g(\ ,t)$ is continuous at x_t, $t = 1, \ldots, T$. If $\{\tilde{x}_t\}_{t=-h}^{T}$ is the optimal trajectory of the problem in Eqs. (4.1) and (4.2), then by Theorem 3.4, there are the vectors

$$\begin{aligned} &w^*(t) \in \tilde{K}_{\tilde{M}_t}, \tilde{w}(t) = 0, 1, \ldots, T;\ \ \hat{w}^* \in K_N(\tilde{w});\ \ \tilde{w}^*(t) \in K^*_{\tilde{\Phi}_t}(\tilde{w}) \\ &t = 1, \ldots, T;\ \ \tilde{w} = (\tilde{x}_{-h}, \tilde{x}_{-h+1}, \ldots, \tilde{x}_T),\ \ \tilde{x}_t = \xi_t,\ \ t = -h,\ -h+1, \ldots, 0 \end{aligned}$$

and number $\lambda \in \{0, 1\}$, not all equal to zero, such that

$$\lambda w^* = \hat{w}^* + \sum_{t=0}^{T} w^*(t) + \sum_{t=1}^{T} \tilde{w}^*(t),\ \ w^* \in \partial\varphi(\tilde{w}). \tag{4.5}$$

Taking into account Eqs. (4.3) and (4.4), the component-wise representation in Eq. (4.5) has the form

$$x^*_{t-h}(t) + \hat{x}^*_{t-h} = 0, \quad t = 0, \ldots h-1; \tag{4.6}$$

$$x^*_0(0) + x^*_0(h) + \hat{x}^*_0 = 0, \tag{4.7}$$

$$\lambda x^*_{t0} - x^*_t(t-1) + x^*_t(t) + x^*_t(t+h) + \tilde{x}^*_t(t), \quad t = 1, \ldots, T-1-h, \tag{4.8}$$

$$\lambda x^*_{t0} = x^*_t(t-1) + x^*_t(t) + \tilde{x}^*_t(t), \quad t = T-h, \ldots, T-1, \tag{4.9}$$

$$\lambda x^*_{T0} = x^*_T(T-1) + x^*_T(T) + \tilde{x}^*_T(T), \quad t = T, \tag{4.10}$$

$$x^*_{t-h}(t) = 0,\ \ t = T; \quad x^*_{t0} \in \partial_x g(\tilde{x}_t, t),\ \ t = 1, \ldots, T. \tag{4.11}$$

Clearly, because of the arbitrariness of $\hat{x}^*_{t-h}$, the relation in Eq. (4.6) always holds. Furthermore, if we take $x^*_{00}=0$, $x^*_0(-1)=\hat{x}^*_0$, $\tilde{x}^*_0=0$, then Eq. (4.8) can be extended to the case $t=0$. By virtue of Eqs. (4.3) and (4.7) to Eq. (4.9) and the definition of LAM, we can write

$$\begin{aligned}
&(\lambda x^*_{t0}-x^*_t(t-1)-x^*_t(t+h)-\tilde{x}^*_t(t),\ x^*_{t-h}(t))\in F^*_t(-x^*_{t+1}(t);\ (\tilde{x}_t,\tilde{x}_{t-h},\tilde{x}_{t+1})),\\
&\qquad t=0,\ldots,T-h-1,\\
&(\lambda x^*_{t0}-x^*_t(t-1)-\tilde{x}^*_t(t),\ x^*_{t-h}(t))\in F^*_t(-x^*_{t+1}(t);\ (\tilde{x}_t,\tilde{x}_{t-h},\tilde{x}_{t+1})),\\
&\qquad t=T-h,\ldots,T-1.
\end{aligned}$$

Thus if in the latter inclusions we introduce the notations

$$\begin{aligned}
&x^*_t\equiv -x^*_t(t-1),\quad x^*\equiv x^*_T(T),\quad \tilde{x}^*_t\equiv\tilde{x}^*_t(t),\quad t=0,\ldots,T-1,\\
&x^*_t(t+h)\equiv\eta^*_{t+h}(t)\equiv\eta^*_{t+h},\quad t=0,\ldots,T-h-1,\\
&x^*_{t-h}(t)\equiv\eta^*_t(t-h)\equiv\eta^*_t,\quad t=T-h,\ldots,T-1
\end{aligned}$$

then by virtue of Eqs. (4.10) and (4.11), we have proved Theorem 4.1.

Theorem 4.1. Let F_t and $g(\ ,t)$, $t=1,\ldots,T$ be a convex multivalued mapping and a convex function, respectively. Moreover, let $\{x_t\}^T_{t=-h}$ be a feasible solution of the problem in Eqs. (4.1) and (4.2) and suppose that $g(\ ,t)$ is continuous at x_t, $t=1,\ldots,T$. Then in order for $\{\tilde{x}_t\}^T_{t=-h}$ to be the optimal solution to the problem in Eqs. (4.1) and (4.2) with the initial values ξ_t and target set M, it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors x^*,x^*_t,η^*_t, $t=0,\ldots,T$, not all equal to zero, such that

i. $(x^*_t-\eta^*_{t+h},\eta^*_t)\in F^*_t(x^*_{t+1},(\tilde{x}_t,\tilde{x}_{t-h},\tilde{x}_{t+1})) + \{K^*_{\Phi_t}(\tilde{x}_t)-\lambda\partial g(\tilde{x}_t,t)\}\times\{0\}$,
$t=0,\ldots,T-1-h$, $\lambda\partial g(\tilde{x}_0,0)=0$, $K^*_{\Phi_0}(\tilde{x}_0)$,

ii. $(x^*_t,\eta^*_t)\in F^*_t(x^*_{t+1};(\tilde{x}_t,\tilde{x}_{t-h},\tilde{x}_{t+1}))+\{K^*_{\Phi_t}(\tilde{x}_t)-\lambda\partial_x g(\tilde{x}_t,t)\}\times\{0\}$
$t=T-h,\ldots,T-1$,

iii. $x^*_T-x^*\in K^*_{\Phi_T}(\tilde{x}_T)-\lambda\partial g(\tilde{x}_T,T)$, $\quad x^*\in K^*_M(\tilde{x}_T)$, $\eta^*_T=0$.

In addition, if $\lambda=1$, then these conditions are sufficient for the optimality of the trajectory $\{\tilde{x}_t\}^T_{t=-h}$. The conditions of this theorem can be rewritten in a more symmetrical form.

Corollary 4.1. Let Eqs. (4.1) and (4.2) be an optimization problem for an ordinary discrete inclusion satisfying the hypotheses of Theorem 4.1. In addition, suppose that F_t is closed for all fixed (x,x_1). Then in order for $\{\tilde{x}_t\}^T_{t=-h}$ to be the optimal solution to problem in Eqs. (4.1) and (4.2), it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors x^*,x^*_t,η^*_t, $t=0,\ldots,T$, not all equal to zero, such that

1. $(x^*_t-\eta^*_{t+h},\eta^*_t)\in\partial_{(x,x_1)}H_t(\tilde{x},\tilde{x}_{t-h},x^*_{t+1})+\{K^*_{\Phi_t}(\tilde{x}_t)-\lambda\partial g(\tilde{x}_t,t)\}\times\{0\}$,
$t=0,\ldots,T-1-h$, $\lambda\partial g(\tilde{x}_0,0)=0$, $K^*_{\Phi_0}(\tilde{x}_0)$,
2. $(x^*_t,\eta^*_t)\in\partial_{(x,x_1)}H_t(\tilde{x}_t,\tilde{x}_{t-h},x^*_{t+1})+\{K^*_{\Phi_t}(\tilde{x}_t)-\lambda\partial_x g(\tilde{x}_t,t)\}\times\{0\}$,
$t=T-h,\ldots,T-1$,

3. $\tilde{x}_{t+1} \in \partial_{y^*} H_t(\tilde{x}_t, \tilde{x}_{t-h}, x^*_{t+1}), \quad t = 0, \ldots, T-1,$
4. $x^*_T - x^* \in K^*_{\Phi_T}(\tilde{x}_T) - \lambda \partial g(\tilde{x}_T, T), \quad x^* \in K^*_M(\tilde{x}_T), \;\; \eta^*_T = 0.$

In addition, if $\lambda = 1$, then these conditions are sufficient for optimality.

□ By Theorem 2.1,

$$F^*_t(y^*; x, x_1, y) = \partial_{(x, x_1)} H_t(x, x_1, y^*)$$

and this set is nonempty if the Argmaximum set $F_t(x, x_1; y^*)$ is nonempty. On the other hand, since $F_t(x, x_1)$ is closed by hypothesis and $H_t(x, x_1)$ is convex, it follows from Theorem 1.30 that

$$\partial_{y^*} H_t(x, x_1; y^*) = F_t(x, x_1; y^*).$$

Thus, taking into account the last two formulas in conditions (i) and (ii), we obtain 1−3. ■

Remark 4.1. Suppose that F_t, $t = 0, \ldots, T-1$ are upper semicontinuous multivalued mappings and $g(\;, t)$ is a lower semicontinuous function. Further assume that Φ_t, M are closed sets and there exists at least one feasible trajectory of the problem in Eqs. (4.1) and (4.2). Then there is an optimal solution of this problem. Indeed, since semicontinuity implies compactness [235], the considered problem can be converted to a minimization problem on a compact set.

Remark 4.2. Note that by analogy, the optimality conditions for nonconvex problems can be proved. We emphasize only that the basic conditions under which a similar result to Theorem 4.1 is true for the nonconvex case are follows.

1. The mappings F_t, $t = 0, \ldots, T-1$ are such that the cones of tangent directions $K_{\mathrm{gph} F_t}(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1})$ are local tents.
2. The cones of tangent directions $K_{\Phi_t}(\tilde{x}_t)$, $K_M(\tilde{x}_T)$ are local tents.
3. The functions $g(\;, t)$, $t = 1, \ldots, T$ admit a CUA $h(\overline{x}, \tilde{x}_t, t)$ at points $\tilde{x}_t$ that is continuous with respect to $\overline{x}$. This implies that the subdifferentials $\partial_x g(\tilde{x}_t, t) = \partial h(0, \tilde{x}_t, t)$ are defined.

Remark 4.3. Because of the importance of the problem in Eqs. (4.1) and (4.2) in various applications, it seems worthwhile to give a second alternative proof. For simplicity, we consider the following problem without delay effect:

$$\begin{aligned}&\text{infimum } g(x_T)\\ &\text{subject to } x_{t+1} \in F_t(x_t), \;\; t = 0, \ldots, T-1; \quad x_0 \in N, \;\; x_T \in M\end{aligned} \tag{4.12}$$

where $F_t: \mathbb{R}^n \to P(\mathbb{R}^n)$ are multivalued mappings and N, $M \subseteq \mathbb{R}^n$ are sets. Obviously, the problem in Eq. (4.12) is equivalent to the following problem:

$$\text{infimum} \quad g(x_T) \;\; \text{subject to} \quad x_T \in F^T(x_0); \quad x_0 \in N, \;\; x_T \in M \tag{4.13}$$

where $F^T \equiv F_{T-1} \circ F_{T-2} \circ \cdots \circ F_0$ is the composition of the mappings F_t, $t=0, \ldots, T-1$. If there exist local tents $K_{\text{gph } F_t}(\tilde{x}_t, \tilde{x}_{t+1})$, $t=0, \ldots, T-1$ and either

1. $(x_t^0, x_{t+1}^0) \in \text{int}\, K_{\text{gph } F_t}(\tilde{x}_t, \tilde{x}_{t+1}), \quad t=0,\ldots,T-2, \ (x_{T-1}^0, x_T^0) \in K_{\text{gph } F_T}(\tilde{x}_T, \tilde{x}_{T+1})$ or

2. $(x_t^0, x_{t+1}^0) \in \text{ri}\, K_{\text{gph } F_t}(\tilde{x}_t, \tilde{x}_{t+1}), \quad t=0,\ldots,T-1$ (4.14)

is satisfied, then it can be shown in a similar way to Theorem 2.10, that

$$(F^T)^*(\cdot;(\tilde{x}_0,\tilde{x}_T)) = F_0^*(\cdot;(\tilde{x}_0,\tilde{x}_1)) \circ F_1^*(\cdot;(\tilde{x}_1,\tilde{x}_2)) \circ \cdots \circ F_{T-1}^*(\cdot;(\tilde{x}_{T-1},\tilde{x}_T)). \tag{4.15}$$

Now by Theorem 3.26, if $\tilde{x}_T$ is a solution of the problem in Eq. (4.13), then

$$x_0^* \in F^*(x_T^*,(\tilde{x}_0,\tilde{x}_T)), x_e^* \in K_M^*(\tilde{x}_T), \ -x_0^* \in K_N^*(\tilde{x}_0), \ \ x_e^* - x_T^* \in \lambda\partial g(\tilde{x}_T).$$

Therefore, applying Eq. (4.15) to this inclusion, we have a chain of inclusions

$$\begin{aligned} &x_t^* \in F^*(x_{t+1}^*;(\tilde{x}_t,\tilde{x}_{t+1})), \quad t=0,\ldots,T-1, \ \ x_e^* \in K_M^*(\tilde{x}_T), \\ &\quad -x_0^* \in K_N^*(\tilde{x}_0), \quad x_e^* - x_T^* \in \lambda\partial g(\tilde{x}_T), \ \ \lambda \geq 0 \end{aligned} \tag{4.16}$$

which for the problem in Eq. (4.12) is the necessary condition for optimality.
Thus, we have proved Theorem 4.2.

Theorem 4.2. Let $\{\tilde{x}_t\}_{t=0}^T$ be the optimal trajectory to the problem in Eq. (4.12). Assume that the cones of tangent directions $K_{\text{gph } F_t}(\tilde{x}_t,\tilde{x}_{t+1}), K_N(\tilde{x}_0)$, $K_M(\tilde{x}_T)$, are local tents and the function g admits a CUA continuous at $\tilde{x}_T$ and the condition in Eq. (4.14) is fulfilled. Then there exist a number $\lambda \geq 0$ and vectors x_e^*, x_t^*, $t=0, \ldots, T$, not all equal to zero, such that Eq. (4.16) is satisfied.

Remark 4.4. Note that for the problem in Eq. (4.12) with objective function $\sum_{t=0}^T g(x_t,t)$ in the alternative proof of Theorem 4.2, we should use Corollary 3.10, which instead of the adjoint discrete inclusion $x_t^* \in F^*(x_{t+1}^*;(\tilde{x}_t,\tilde{x}_{t+1}))$, implies that

$$x_t^* \in F^*(x_{t+1}^*;(\tilde{x}_t,\tilde{x}_{t+1})) - \lambda\partial_x g(\tilde{x},t).$$

Now let us consider the following problem:

$$\text{infimum } \ g(x_T) \ \text{ subject to } \ x_{t+1} = \psi(x_t), \ \ t=0,\ldots,T-1; \quad x_0 \in N, \ \ x_T \in M, \tag{4.17}$$

where $\psi:\mathbb{R}^n \to \mathbb{R}^n$ and $g:\mathbb{R}^n \to \mathbb{R}$ are differentiable functions and N, M are convex sets.

Corollary 4.2. Let ψ and g be smooth functions and suppose N, M are convex sets. Then in order for $\{\tilde{x}_t\}_{t=0}^T$ to be the optimal solution of the problem in

Eq. (4.17), it is necessary that there exist a number $\lambda \geq 0$ and vectors x_e^*, x_t^*, $t = 0, \ldots, T$, not all equal to zero, such that

$$x_t^* = \psi'^*(\tilde{x}_t)x_{t+1}^*, \quad t = 0, \ldots, T-1, \quad x_e^* \in K_M^*(\tilde{x}_T),$$
$$x_0^* \in K_N^*(\tilde{x}_0), \quad x_e^* - x_T^* \in \lambda \partial g(\tilde{x}_T), \quad \lambda \geq 0$$

□ It is enough to use Eq. (3.69) in the conditions of Eq. (4.16) of Theorem 4.2. ■

Example 4.1. Let $F(x) = Ax + U$, where U is a convex set in $\mathbb{R}^n$ and A is an $n \times n$ matrix. It is required to choose control vectors $u_t \in U$, $t = 0, \ldots, T-1$ that minimize $\sum_{t=1}^{T} g(x_t, t)$, where we assume that the $g(\;, t)$ are smooth functions. Let $\{\tilde{x}_t\}_{t=0}^{T}$ be an optimal trajectory corresponding to $\tilde{u}_t \in U$, $t = 0, \ldots, T-1$; i.e.,

$$\tilde{x}_{t+1} = A\tilde{x}_t + \tilde{u}_t, \quad t = 0, \ldots, T-1.$$

According to Example 2.2, if $y_0 = Ax_0 + u_0$, $u_0 \in U$, then the LAM is

$$F^*(y^*; z_0) = \begin{cases} A^*y^*, & \text{if } -y^* \in [\operatorname{cone}(U - u_0)]^*, \\ \varnothing, & \text{if } -y^* \notin [\operatorname{cone}(U - u_0)]^*. \end{cases}$$

We assume that the LAM F^* is nonempty. The condition $-y^* \in [\operatorname{cone}(U - u_0)]^*$ is equivalent to $\langle u - u_0, -y^* \rangle \geq 0$, $u \in U$ or $\langle u_0, y^* \rangle = \sup_u\{\langle u, y^* \rangle : u \in U\}$. Thus, taking into account that $K_M^*(\tilde{x}_T) = \{0\}$, we can write

$$\begin{aligned} x_t^* &= A^*x_{t+1}^* - \lambda g_x'(\tilde{x}_t, t), \quad t = 0, \ldots, T-1, \\ \langle \tilde{u}_t, x_{t+1}^* \rangle &= \sup_u\{\langle u, x_{t+1}^* \rangle : u \in U\}, \quad x_T^* = -\lambda g_x'(\tilde{x}_T, T). \end{aligned} \tag{4.18}$$

By Eq. (4.18), we have $\lambda = 1$. Indeed, if $\lambda = 0$, then $x_T^* = -\lambda g_x'(\tilde{x}_T, T) = 0$ and $x_t^* = A^*x_{t+1}^*$, which implies that $x_t^* = 0$, $t = 0, \ldots, T-1$. Consequently, necessary and sufficient conditions for optimality are the conditions in Eq. (4.18), where $\lambda = 1$.

Example 4.2. (a) Consider an optimization problem

$$\text{infimum} \quad \sum_{t=1}^{r} g(x, t) \tag{4.19}$$

$$\text{subject to} \quad \begin{aligned} &\varphi(x_t, x_{t-h}, x_{t+1}) \leq 0, \quad t = 0, \ldots, T-1, \\ &x_t = \xi_t, \quad t = -h, \ldots, 0, \end{aligned} \tag{4.20}$$

where $g(\;, t)$ is differentiable and $\varphi(z)$, $z = (x, x_1, y)$ is continuous and convex, and there is a point z_1 such that $\varphi(z_1) < 0$. The problem is to find a solution $\{\bar{x}_t\}_{t=-h}^{T}$ satisfying Eq. (4.20) that minimizes Eq. (4.19). In this case,

$$\text{gph } F = \{z : \varphi(z) < 0\}, \quad F(x) = \{y : \varphi(x, x_1, y) < 0\}.$$

From Lemma 2.10, it is easy to see that

$$F^*(y^*;z) = \{(-\lambda x_0^*, -\lambda x_1^{0*}) : y^* = \lambda y_0^*, \ (x_0^*, x_1^{0*}, y_0^*) \in \partial_z \varphi(z), \ \lambda \geq 0, \lambda\varphi(z) = 0\},$$

if $\varphi(z) = 0$. Note that in the considered case, $M = \Phi_t \equiv \mathbb{R}^n$, $t = 1, \ldots, T$ and so $K_M^*(\tilde{x}_T) = \{0\}$, $K_\Phi^*(\tilde{x}_t) = \{0\}$. Then by Theorem 4.1, we can write

$$(x_t^* - \eta_{t+h}^*, \eta_t^*) \in (-\lambda_t x_{0t}^*, -\lambda_t x_{1t}^{0*}) + (-\lambda g_x^t(\tilde{x}_t, t)) \times \{0\},$$

$$x_{t+1}^* = \lambda_t y_{0t}^*, (x_{0t}^*, x_{1t}^{0*}, y_{0t}^*) \in \partial_z \varphi(\tilde{x}, \tilde{x}_{t-h}, \tilde{x}_{t+1}),$$

$$\lambda_t \varphi(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1}) = 0, \quad \lambda_t \geq 0, \ t = 0, \ldots, T-1-h,$$

$$(x_t^*, \eta_t^*) \in (-\lambda_t x_{0t}^*, -\lambda_t x_{1t}^{0*}) + (-\lambda g_x^t(\tilde{x}_t, t)) \times \{0\},$$

$$x_{t+1}^* = \lambda_t y_{0t}^*, (x_{0t}^*, x_{1t}^{0*}, y_{0t}^*) \in \partial_z \varphi(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1}),$$

$$\lambda_t \varphi(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1}) = 0, \quad \lambda_t \geq 0, \ t = T-h, \ldots, T-1,$$

$$-x_T^* \in \partial_x g(\tilde{x}_T, T), \quad \eta_T^* = 0.$$

Thus, conditions (i) to (iii) of Theorem 4.1 take the form

1. $(x_t^* - \eta_{t+h}^* + \lambda g_x'(\tilde{x}_t, t), \eta_t^*, x_{t+1}^*) \in \partial_z \varphi(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1}), \quad t = 0, \ldots, T-1-h;$
2. $(x_t^* + \lambda g_x'(\tilde{x}_t, t), \eta_t^*, x_{t+1}^*) \in \partial_z \varphi(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1}), \quad t = T-h, \ldots, T-1;$
3. $\lambda_t \varphi(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t+1}) = 0, \ \lambda_t \geq 0, \ t = 0, \ldots T-1, \ -x_T^* \in \partial_x g(\tilde{x}_T, T), \eta_T^* = 0.$

(b) In particular, let us now investigate the so-called von Neumann economic dynamics model [172] with delay. Suppose we have m technological capacity to manufacture output with unit commodity intensity leads manufacture of a_j, $j = 1, \ldots, m$ commodity, $a_j \in \mathbb{R}^n$. Thus the number of different manufactured goods is n and under the unit commodity intensity utilization of jth technological capacity of manufacture of ith commodity is produced amount of a_j^i, $i = 1, \ldots, n$ goods. Naturally, we let $a_j^i \geq 0$. Here under the unit commodity intensity employment is emitted $b_j \in \mathbb{R}^n$ and $c_j \in \mathbb{R}^n$ commodities. Now, if at the given instant time and in the past there are x and x_1 outputs, correspondingly then intensity λ_j of each manufacture capacity, obviously must satisfy the inequality $x \geq \sum_{j=1}^m b_j \lambda_j$, $x_1 > c_j \lambda_j$, $x_1 > c_j \lambda_j$, $\lambda_j > 0$, where is emitted commodity vector y, satisfying the equation $y = \sum_{j=1}^m a_j \lambda_j$. Finally, taking the matrices A, B, and C with columns a_j, b_j, and c_j, respectively we define a multivalued mapping, the graph of which is a polyhedral cone:

$$K = \{(x, x_1, y) : \ x \geq B\lambda, \ \ x_1 \geq C\lambda, \ \ y = A\lambda, \ \ \lambda \geq 0, \ \ \lambda \in \mathbb{R}^m\},$$

where from our above-stated interpretation it follows that A, B, and C are $n \times m$ matrices with nonnegative elements and λ is a vector with components λ_j, $j = 1, \ldots, m$. We formulate the following problem:

$$\text{infimum } g(x_T, T) \text{ subject to } x_{t+1} \in F(x_t, x_{t-h}), t = 0, 1, \ldots, T-1;$$
$$x_t = \xi_t, \ t = -h, -h+1, \ldots, 0,$$

where

$$\text{gph } F = K, g(x_T, T) = -\langle x_T, p^* \rangle, p^* \geq 0.$$

Obviously, $g(x_T, T)$ can be interpreted as the cost of the commodity vector x_T. It can be easily seen that

$$K^* = \{(x^*, x_1^*, y^*) : \ B^* x^* + C^* x_1^* + A^* y^* \geq 0, \ \ x^* \geq 0, \ \ x_1^* \geq 0\}.$$

Then taking into account Example 2.7, we can conclude that

$$F^*(y; (x, x_1, y)) = \{(x^*, x_1^*) : \ B^* x^* + C^* x_1^* - A^* y^* = 0, \ x^* \geq 0, x_1^* \geq 0\}.$$

Let $g(x_T, T) = -\langle x_T, p^* \rangle$, where $p^* \geq 0$ can be interpreted as a cost of the commodity vector x_T. Thus, by Theorem 4.1, from the last form of the LAM, we have

$$B^*(x_t^* - \eta_{t+h}^*) + C^* \eta_t^* - A^* x_{t+1}^* = 0, \ x_t^* - \eta_{t+h}^* \geq 0, \ \eta_t^* \geq 0, \ t = 0, 1, \ldots, T-1-h,$$

$$B^* x_t^* + C^* \eta_t^* - A^* x_{t+1}^* = 0, \quad x_t^* \geq 0, \ \eta_t^* \geq 0, \ t = T-h, \ldots, T-1; \quad \eta_T^* = 0,$$
$$x_T^* = \lambda p^*, \ \ \lambda \in \{0, 1\}.$$

At once observe that since F is polyhedral, $\lambda = 1$. Moreover, $p_t^* = x_t^*$ and $q_t^* = \eta_t^* (t = 0, \ldots, T)$ as above can be interpreted as the costs of the commodities at t and $t - h$, respectively. Consequently, in order that $\{\tilde{x}_t\}_{t=-h}^{T}$ be an optimal trajectory of the von Neumann problem, maximizing the finite profit, it is necessary and sufficient that there exist costs p_t^* and q_t^* such that

$$B^*(p_t^* - q_{t+h}^*) + C^* q_t^* - A^* p_{t+1}^* = 0, \ \ p_t^* - q_{t+h}^* \geq 0, \ q_t^* \geq 0, \ t = 0, 1, \ldots, T-1-h,$$

$$B^* q_t^* + C^* q_t^* - A^* p_{t+1}^* = 0, \ p_t^* \geq 0, \ q_t^* \geq 0, \ t = T-h, \ldots, T-1; \ \ q_T^* = 0, \ p_T^* = p^*.$$

4.3 Polyhedral Optimization of Discrete and Differential Inclusions

This section concerns optimal control problems for polyhedral discrete and differential inclusions. We derive necessary and sufficient conditions for such problems.

Then we study in detail the main properties of polyhedral mappings. Moreover, in all the theorems proved in this section, we assume satisfaction of the so-called condition of generality of position, imposed on the coefficients of the polyhedral map and on the disposition of the given polyhedron. The result obtained can be characterized briefly as the theorem of the number of switchings. Note that we shall describe as a switching point a discontinuity of the optimal trajectory. Consider a multivalued (set-valued) mapping $F:\mathbb{R}^n \times [0,1] \to P(\mathbb{R}^n)$ and define the nonautonomous differential inclusion

$$\dot{x}(t) \in F(x(t),t) \quad \text{a.e. } t \in [0,1] \tag{4.21}$$

generated by F, where $\dot{x}(t)$ stands for the time derivative of $x(t)$. This means that by a solution to a differential inclusion we understand a function absolutely continuous on [0, 1]. On the other hand, it has been widely recognized that the differential inclusion in Eq. (4.21) provides a useful generalization of the control systems described by differential equations with control parameters:

$$\dot{x} = f(x,u,t), \quad u \in U, \tag{4.22}$$

where $F(x, t) = f(x, U, t)$. In some cases, the differential inclusion in Eq. (4.21) admits a parametric representation of the type shown in Eq. (4.22), and a solution of inclusion in Eq. (4.21) is also a solution of the differential equation in Eq. (4.22) for some control function $u(t) \in U$. Now, consider an optimization problem for a convex differential inclusion:

$$\text{infimum } J(x(\cdot)) = \int_0^1 g(x(t),t)dt + \varphi_0(x(1)), \tag{4.23}$$

$$\begin{aligned} \text{subject to} \quad & \dot{x}(t) \in F(x(t),t), \quad \text{a.e. } t \in [0,1], \\ & x(0) \in N_0, \quad x(1) \in M_1, \end{aligned} \tag{4.24}$$

where $g(\ ,t)$ and $\varphi_0(\cdot)$ are convex continuous functions, F is a convex set-valued mapping, and $N_0, M_1 \subseteq \mathbb{R}^n$ are convex sets. It is required to find a solution $x(t)$, $t \in$ [0, 1] to the differential inclusion in Eq. (4.24) with boundary conditions $x(0) \in N_0$, $x(1) \in M_1$ minimizing Eq. (4.23).

Theorem 4.3. Let $g(\ ,t)$ and $\varphi_0(\cdot)$ be convex continuous functions. Moreover, let F be a closed, convex, and bounded set-valued mapping, and let N_0, M_1 be convex sets. Further assume that the mapping $t \to \operatorname{gph} F(\ ,t)$ is lower semicontinuous and $\tilde{x}(t) \in \operatorname{int} \operatorname{dom} F$ is an optimal solution to the problem in Eqs. (4.23) and (4.24). Then there exist a number $\lambda_0 \geq 0$, a vector x_e^*, and an adjoint trajectory (i.e., an absolutely continuous function $x^*(\cdot)$ not identically equal to zero) such that

(i) $-\dot{x}^*(t) \in F^*(x^*(t); (\tilde{x}(t), \dot{\tilde{x}}(t)), t) - \lambda_0 \partial g(\tilde{x}(t), t), \quad \text{a.e. } t \in [0,1],$
(ii) $\dot{\tilde{x}}(t) \in F(x^*(t); x^*(t), t) \quad \text{a.e. } [0,1],$
(iii) $x_e^* - x^*(1) \in \lambda_0 \partial \varphi_0(\tilde{x}(1)), \quad x_e^* \in K_{M_1}^*(\tilde{x}(1)), \quad -x^*(0) \in K_{N_0}^*(\tilde{x}(0)).$

In addition, if $\lambda_0 > 0$, then these conditions are sufficient for optimality.

□ This proof is analogous to the proof of the autonomous problem in Ref. [226] in the nonconvex case. To avoid long calculations we omit it. The difference is only that in the process of the proof, the condition of the lower semicontinuity of the mapping $t \to \operatorname{gph} F(\ ,t)$ provides the uniform boundedness of the LAM in the sense of the terminology of Chapter 2, which for autonomous problems is unnecessary. Our principal method is to formulate necessary conditions for the discrete-approximation problem, where

$$x_\delta(t+\delta) \in x_\delta(t) + \delta F(x_\delta(t), t), \quad t = 0, 2\delta, \ldots, (2^m - 1)\delta$$

and to prove uniform convergence $\tilde{x}_\delta(t) \to \tilde{x}(t), x_\delta^*(t) \to x^*(t)$ as $\delta \to 0$. ■

Corollary 4.3. If the assumptions of Theorem 4.3 are satisfied, then conditions (i) and (ii) can be rewritten as

$$-\dot{x}^*(t) \in \partial_x H(\tilde{x}(t), x^*(t), t) - \lambda_0 \partial g(\tilde{x}(t), t) \quad \text{a.e. } t \in [0,1],$$

$$\dot{\tilde{x}}(t) \in \partial_{y^*} H(\tilde{x}(t); x^*(t), t) \quad \text{a.e. } t \in [0,1]$$

□ The proof is analogous to the proof of Corollary 4.1. ■

Remark 4.5. On the whole, in Chapters 5 and 6, we use optimality conditions for convex differential inclusions. Nevertheless, under more general assumptions, necessary conditions can be formulated in terms of the LAM to F at a point $x \in \operatorname{dom} F$. On the optimization of different nonconvex differential inclusions, also consult Aubin and Frankowska [17,18,21], Aubin and Ekeland [19], Clarke [54–59], Mordukhovich [187–190,193–195,197,198], and Rockafellar [230–232] for related and additional material.

First, let us consider the optimization of polyhedral discrete inclusions

$$\begin{aligned} &\text{infimum} \sum_{t=0}^{T} g(x_t, t) \\ &\text{subject to } \begin{array}{c} x_{t+1} \in F_t(x_t), \quad t = 0, \ldots, T-1 \\ x_0 \in N_0, \quad x_T \in M_T, \end{array} \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} &g(x_t, t) = \max_{i \in I_t} \left\{ \langle x_t, b_i^t \rangle + \beta_i^t \right\}, \quad t = 0, \ldots, T, \\ &F(x) = \{ y : Ax - By \le d \}, \\ &N_0 = \{ x_0 : N x_0 \le p \}, \\ &M_T = \{ x_T : M x_T \le q \}. \end{aligned} \tag{4.26}$$

Here T is a natural number, I_t, $t = 0, 1, \ldots, T$ are finite index sets, β_i^t are real numbers, $b_i^t \in \mathbb{R}^n$, A, B are $m \times n$ matrices, N, M are rectangular matrices with n columns, and d, p, q are column vectors with corresponding dimensions. We must find an optimal trajectory $\{\tilde{x}_t\}_{t=0}^T$ of Eq. (4.25) that minimizes $\sum_{t=0}^{T} g(x_t, t)$. The problem in Eqs. (4.25) and (4.26) is labeled (PD) and called the polyhedral optimization for discrete inclusions.

Theorem 4.4. In order for $\{\tilde{x}_t\}_{t=0}^T$ to be an optimal trajectory of problem (PD), it is necessary and sufficient that there exist vectors x^*, $x_t^* t = 0, 1, \ldots, T$ satisfying

$$x_t^* = A^* \lambda_t + u_t^*, \quad u_t^* \in \partial g(\tilde{x}_t, t), \quad x_T^* + x^* \in \partial g(\tilde{x}_T, T). \tag{4.27}$$

$$x_{t+1}^* = B^* \lambda_t, \quad \lambda_t \geq 0, \quad \langle \lambda_t, A\tilde{x}_t - B\tilde{x}_{x+1} - d = 0 \rangle, \quad t = 0, \ldots, T-1$$

$$x_0^* = -N^* \gamma_0, \gamma_0 \geq 0, \quad \langle \gamma_0, N\tilde{x}_0 - p = 0 \rangle,$$

$$x^* = -M^* \gamma_T, \gamma_T \geq 0, \quad \langle \gamma_T, M\tilde{x}_T - q = 0 \rangle,$$

$$\begin{aligned} \partial g(\tilde{x}_t, t) = \operatorname{conv}\left\{ \bigcup_{i \in I_t(\tilde{x}_t)} b_i^t \right\}, \quad I_t(\tilde{x}_t) = \{ i \in I_t : g(\tilde{x}_t, t) \\ = \langle \tilde{x}_t, b_i^t \rangle + \beta_i^t \}, \quad t = 0, \ldots, T. \end{aligned} \tag{4.28}$$

□ Remember that by Eq. (2.51), the LAM for polyhedral mapping is

$$\begin{aligned} F^*(y^*, z) = \{ -A^* \lambda : y^* = -B^* \lambda, \quad \lambda \geq 0, \ \langle Ax - By - d, \lambda \rangle = 0 \}, \\ z = (x, y) \in \operatorname{gph} F \end{aligned} \tag{4.29}$$

Moreover, by Theorem 1.13 (Farkas) it is easy to conclude that if $\tilde{x}_0 \in N_0$, $\tilde{x}_T \in M_T$, then

$$\begin{aligned} K_{N_0}^*(\tilde{x}_0) = \{ x_0^* = -N^* \gamma_0, \langle \gamma_0, N\tilde{x}_0 - p \rangle = 0, \gamma_0 \geq 0 \} \\ K_{M_T}^*(\tilde{x}_T) = \{ x_T^* = -M^* \gamma_T, \langle \gamma_T, M\tilde{x}_T - q \rangle = 0, \gamma_T \geq 0 \}, \end{aligned} \tag{4.30}$$

where $K_{N_0}^*(\tilde{x}_0)$ and $K_{M_T}(\tilde{x}_T)$ are the cones of tangent directions at $\tilde{x}_0 \in N_0$, $\tilde{x}_T \in M_T$, respectively. Then taking into account Eqs. (4.29) and (4.30), by Theorem 4.2 and Remark 4.4, we have the relations in Eq. (4.27). It should be noted, however, that in the mentioned theorem, $\lambda = 1$. This occurs because in the proof of Theorem 4.2, we use Lemma 1.22 for the dual cone to the algebraic sum of polyhedral cones. Note that for the validity of Eq. (4.28), it is enough to apply Theorem 1.32. ■

Consider now the autonomous optimization problem in Eqs. (4.23) and (4.24):

$$\begin{aligned} &\text{infimum} && J(x(\cdot)) = \int_0^1 g(x(t),t)dt + \varphi_0(x(1)), \\ &\text{subject to} && \dot{x}(t) \in F(x(t)), \quad \text{a.e. } t \in [0,1], \\ &&& x(0) \in N_0, \quad x(1) \in M_1, \end{aligned} \tag{4.31}$$

generated by the polyhedral set-valued mapping

$$F(x) = \{y : Ax - By \le d\}, \tag{4.32}$$

where A, B, and d are matrices and vector column with dimensions as indicated above.

Theorem 4.5. Let F be a bounded set-valued polyhedral mapping and $\tilde{x}(t) \in \text{int dom}\, F$ be an optimal solution to the problem in Eqs. (4.31) and (4.32). Then there exist a number $\lambda_0 \ge 0$, a vector x_e^*, and an absolutely continuous function $x^*(\cdot)$, not all zero, such that

1. $x^*(1) + x_e^* \in \lambda_0 \partial \varphi_0(\tilde{x}(1)), \quad x_e^* = -M^* \gamma_1, \; \gamma_1 \ge 0, \quad x^*(0) = -N^* \gamma_0, \; \gamma_0 \ge 0,$
   $\langle \gamma_0, N\tilde{x}(0) - p \rangle = 0, \quad \langle \gamma_1, M\tilde{x}(1) - q \rangle = 0;$
2. $-\dot{x}^*(t) = A^* \lambda(t) + \lambda_0 u^*(t), \quad u^*(t) \in \partial g(\tilde{x}(t), t),$
   $x^*(t) = B^* \lambda(t), \quad \lambda(t) \ge 0, \langle \lambda(t), A\tilde{x}(t) - B\tilde{x}(t) - d \rangle = 0, \quad \text{a.e. } t \in [0,1].$

□ It remains only to take into account Eqs. (4.29) and (4.30) in conditions (i) to (iii) of Theorem 4.3. ■

Theorem 4.6. Let the hypotheses of Theorem 4.5 be fulfilled and $\lambda_0 > 0$. Then conditions 1 and 2 of Theorem 4.5 are sufficient for optimality of the trajectory $\tilde{x}(t), \; t \in [0,1]$.
□ Without loss of generality, we set $\lambda_0 = 1$. It follows from condition (2) of Theorem 4.5 that

$$\begin{aligned} g(x(t),t) - g(\tilde{x}(t),t) &\ge \langle x(t) - \tilde{x}(t), -B^* \dot{\lambda}(t) - A^* \lambda(t) \rangle, \\ g(x(t),t) - g(\tilde{x}(t),t) &\ge \langle -Bx(t), \dot{\lambda}(t) \rangle + \langle -B\tilde{x}(t), \dot{\lambda}(t) \rangle - \langle Ax(t), \lambda(t) \rangle \\ &\quad + \langle A\tilde{x}(t), \lambda(t) \rangle \end{aligned} \tag{4.33}$$

Moreover, for all feasible solutions $x(t)$, $t \in [0,1]$, we have $\langle \lambda(t), Ax(t) - Bx(t) - d \rangle \le 0$. Hence, from condition (2), we obtain

$$-\langle Ax(t), \lambda(t) \rangle + \langle A\tilde{x}(t), \lambda(t) \rangle \ge \langle -B\dot{x}(t) - d, \lambda(t) \rangle + \langle B\dot{\tilde{x}}(t) + d, \lambda(t) \rangle,$$

$$-\langle Ax(t), \lambda(t) \rangle + \langle A\tilde{x}(t), \lambda(t) \rangle \ge \langle \dot{\tilde{x}}(t), B^* \lambda(t) \rangle + \langle \dot{x}(t), B^* \lambda(t) \rangle.$$

Then it follows from Eq. (4.33) that

$$g(x(t),t)-g(\tilde{x}(t),t)\geq -\langle Bx(t),\dot{\lambda}(t)\rangle+\langle B\tilde{x}(t),\dot{\lambda}(t)\rangle \\ +\langle \dot{\tilde{x}}(t),B^*\lambda(t)\rangle-\langle \dot{x}(t),B^*\lambda(t)\rangle$$

or

$$\frac{d}{dt}\langle Bx(t),\lambda(t)\rangle\geq g(\tilde{x}(t),t)-g(x(t),t)+\frac{d}{dt}\langle B\tilde{x}(t),\lambda(t)\rangle.$$

Now, integrating this inequality over [0, 1], we obtain

$$\langle x(1)-\tilde{x}(1),B^*\lambda(1)\rangle\geq\langle x(0)-\tilde{x}(0),B^*\lambda(0)\rangle+\int_0^1[g(\tilde{x}(t),t)-g(x(t),t)]dt. \tag{4.34}$$

On the other hand, taking into account condition (1) of Theorem 4.5, we establish that

$$\varphi_0(x(1))-\varphi_0(\tilde{x}(1))\geq\langle\tilde{x}(1)-x(1),B^*\lambda(1)-M^*\gamma_1\rangle,$$

$$\langle x(1)-\tilde{x}(1),-M^*\gamma_1\rangle\geq 0,\quad \langle x(0)-\tilde{x}(0),B^*\lambda(0)\rangle.$$

Thus, from these inequalities and Eq. (4.34), we conclude that for all feasible solutions $x(t)$, $t\in[0, 1]$, we have

$$\int_0^1[g(x(t),t)-g(\tilde{x}(t),t)]dt+\varphi_0(x(1))-\varphi_0(\tilde{x}(1))\geq 0.$$

The proof is completed. ■

Remark 4.6. Theorem 4.6 is valid for the more general case where F and N_0, M_1 are arbitrary autonomous convex mapping and sets, respectively, satisfying the hypotheses (i)−(iii) of Theorem 4.3 for autonomous adjoint DFI. In Theorem 4.7, the existence of feasible solutions $\{x_t\}_{t=0}^T$ is proved.

Theorem 4.7. The sequence of points $\{x_t\}_{t=0}^T$ is a feasible solution of the problem in Eq. (4.25) if and only if there are no vectors $u_i\geq 0$, $\gamma_0\geq 0$, $\gamma_T\geq 0$ such that

$$\sum_{i=0}^{T-1}\langle u_i,d\rangle+\langle\gamma_0,p\rangle+\langle\gamma_T,q\rangle=-1, N^*\gamma_0+A^*u_0=0,$$

$$M^*\gamma_T-B^*u_{T-1}=0,\quad A^*u_i-B^*u_{i-1}=0,\ \ i=1,\ldots,T-1.$$

☐ Let us denote

$$Q = \begin{pmatrix} N & 0 & 0 & . & . & . & 0 & 0 \\ A & -B & 0 & . & . & . & 0 & 0 \\ 0 & A & -B & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & A & -B \\ 0 & 0 & 0 & . & . & . & 0 & M \end{pmatrix}, \quad b = \begin{pmatrix} p \\ d \\ . \\ . \\ . \\ d \\ q \end{pmatrix},$$

where 0 is the zero matrix partitioned into submatrices Q and the number of m-dimensional vector columns d contained in b is T. Let also denote $w = (x_0, \ldots, x_T) \in \mathbb{R}^{n(T+1)}$. Then it is not hard to see that the inequalities

$$Ax_t - Bx_{t+1} \leq d, \quad t = 0, \ldots, T-1, \;\; Nx_0 \leq p, \;\; Mx_T \leq q$$

can be rewritten as

$$Qw \leq b. \tag{4.35}$$

Thus, the existence problem of feasible solutions of Eq. (4.25) is reduced to the existence of solutions of the inequality in Eq. (4.35). Therefore, applying the solvability theorem for systems of linear inequalities, we obtain that Eq. (4.35) has a solution if and only if there is no $u \geq 0$ such that

$$Q^* u = 0, \quad \langle b, u \rangle = -1. \tag{4.36}$$

Now, taking into account the form of the matrix Q and vector column b, rewriting Eq. (4.36), we obtain the required result. ■

Furthermore, we consider F, a bounded polyhedral map. Let J be some subset of the index set $I = \{1, \ldots, m\}$. Then B_J is a matrix consisting of the rows B_i, $i \in J$. Next, A_J is defined by analogy. Similarly, if $d(x) = d - Ax$, then $d_i(x)$ is the ith component of the vector $d(x)$ and $d_J(x)$ is the vector with components $d_i(x)$, $i \in J$. It is known that a polytope $F(x)$ is a convex hull of its vertices (which are finite in number). Here a vertex $y \in F(x)$ is characterized by exactly n equalities

$$-B_i y = d_i(x),$$

for which $\{B_i\}$ is linearly independent. Now we take all possible collections J of n indexes for which B_j is nonsingular and construct a point

$$y^J(x) = -B_J^{-1} d_J(x). \tag{4.37}$$

Each point $y^J(x)$ is a vertex of $F(x)$ if $y^J(x) \in F(x)$.

Definition 4.1. A vertex $y^J(x) \in F(x)$ is said to be degenerate if

$$- B_J y^J(x) < d_J(x), \tag{4.38}$$

where $\overline{J} = I \backslash J$. Thus, at a nondegenerate vertex, the set of equalities of the system of inequalities $Ax - By \leq d$ is linearly independent, the number of which is exactly n. A set of points $y^J(x)$ satisfying Eq. (4.38) and denoted by D_J obviously is an open set and is called the nondegeneracy domain of the vertex $y^J(x)$. There is a cone

$$K_J = \{\overline{y} \in \mathbb{R}^n : \ -B_J\overline{y} \ \leq \ 0\} \tag{4.39}$$

connected with each nondegenerate vertex $y^J(x)$. It is clear that this is the cone of directions $\overline{y}$ for which $y^J(x) + t\overline{y} \in F(x)$ for sufficiently small $t > 0$. Indeed, let $\overline{y} \in K_J$. Since $y^J(x)$ is a vertex, then for a small $t > 0$ there exist the inequalities

$$- B_J(y^J(x) + t\overline{y}) \ \leq \ d_J(x),$$

$$- B_{\overline{J}}(y^J(x) + t\overline{y}) \ \leq \ d_J(x)$$

or what is the same

$$- B(y^J(x) + t\overline{y}) \ \leq \ d(x).$$

It follows from Eq. (4.38) that the cone K_J depends only on the index set J and does not depend on the point x of the nondegeneracy domain. By Theorem 1.13, the dual cone to Eq. (4.39) is

$$K_J^* = \{y^* = B_J^* u_J : u_J \geq 0\},$$

where, in the preceding notation, u_J is a vector with components u_i, $i \in J$. We shall call that vector column w_{Ji}, $i \in J$, the rib of the polytope $F(x)$ at the vertex $y^J(x)$, if

$$B_j w_{Ji} = \begin{cases} 0, & \text{if } j \neq i, j \in J, \\ 1, & \text{if } j = i, \end{cases}$$

where $B_j w_{Ji}$ is understood to be the product of the row vector and the column vector. If W_J is a matrix with the columns w_{Ji}, $i \in J$, then

$$B_J W_J = E,$$

where E is the $n \times n$ identity matrix. Obviously, $W_J = B_J^{-1}$.

Lemma 4.1. $K_J = \{\overline{y} = W_J v, v \geq 0\}$.

□ Indeed, if $\overline{y} \in K_J, B_J\overline{y} = v \geq 0$, then $\overline{y} = W_J v$. Conversely, if $\overline{y} = W_J v, v \geq 0$, then $B_J\overline{y} = v \geq 0$; i.e., $\overline{y} \in K_J$. ■

Let w_{Ji}, $i \in J$ be a rib. Consider the ray $y(t) = y^J(x) + tw_{Ji}$, $t > 0$. Then for $j \in J$, we can write

$$-B_j(y^J(x) + tw_{Ji}) = \begin{cases} d_j(x), & \text{if } j \neq i, \\ d_j(x) - t < d_i(x), & \text{if } j = i. \end{cases}$$

Therefore, at $y(t)$, $t \geq 0$, the inequality $-B_J y(t) \leq d_J(x)$ holds for arbitrary $t > 0$, and exactly one of the inequalities holds strictly. By virtue of nondegeneracy (Eq. (4.38)), for small $t > 0$ the remaining inequalities

$$-B_j y(t) = -B_j y^J(t) - tB_j w_{Ji} \leq d_j(x), \quad j \in \overline{J}$$

are satisfied. However, there exists an index $j \in \overline{J}$ such that $B_j w_{Ji} < 0$, because, on the contrary a point $y(t)$ belongs to $F(x)$ for every $t > 0$, that contradicts the boundedness of the polytope $F(x)$. Thus, there is a smallest number $t = t_0 > 0$ such that for some $j_0 \in \overline{J}$, $B_j w_{Ji} < 0$,

$$-B_{j_0} y(t_0) = d_{j_0}(x).$$

Moreover, since $B_j w_{Ji} = 0$, $j \in J \backslash \{i\}$ by construction and $B_j w_{Ji} < 0$, then $\{B_{j_0}; B_j j \in J \backslash \{i\}\}$ are linearly independent vectors. Thus, $y(t_0) \in F(x)$ satisfies exactly n linearly independent equalities

$$-B_j y(t_0) = d_j(x), j \in (J \backslash \{i\}) \cup \{j_0\}.$$

So $y(t_0)$ is the vertex corresponding to the index set $J_1 = (J \backslash \{i\}) \cup \{j_0\}$. Therefore, if $y^J(x)$ is a nondegenerate vertex and w_{Ji} is the rib connected with this vertex, then there is a vertex $y(t_0)$ such that

$$y(t_0) = y^J(x) + t_0 w_{Ji}, \quad t_0 > 0, \tag{4.40}$$

where $-B_i y(t_0) < d_i(x)$. This fact will be used in the next investigations.

Definition 4.2. A polytope $F(x)$ is called nondegenerate at a point x, if every vertex of $F(x)$ is nondegenerate at this point.

Lemma 4.2. Let $F(x)$ be a nondegenerate polytope on a connected set D. Then the number of vertices on D is constant, which are defined by the same index sets.

□ Suppose that $y^J(x_1)$, $x_1 \in D$ is a vertex of the polytope $F(x_1)$, but $y^J(x_2) \notin F(x_2)$, $x_2 \in D$. Then for some $j \in \overline{J}$

$$-B_j y^J(x_1) < d_j(x_1),$$

$$-B_j y^J(x_2) > d_j(x_2).$$

Hence by continuity, at some point of the continuous curve $x(\cdot)$ joining x_1 and x_2, we have

$$-B_j y^J(x(t)) = d_j(x(t)),$$

i.e., the vertex $y^J(x(t))$ is degenerate at point $x(t) \in D$, which contradicts that $F(x)$ is nondegenerate in D. ■

Suppose that D is a fixed connected set, where the polytope $F(x)$ is nondegenerate. According to what was mentioned above, its vertices in D are defined by fixed collections in the index set J, which can be enumerated $k = 1, \ldots, q$. Briefly, we set

$$y^k(x) \equiv y^J(x) = -B_J d_J(x).$$

Lemma 4.3. If $y^* \in \mathbb{R}^n$ and the minimum $\langle y, y^* \rangle$ is attained at the vertex $y^k(x) \in F(x)$, at some point $x \in D$, then $y^k(x)$ has the same property at every point of the nondegeneracy set D.

□ Indeed it is not hard to see that if $y^k(x)$ is a solution of problem

$$\text{infimum } \langle y, y^* \rangle \text{ subject to } y \in F(x),$$

then $y^* \in K^*_{J_k}$. In turn, the cone $K^*_{J_k}$ depends only on J_k but not on x. This ends the proof of the lemma. ■

As before, let D be a connected nondegeneracy domain of a polytope $F(x)$ with vertices $y^k(x)$, $k = 1, \ldots, q$. Let $I_1(y^*)$ be the set of vertices where the minimum of $\langle y, y^* \rangle$ relative to $F(x)$ is attained. By Lemma 4.3, this minimum does not depend on $x \in D$. Observe that $I_1(\cdot)$ is upper semicontinuous; i.e., $I_1(y^*) \subseteq I_1(y_0^*)$ when y^* belongs to some small neighborhood of y_0^*. Setting

$$J_1(y^*) = \bigcap_{k \in I_1(y^*)} J_k,$$

consider a point of the form

$$y = \sum_{k \in I_1(y^*)} \gamma_k y^k(x), \quad \gamma_k \geq 0, \quad \sum_{k \in I_1(y^*)} \gamma_k = 1. \tag{4.41}$$

The minimum of $\langle y, y^* \rangle$ relative to $F(x)$ at this point also is attained. And, as easily follows from the nondegeneracy condition, at the point y, the equalities

$$-B_i y = d_i(x), \quad i \in J_1(y^*)$$

hold and the remaining inequalities

$$-B_j y \le d_j(x), \quad j \in J_1(y^*)$$

are satisfied strictly. Applying the necessary and sufficient conditions for a minimum at y and $y^k(x)$, we obtain that there exist vectors $u_{J_1(y*)}$ and $u_J^k, k \in I_1(y^*)$ such that

$$\begin{aligned} y^* &= B^*_{J_1(y*)} u_{J_1(y*)}, \quad & u_{J_1(y*)} &\ge 0, \\ y^* &= B^*_{J_k} u^k_{J_k}, & u^k_{J_k} &\ge 0. \end{aligned} \tag{4.42}$$

If now $i \in J_1(y^*)$, then by scalar multiplying these relations with $w_{ki}(w_{ki} \equiv w_{J_k i})$, we have

$$u_i = \langle w_{ki}, y^* \rangle, \quad u_i^k = \langle w_{ki}, y^* \rangle,$$

where u_i and u_i^k are the ith components of the vectors $u_{J_1(y*)}$ and $u^k_{J_k}$, $k \in I_1(y^*)$, respectively. Thus,

$$u_i = u_i^k, \quad i \in J_1(y^*), \ \ k \in I_1(y^*).$$

Now subtracting the equalities in Eq. (4.42), we obtain that

$$B^*_{J_k \setminus J_1(y^*)} u^k_{J_k \setminus J_1(y^*)} = 0.$$

By virtue of the linear independence of the set of vectors B_i, $i \in J_k$, it follows that $u_i^k = 0$, $i \notin J_1(y^*)$. Moreover, as we will show, $u_i > 0$, $i \in J_1(y^*)$. Suppose that this is not so and that $u_{i_1} = \langle w_{ki_1}, y^* \rangle = 0$ for some $i_1 \in J_1(y^*)$. Then according to what was stated above, there is a vertex

$$y^0 = y^k(x) + t w_{ki_1}, \quad k \in I_1(y^*), \ \ t > 0$$

such that $-B_{i_1} y^0 < d_{i_1}(x)$; i.e., i_1 is not contained in its index set. On the other hand,

$$\langle y^0, y^* \rangle = \langle y^k(x), y^* \rangle + t \langle w_{ki_1}, y^* \rangle = \langle y^k(x), y^* \rangle,$$

i.e., the minimum of $\langle y, y^* \rangle$ is attained also at y^0, too. Since i_1 does not belongs to the index set defining this vertex, then $i_1 \notin J_1(y^*)$. This occurs because $J_1(y^*)$ is the intersection of such sets. This contradicts the choice of i_1. Thus, we have proved the following result.

Lemma 4.4. All the vectors

$$u^k_{J_k} = B^{*\,-1}_{J_k} y^*, \quad k \in I_1(y^*)$$

are equal and

$$u_i > 0, \ \ i \in J_1(y^*); \quad u_i = 0, \ \ i \notin J_1(y^*).$$

Corollary 4.4. All vectors of the form $A^*_{J_k} B^{*\,-1}_{J_k} y^*, \ \ k \in I_1(y^*)$ are equal.
☐ Indeed, it follows from Lemma 4.4 that

$$A^*_{J_k} B^{*\,-1}_{J_k} y^* = A^*_{J_k} u^k_{J_k} = \sum_{i \in J_k} A^*_i u_i = \sum_{i \in J_1(y^*)} A^*_i u_i,$$

where A_i is a row vector of A and so A^*_i is a column vector. Moreover, it is taken into account that u^k_i does not depend on k and that $u_i = 0$ for every $i \notin J_1(y^*)$. ■

Lemma 4.4 and its corollary implies that by the formula

$$f(y^*) = -A^*_{J_k} B^{-1*}_{J_k} y^*, \quad k \in I_1(y^*) \tag{4.43}$$

the vector $f(y^*)$ is defined uniquely by the vector y^*.

Note that on the one hand, the set of points where $\langle y, y^* \rangle$ attains its minimum relative to $F(x)$ coincides with a set of the form in Eq. (4.41), and on the other hand this set is the set of points satisfying the following linear equalities and inequalities:

$$-B^*_{J_1(y^*)} y = d_{J_1(y^*)}(x),$$

$$-B_i y \le d_i(x), \quad i \notin J_1(y^*). \tag{4.44}$$

Therefore, setting $y^*_1 = f(y^*)$, we deduce that minimizing $\langle y, y^*_1 \rangle$ over the set in Eq. (4.41) is the same as over that in Eq. (4.44).

From Eq. (4.41), it follows that this minimum is attained at the set of vertices $I_2(y^*) \subseteq I_1(y^*)$ and that all points of the set of solutions are convex combinations of these vertices. Now we set

$$J_2(y^*) = \bigcap_{k \in I_2(y^*)} J_k.$$

Clearly, $J_2(y^*) \supseteq J_1(y^*)$. If on the basis of Eq. (4.44) we analyze the necessary condition of an extremum as above, we can conclude that all the vectors

$$u^k_{J_k} = B^{-1*}_{J_k} y^*_1, \quad k \in I_2(y^*)$$

are independent of k in the sense that $u^k_i, \ \ i \in J_2(y^*)$ are equal for every $k \in J_2(y^*)$ and $u^k_i = 0, \ \ \forall i \in J_2(y^*)$. Furthermore,

$$u_i^k > 0, \quad \forall i \in Q_1(y^*),$$

where

$$Q_1(y^*) = J_2(y^*) \backslash J_1(y^*).$$

Hence, the vectors

$$y_2^* = -A_{J_k}^* B_{J_k}^{-1*} y_1^*, \quad k \in I_2(y^*) \tag{4.45}$$

do not depend on k. With regard to $I_2(y^*) \subseteq I_1(y^*)$, comparing Eqs. (4.43) with (4.45), we derive that

$$y_2^* = (-A_{J_k}^* B_{J_k}^{-1^*})^2 y^*. \tag{4.46}$$

Now it is understood that the construction process of the vectors y_1^*, y_2^* can be continued. Namely, if a vector y_l^* already is constructed, then $I_{l+1}(y^*)$ is the set of vertices $k \in I_l(y^*)$ where $\langle y, y_1^* \rangle$ attains its minimum over the set of points of the form

$$y = \sum_{k \in I(y^*)} \gamma_k y^k(x), \quad \gamma_k \geq 0, \qquad \sum_{k \in I(y^*)} \gamma_k = 1,$$

$$I_{l+1}(y^*) \subseteq I_l(y^*),$$

$$J_{l+1}(y^*) = \bigcap_{k \in I_{l+1}(y^*)} J_k \supseteq J_1(y^*).$$

In addition, minimization of $\langle y, y_1^* \rangle$ over this set is equivalent to its minimization over the set of solutions of the linear equalities and inequalities

$$-B_{J_l(y^*)}^* y = d_{J_l(y^*)}(x),$$

$$-B_i y \leq d_i(x), \quad i \notin J_l(y^*),$$

whence it follows that the vectors $B_{J_k}^{-1*} y_l^*$ do not depend on k, $k \in I_{l+1}(y^*)$; i.e.,

$$(B_{J_k}^{-1*} y_l^*)_i > 0, \quad i \in Q_l(y^*) = J_{l+1}(y^*) \backslash J_l(y^*),$$

$$(B_{J_k}^{-1*} y_l^*)_i = 0, \quad i \in J_{l+1}(y^*) \tag{4.47}$$

and

$$y_{l+1}^* = -A_{J_k}^* B_{J_k}^{-1*} y_1^* = (-A_{J_k}^* B_{J_k}^{-1*})^{l+1} y^*, \quad k \in I_{l+1}(y^*).$$

Since $I_{l+1}(y^*) \subseteq I_l(y^*)$, it is obvious that beginning with some number l, all the sets coincide. We can formulate the obtained result as follows.

Lemma 4.5. Let a connected set D be the nondegeneracy domain of a polytope $F(x)$. Then there exists an index k such that

$$y_l^* = (-A_{J_k}^* B_{J_k}^{-1*})^l y^*; \quad k \in J_l(y^*), \ \ l = 1, 2, \ldots$$

Let $Y_0(x, y^*)$ be the set of points of $\langle y, y^* \rangle$ over $F(x)$ and, analogously, $Y_l(x, y^*)$ the set of points of $\langle y, y_l^* \rangle$ over $Y_{l-1}(x, y^*)$,

$$y^k(x) \in Y_l(x, y^*), \ \ l = 0, 1, \ldots$$

Moreover, the components of the vectors $B_{J_k}^{-1^*} y_l^*$corresponding to $i \in Q_l(y^*)$ is strictly positive.

Theorem 4.8 is an important case of Theorem 2.16 for a nondegeneracy domain.

Theorem 4.8. Let D be the nondegeneracy domain of a polytope $F(x)$ and $z = (x, y) \in \text{gph}\, F$. Then for the LAM, we have

$$F^*(y^*; (x, y)) = \begin{cases} -f(y^*), & \text{if } y \in F(x, y^*), \\ \varnothing, & \text{if } y \notin F(x, y^*). \end{cases}$$

☐ Let x be a nondegeneracy point of $F(x)$. Then the minimum of $\langle y, y^* \rangle$ over $F(x)$ is attained at one of the vertices y^k of $F(x)$, defined by an index set J_k:

$$\begin{aligned} &A_{J_k} x - B_{J_k} y^k = d_{J_k}, \\ &A_{\overline{J}_k} x - B_{\overline{J}_k} y^k < d_{\overline{J}_k}, \end{aligned}$$

and in this case the matrix B_{J_k} is invertible. On the other hand, since

$$K_{\text{gph}\ F}(z) = \{(\overline{x}, \overline{y}) : A_{J(z)}\overline{x} - B_{J(z)}\overline{y} \ \leq \ 0\}$$

(see Example 2.8), then

$$K_{\text{gph}\ F}^*(z) = \{(x^*, y^*) : x^* = -A_{J(z)}^* \lambda_{J(z)}, y^* = B_{J(z)}^* \lambda_{J(z)}, \lambda_{J(z)} \geq 0\}.$$

By applying the latter formula at the point (x, y^k), we deduce that $y^* = B_{J_k}^* \lambda_{J_k}, \ \ \lambda_{J_k} \geq 0$ and hence by Lemma 4.4 $\lambda_{\overline{J}_k} = 0$. Therefore,

$$x^* = -A_{J(z)}^* \lambda_{J(z)} = -A_{J(z)}^* B_{J(z)}^{*} y^*;$$

i.e., x^* is defined uniquely and the formula for it coincides with the formula in Eq. (4.43). ■

Remark 4.7. Theorem 4.8 says that in fact the LAM in the nondegeneracy domain does not depend on x, by virtue of the importance of Theorem 4.8, it is interesting that another useful alternative proof can also be deduced by applying Theorem 2.1. Indeed, if the minimum $\langle y, y^* \rangle$ over $F(x)$ is attained at some vertex

$$y^k = B_{J_k}^{-1} A_{J_k} x + B_{J_k}^{-1} d_{J_k}, \quad y^k \in F(x; y^*), \;\; k \in I_1(y^*),$$

then clearly

$$H(x, y^*) = \langle y^k, y^* \rangle = \langle x, A_{J_k}^* B_{J_k}^{*\,-1} y^* \rangle + \langle B_{J_k}^{-1} d_{J_k}, y^* \rangle$$

and so

$$\partial_x H(x, y^*) = A_{J_k}^* B_{J_k}^{*\,-1} y^*, \;\; y^k \in F(x; y^*),$$

where the same formula is true for all $k \in I_1(y^*)$.

Theorem 4.9. If an optimal trajectory $\tilde{x}(t), \;\; t \in [t_1, t_2]$ entirely lies in the nondegeneracy domain D of a polytope $F(x)$, then condition (2) of Theorem 4.5 can be rewritten as follows:

$$\begin{aligned} &\dot{x}^*(t) = f(x^*(t)) + \lambda_0 u^*(t), \quad u^*(t) \in \partial g(\tilde{x}(t), t), \\ &\langle \dot{\tilde{x}}(t), x^*(t) \rangle = \min_y \{ \langle y, x^*(t) \rangle : y \in F(\tilde{x}(t)) \}. \end{aligned} \tag{4.48}$$

□ Actually, condition (2) of Theorem 4.5 expresses the fact that

$$-\dot{x}^*(t) \in F^*(x^*(t); (\tilde{x}(t), \dot{\tilde{x}}(t)), t) - \lambda_0 u^*(t), \quad u^*(t) \in \partial g(\tilde{x}(t), t) \quad \text{a.e } \; t \in [0, 1].$$

Hence, Eq. (4.48) is an immediate consequence of Theorem 4.8. ■

4.4 Polyhedral Adjoint Differential Inclusions and the Finiteness of Switching Numbers

In this section, the basic results for polyhedral optimization in the nondegenerate case will be given. The main result is Theorem 4.12, where we assume satisfaction of the so-called condition for generality of position. The obtained result can be called the theorem of the number of switchings. Note that we shall describe a point of discontinuity of the optimal trajectory as a switching. The proof is based on the lemmas of Section 4.3.

Let D be the nondegeneracy domain of a polytope $F(x)$. In Section 4.3 it was proved that for all $x \in D$, the formula

$$f(y^*) = -A^*_{J_k} B^{-1*}_{J_k} y^*, \quad k \in I_1(y^*)$$

uniquely defines the vector $f(y^*)$ for a given y^*. It is not hard to conclude that $f(\cdot)$ satisfies a Lipschitzian condition with a constant depending only on D. Here, intuitively, it is natural that the constant should be $L = \max_{1 \le k \le q} \left\| A^*_{J_k} B^{*-1}_{J_k} \right\|$. As a preliminary for this intuitive conviction, we prove Lemma 4.6.

Lemma 4.6. Let $K^*_i \equiv K^*_{J_i}$ be the dual cone to the cone K_i at a vertex $y^i(x) \equiv y^{J_i}(x)$ of a polytope $F(x)$. Then

a. $\operatorname{int} K^*_i \cap \operatorname{int} K^*_j = \varnothing, \;\; i \ne j; \;\; i,j = 1,2,\ldots,q,$
b. $\cup_i K^*_i = \mathbb{R}^n$.

☐ Let us prove a. Suppose that this is not so; i.e., $\operatorname{int} K^*_i \cap \operatorname{int} K^*_j \ne \varnothing, \;\; i \ne j$. Taking $y^*_0 \in \operatorname{int} K^*_i \cap \operatorname{int} K^*_j$, by Lemma 1.18 we derive that

$$\begin{aligned} &\left\langle y^*_0, \bar{y}_1 \right\rangle > 0, \quad \bar{y}_1 \in K_i, \bar{y}_1 \ne 0, \\ &\left\langle y^*_0, \bar{y}_2 \right\rangle > 0, \quad \bar{y}_1 \in K_i, \bar{y}_2 \ne 0. \end{aligned}$$

Then setting $\bar{y}_1 = y^j(x) - y^i(x)$ and $\bar{y}_2 = y^i(x) - y^j(x)$,

$$\begin{aligned} &\left\langle y^*_0, y^j(x) - y^i(x) \right\rangle > 0, \\ &\left\langle y^*_0, y^i(x) - y^j(x) \right\rangle > 0. \end{aligned}$$

From these inequalities, it follows that $\langle y^*_0, 0 \rangle > 0$; i.e., $0 > 0$. This contradiction gives us a. For b it is enough to show that if $y^* \in \mathbb{R}^n$, then $y^* \in \cap_i K^*_i$; i.e., $y^* \in K^*_i$ for some i. Actually, at a vertex $y^i(x)$, the minimum of $\langle y, y^* \rangle$ is attained if and only if $y^* \in K^*_i$. ■

Let now $\bar{y}^* \in \mathbb{R}^n, \;\; \bar{\bar{y}}^* \in \mathbb{R}^n$ be arbitrary points. It is clear that if $\bar{y}^*, \bar{\bar{y}}^*$ belong to one and only one cone $K^*_k \equiv K^*_{J_k}$, then $f(\cdot)$ is Lipschitzian with a Lipschitz constant L. Hence, assume that $\bar{y}^* \in K^*_{i_0}, \;\; \bar{\bar{y}}^* \in K^*_{j0}$ for some i_0, j_0 ($i_0 \ne j_0$). We join the points $\bar{y}^*, \;\; \bar{\bar{y}}^*$ with a line segment and the points of intersections with the boundaries of cones having common faces (Lemma 4.6) and we denote by $y^*_1, y^*_2, \ldots, y^*_{s-1}$ (s is some natural number) so that $y^0_a \ne \bar{y}^*, \;\; y^0_a \ne \bar{\bar{y}}^*, \;\; \alpha = 1, \ldots, s-1$. Enumerating these cones we can write

$$\bar{y}^* = y^*_0 \in K^*_1 \equiv K^*_{i_0}, \;\; y^*_1 \in K^*_2, \ldots, y^*_{s-1} \in K^*_s, \;\; \bar{\bar{y}}^* = y^*_s \in K^*_s \equiv K^*_{j_0}. \tag{4.49}$$

By Lemma 4.4, the vectors

$$f(y^*_\alpha) = -\, C^*_k y^*_\alpha, \;\; k \in I_1(y^*_\alpha); \quad C^*_k = A^*_{J_k} B^{*\,-1}_{J_k}, \quad \alpha = 0, 1, \ldots, s$$

coincide. Then by Eq. (4.49), $C_k^* y_k^* = C_{k+1}^* y_{k+1}^*$, $k = 1, \ldots, s-1$. By using these relations we observe that

$$f(\overline{y}^*) - f(\overline{\overline{y}}^*) = \sum_{\alpha=1}^{s} C_\alpha^* (y_{\alpha-1}^* - y_\alpha^*).$$

Now, since $\|\overline{y}^* - \overline{\overline{y}}^*\| = \sum_{\alpha=1}^{s} \|y_{\alpha-1}^* - y_\alpha^*\|$, we have

$$\|f(\overline{y}^*) - f(\overline{\overline{y}}^*)\| \leq \sum_{\alpha=1}^{s} \|C_\alpha^*\| \|y_{\alpha-1}^* - y_\alpha^*\| \leq L\|\overline{y}^* - \overline{\overline{y}}^*\|.$$

Let us return to the polyhedral Mayer problem (i.e., the optimization problem in Eqs. (4.31) and (4.32), where $g \equiv 0$ and N_0, M_1 are arbitrary convex sets:

$$\begin{array}{ll} \text{minimize} & \varphi_0(x(1)), \\ \text{subject to} & \dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [0,1], \\ & x(0) \in N_0, \ x(1) \in M_1 \end{array} \tag{4.50}$$

$$F(x) = \{y : Ax - By \leq d\}.$$

Then, obviously, the adjoint differential inclusion in Eq. (4.48) of Theorem 4.9 has the form

$$\frac{d}{dt} y^*(t) = f(y^*(t)). \tag{4.51}$$

We shall investigate this differential equation for small $t \geq 0$. At once we note that since $f(\cdot)$ is Lipschitzian, then it has a unique solution for every initial condition $y^*(0) = y^*$. By Lemma 4.5, we choose an index k, corresponding to y^* and consider the equation

$$\frac{d}{dt} z^*(t) = -C_k^* z^*(t), \ z^*(0) = y^*, \ C_k^* = A_{J_k}^* B_{J_k}^{-1*}.$$

As a linear equation, its solution has the form

$$z^*(t) = e^{-C_k^* t} y^* = \left(\sum_{l=0}^{\infty} \frac{(-C_k^*)^l}{l!} t^l \right) y^*. \tag{4.52}$$

By Cayley's theorem [174] (α_j are are real numbers),

$$(-C_k^*)^n = \sum_{j=0}^{n-1} \alpha_j (-C_k^*)^j.$$

The formula in Eq. (4.52) can be rewritten in the form

$$z^*(t) = \sum_{l=0}^{n-1} \frac{1}{l!}(t^l + \varphi_l(t))(-C_k^*)^l y^* = \sum_{l=0}^{n-1} \frac{1}{l!}(t^l + \varphi_l(t)) y_l^*, \tag{4.53}$$

where $\varphi_l(\cdot)$ are analytic functions and $\varphi_l(t)t^{-1} \to 0$ whenever $t \downarrow 0$ and $y_0^* = y^*$.

Let us now set $Q_0(y^*) = J_1(y^*)$. If we remember from Eq. (4.47) that $Q_l(y^*) = J_{l+1}(y^*) \backslash J_l(y^*)$, then we get

$$J_{l+1}(y^*) = Q_0(y^*) \cup Q_1(y^*) \cup \cdots \cup Q_l(y^*).$$

Moreover, according to Eq. (4.49), the vectors $u_{J_k}^l = B_{J_k}^{-1*} y_l^*$ have strictly positive components u_i^l if $i \in Q_l(y^*)$ and $u_i^l = 0$, if $i \notin J_{l+1}(y^*)$. By construction, $J_{l+1} \subseteq J_k$. Furthermore,

$$\begin{aligned} y_l^* &= B_{J_k}^* u_{J_k}^l = B_{Q_l(y^*)}^* u_{Q_l(y^*)}^l + B_{J_l(y^*)}^* u_{J_l(y^*)}^l \\ &= B_{Q_l(y^*)}^* u_{Q_l(y^*)}^l + \sum_{i=0}^{l-1} B_{Q_i(y^*)}^* u_{Q_i(y*)}^l, \quad u_{Q_l(y^*)}^l > 0. \end{aligned}$$

Thus,

$$\begin{aligned} z^*(t) &= \sum_{l=0}^{n-1} \frac{1}{l!}(t^l + \varphi_l(t)) \left[B_{Q_l(y^*)}^* u_{Q_l(y^*)}^l + \sum_{i=0}^{l-1} B_{Q_i(y*)}^* u_{Q_i(y^*)}^l \right] \\ &= \sum_{i=0}^{n-1} B_{Q_i(y^*)}^* \left[\frac{1}{i!}(t^i + \varphi_i(t)) u_{Q_i(y^*)}^i + \sum_{l=i+1}^{n-1} \frac{1}{l!}(t^l + \varphi_l(t)) u_{Q_i(y^*)}^l \right]. \end{aligned} \tag{4.54}$$

Let $u_{Qi(y^*)}(t)$ denote the expression in braces in Eq. (4.54). Then we get

$$z^*(t) = \sum_{i=0}^{n-1} B_{Q_i(y^*)}^* u_{Q_i(y^*)}(t) = B_{J_n(y^*)}^* u_{J_n(y^*)}(t). \tag{4.55}$$

Thus, according to Eq. (4.55), we conclude that $u_{J_n(y^*)}(t) > 0$ for small t. Therefore, since $J_n \subseteq J_k$, it follows from Eq. (4.55) that $z^*(t) \in K_{J_k}^*$ for small t (see Eq. (4.39), determining K_J and the dual cone K_J^* in the previous section). Then by virtue of the necessary and sufficient conditions, it can be concluded that $y^k(x)$ minimizes $\langle y, z^*(t) \rangle$ over $F(x)$ for small t; i.e., $k \in I_1[z^*(t)]$ for small t.

Now we can formulate an important property of the differential inclusion (equation) in Eq. (4.51).

Lemma 4.7. There is a vertex defined by the index k, such that the solution of the differential equation in Eq. (4.51) with the initial condition $y^*(0)=y^*$ and the solution of

$$\frac{d}{dt}y^*(t) = -C_k^* z^*(t), \quad y^*(0)=y^*, \quad k \in I_1(y^*) \tag{4.56}$$

coincide for sufficiently small t.

☐ In fact, the solution of the differential equation in Eq. (4.56) as denoted above by $z^*(\cdot)$ is such that $k \in I_1[z^*(t)]$. Therefore, by definition of Eq. (4.43) of $f(y^*)$,

$$f(z^*(t)) = -C_k^* z^*(t);$$

i.e.,

$$\frac{d}{dt}z^*(t) = f(z^*(t)).$$

Thus, $z^*(\cdot)$ is the solution of the differential equation in Eq. (4.51). Since the solution of this equation is unique, it follows that $y^*(t)=z^*(t)$ as was to be proved. ■

Now, with the help of Lemma 4.7, we prove the piecewise linearity of the differential equation contained in Eq. (4.48), where $u^*(t) \equiv 0$ (remember that $g \equiv 0$ by hypothesis).

Theorem 4.10. Let $\tilde{x}(t)$, $t \in [t_1, t_2]$ be an optimal trajectory lying entirely in the nondegeneracy domain D of a polytope $F(x)$. Then the differential inclusion in Eq. (4.48) $[u^*(t) \equiv 0]$ has a unique solution on the interval $[t_1, t_2]$. Furthermore, the interval $[t_1, t_2]$ can be divided into a finite number of half intervals of the form $[\tau_j, \tau_{j+1})$, $j=0,1,\ldots M-1$, $\tau_0 = t_1$, $\tau_j < \tau_{j+1}$, $\tau_M = t_2$ such that in each of them, $x^*(t)$ is a solution of the differential equation

$$-\dot{x}^*(t) = C_{k(j)}^* x^*(t), \quad C_{k(j)}^* = A_{J_{k(j)}}^* B_{J_{k(j)}}^{*\,-1}, \tag{4.57}$$

where $k(j)$ are vertices of the polytope $F(\tilde{x}(t))$ on which are attained the minimum of $\langle y, x^*(t)\rangle$ over $y \in F(\tilde{x}(t))$.

☐ The uniqueness assertion was proved above. We prove the second assertion. By Lemma 4.7, each $\tau \in [t_1, t_2]$ corresponds to some half interval $[\tau, \tau+\Delta)$, $\Delta > 0$ such that the solutions of the equations

$$\dot{x}^*(t) = f(x^*(t)) \tag{4.58}$$

and

$$\dot{x}^*(t) = -C_k^* x^*(t), \quad k \in I_1(x^*(\tau)) \tag{4.59}$$

coincide. By analogy it can be shown that there exists $\Delta_1 > 0$ so that the differential equations in Eqs. (4.58) and (4.59) (maybe for another k) on the half interval of the form $(\tau - \Delta_1, \tau]$ have the same solutions. Thus, every point $\tau \in [t_1, t_2]$ corresponds to an interval $(\tau - \Delta_1, \tau + \Delta)$ and in each half interval of this interval, Eq. (4.58) is equivalent to Eq. (4.59) for some k. Thus, taking all $\tau \in [t_1, t_2]$ from the collection of corresponding intervals, we can choose a finite covering of $[t_1, t_2]$. Denoting now the end points and midpoints of the intervals contained in this covering by τ_j and enumerating them in ascending order, we find that in each interval (τ_j, τ_{j+1}), Eq. (4.58) is equivalent to Eq. (4.59) for some k. Since it is clear that Eq. (4.59) is satisfied also at a point $\tau = \tau_j$, the desired result is proved completely. ■

Let us return to the problem in Eq. (4.50), where $N_0 = \mathbb{R}^n$ and φ_0 is a smooth convex function.

Theorem 4.11. Let φ_0 be a smooth convex function and $N_0 = \mathbb{R}^n$. Then if $\tilde{x}(t)$, $t \in [0,1]$ is an optimal trajectory lying entirely in the nondegeneracy domain D of a polytope $F(x)$ and satisfying $\tilde{x}(t) \in \operatorname{int} M_1$, then the conditions

$$\begin{array}{ll} 1. & x^*(1) + x_e^* = \lambda_0 \varphi_0'(\tilde{x}(1)), \quad x_e^* \in K_{M_1}^*(\tilde{x}(1)), \\ 2. & -\dot{x}^*(t) = A^*\lambda(t), \quad x^*(t) = B^*\lambda(t), \quad \lambda(t) \geq 0, \\ & \langle \lambda(t), A\tilde{x}(t) - B\tilde{x}(t) - d \rangle = 0, \quad \text{a.e. } t \in [0,1] \end{array} \tag{4.60}$$

$$x^*(0) \in K_{N_0}^*(\tilde{x}(0))$$

are sufficient for the optimality of $\tilde{x}(t)$. Here, the real number $\lambda_0 \geq 0$, vector x_e^*, and function $x^*(\cdot)$ are not all zero.

☐ It follows from the hypotheses that $K_{M_1}(\tilde{x}(1)) = \mathbb{R}^n$ and so $K_{M_1}^*(\tilde{x}(1)) = \{0\}$. Then by hypothesis 1 in Eq. (4.60), $x_e^* = 0$. Consequently, we find that $\lambda_0 \neq 0$. Actually, if we suppose that this is not so (i.e., $\lambda_0 = 0$), then from the first hypothesis in Eq. (4.60), we derive that $x^*(1) = 0$. Thus, the piecewise linearity of Eq. (4.58) (see also Eq. (4.57)) implies that $x^*(t) \equiv 0$. This contradicts the fact that the real number $\lambda_0 \geq 0$, vector x_e^*, and function $x^*(\cdot)$ are not all zero. Therefore, we have $\lambda_0 > 0$. Finally, taking into account Theorem 4.6 and Remark 4.6, we get the desired result. ■

Now, with the use of Theorem 4.10 we prove the main result, which can be called the theorem on the finiteness of the number of switchings. First, we assume satisfaction of the following condition for generality of position: let the set of vectors

$$w_{ki}, C_k w_{ki}, \ldots, C_k^{n-1} w_{ki} \tag{4.61}$$

be linearly independent for each vertex of number k and rib w_{ki}, $i \in J_k$.

Theorem 4.12. Let $\tilde{x}(t)$, $t \in [0,1]$ be an optimal trajectory lying entirely in the nondegeneracy domain D of a polytope $F(x)$. Furthermore, suppose that $x^*(t) \not\equiv 0$

and the condition for generality of position is satisfied. Then the interval $[t_1, t_2]$ can be divided into a finite number of half intervals $[\tau_j, \tau_{j+1})$, $j = 0, 1, \ldots, M-1$ such that in each of them an optimal trajectory $\tilde{x}(t)$ and an adjoint trajectory $x^*(t)$ are solutions of the differential equations

$$\dot{\tilde{x}}(t) = -C_k\tilde{x}(t) - B_{J_k}^{-1}d_{J_k}, \quad C_k = B_{J_k}^{-1}A_{J_k}, \qquad \dot{x}^*(t) = -C_k^* x^*(t),$$

respectively.

□ According to Theorem 4.11, the adjoint equation for $x^*(t)$ on half interval $[\tau_j, \tau_{j+1})$ is described in Eq. (4.59). In turn, by Theorem 4.49, $\dot{\tilde{x}}(t)$ almost everywhere minimizes $\langle y, x^*(t)\rangle$ over $F(\tilde{x}(t))$. In particular, at the vertex $y^k(x)$, the minimum of $\langle y, x^*(t)\rangle$ relative to $F(\tilde{x}(t))$ is attained, where

$$y^k(x) = B_{J_k}^{-1}(A_{j_k}x - d_{j_k}) = C_k x - B_{j_k}^{-1}d_{j_k}.$$

We shall show that this minimum cannot be attained at another vertex by infinitely many points t (see also Refs. [140,223]). On the contrary, if in addition to $y^k(\tilde{x}(t))$ there is also a vertex $y^i(\tilde{x}(t))$ at which the indicated minimum is attained on infinitely many points t, then

$$\left\langle w_{ki}, x^*(t)\right\rangle = 0 \tag{4.62}$$

on the infinitely many points t, too. Since the function $x^*(\cdot)$ is the solution of Eq. (4.59) with constant coefficients, it is analytic; i.e., function $\langle w_{ki}, x^*(t)\rangle = 0$ is also analytic. This analytic function of the variable t by Eq. (4.62) vanishes for an infinite set of values t, so that we have throughout the interval $[\tau_j, \tau_{j+1})$ the validity of Eq. (4.62). On differentiating the relationship successively with respect to t, we get

$$\left\langle C_k^l w_{ki}, x^*(t)\right\rangle = 0, \quad l = 0, 1, \ldots, n-1. \tag{4.63}$$

Since by the hypothesis of generality of position, the vectors in Eq. (4.61) form a basis of the space $\mathbb{R}^n$, Eqs. (4.62) and (4.63) imply that given any t, $t \in [\tau_j, \tau_{j+1}]$ the vector $x^*(t)$ is orthogonal to all the vectors of a basis [i.e., $x^*(t) \equiv 0$], which contradicts the assumption of the nontriviality of the solution $x^*(t)$, $t \in [\tau_j, \tau_{j+1}]$. This completes the proof of the theorem. ■

Finally in this section, by applying the method described in Sections 4.3 and 4.4, we can investigate a Mayer optimization problem of optimal control on a linear manifold:

$$\begin{aligned}
&\text{infimum} \quad \varphi_0(x(1)) \\
&\dot{x} = Ax + Bu, \quad x(0) = x_0 \in L, \\
&x(t) \in L \subseteq \mathbb{R}^n \quad \forall t \in [t_0, t_1], \\
&u(t) \in U \subset \mathbb{R}^r \quad \forall t \in [0, 1] \\
&L = \{x : Cx - d = 0\}, \quad U = \{u : Wu \le p\}.
\end{aligned} \tag{4.64}$$

Here A, B, C, W are $n \times n$, $n \times r$, $k \times n$, $m \times n$ matrices, respectively; φ_0, φ are as before, continuous convex functions; and we shall regard as any permissible piecewise-continuous controls with values in the control domain U. For definiteness, at each point of discontinuity $u(\tau) = u(\tau - 0)$. Moreover, d and p are column vectors of corresponding dimensions. Assume that rank $C = k$, i.e., the dimension of the linear manifold L is dim $L = n - k$. Obviously, for the trajectory $x(t)$, $t \in [0, 1]$ associated with a permissible control $u(t) \in U$ to lie on a linear manifold L, we require that $C\dot{x}(t) = 0,\ t \in [0, 1]$; i.e.,

$$C(Ax(t) + Bu(t)) = 0, \quad u(t) \in U, \quad t \in [0, 1].$$

Then it is easy to see that the problem in Eq. (4.64) is equivalent to the following Mayer problem for polyhedral differential inclusion:

$$\begin{aligned}
&\text{infimum} \quad \varphi_0(x(1)) \\
&\text{subject to} \quad \varphi(x(1)) \leq 0, \\
&\dot{x}(t) \in F(x(t)), \quad t \in [0, 1], \\
&F(x) = \{y = Ax + Bu : u \in F_0(x)\}, \\
&F_0(x) = \{u : Wu \leq p, \ \ C(Ax + Bu) = 0\}.
\end{aligned} \tag{4.65}$$

Taking into account a concrete form of LAM, the next theorem is almost a word-for-word repetition of Theorem 4.11. In what follows, we assume that $F(x) = Ax + BF_0(x)$ is a polytope (or bounded polyhedron).

Theorem 4.13. Let $\tilde{x}(t)$, $t \in [0, 1]$ be a feasible solution of the problem in Eq. (4.65) and $\tilde{x}(t) \in \text{int dom } F$. The necessary condition for the trajectory $\tilde{x}(\cdot)$ to be optimal is that there exist a control function $\tilde{u}(\cdot)$, number $\lambda_0 \geq 0$, vector $x_0^* \in K_{M_1}^*(\tilde{x}(1))$, and nontrivial adjoint trajectory $x^*(\cdot)$, not all zero, such that

1. $x^*(1) + x_e^* \in \lambda_0 \partial \varphi_0(\tilde{x}(1))$,

2. $\begin{aligned} &-\dot{x}^*(t) = A^* x^*(t) - A^* C^* \lambda(t), \ \ B^* x^*(t) = B^* C^* \lambda(t) - W^* \gamma(t), \\ &\langle \gamma(t), W\tilde{u}(t) - p \rangle = 0, \quad \lambda(t) \geq 0, \ \ \gamma(t) \geq 0, \end{aligned}$

where $M_1 = \{x : \varphi(x) \leq 0\}$.

Obviously, if $L = \mathbb{R}^n$ and U is a bounded polyhedron, then hypothesis 2 of Theorem 4.13, as in linear optimal control theory, consists of the following:

$$\begin{aligned}
&\dot{x}^*(t) = -A^* x^*(t), \\
&\langle B\tilde{u}(t), x^*(t) \rangle = \max_{u \in U} \langle Bu, x^*(t) \rangle
\end{aligned}$$

Let $u^l(x)$, $l = 1, \ldots, q$ be the set of vertices of a polytope $F_0(x)$ and $\hat{w}_{li}$, $i \in J_l \cup \{1, ..., k\}$, $l = 1, ..., q$ be the collection of ribs of $F_0(x)$. Then we can formulate the condition for generality of position: the vectors

$$B\hat{w}_{l\,i},\ (A + H_l)B\hat{w}_{l\,i},\ (A + H_l)^2 B\hat{w}_{l\,i}, \ldots, (A + H_l)^{n-1} B\hat{w}_{l\,i} \tag{4.66}$$

are linearly independent in space $\mathbb{R}^n$ for any fixed l and all $i \in J_l \cup \{1, \ldots, k\}$. Note that here by using the Frobenius formula [175] for determining the inverse of a matrix that has been partitioned into submatrices, the matrix H_l has a concrete form. It should be noted that H_l is the zero matrix if $L = \mathbb{R}^n$ (for more detailed information, see Ref. [143]) and then Eq. (4.66) implies the well-known condition for generality of the classical linear theory of optimal control. We shall describe as a *switching point* a point of discontinuity of $\tilde{u}(t)$; i.e., if τ is a point of discontinuity of $\tilde{u}(t)$ and $\tilde{u}(\tau - 0) = u^i(\tilde{x}(\tau))$, $\tilde{u}(\tau + 0) = u^j(\tilde{x}(\tau))$, $i \neq j$, we shall say that a switching of the optimal control $\tilde{u}(t)$ occurs at $t = \tau$ from the vertex $u^i(\tilde{x}(\tau))$ to the vertex $u^j(\tilde{x}(\tau))$. We then formulate Theorem 4.14 on the finiteness of the number of switchings.

Theorem 4.14. Let $\tilde{x}(t)$, $t \in [t', t'']$ be an optimal trajectory lying entirely in the nondegeneracy domain D of a polytope $F_0(x)$. Moreover, let $x^*(t)$ be a nontrivial adjoint trajectory and suppose that the hypothesis for generality of position in Eq. (4.66) is satisfied. Then the interval $[t', t'']$ can be divided into a finite number of subintervals $[\vartheta_j, \vartheta_{j+1})$, $j = 0, 1, \ldots, N-1$ (N is natural number) such that in each of them

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t), \quad \tilde{u}(t) = u^l(\tilde{x}(t)), \dot{x}^*(t) = -(A^* + H_l^*)x^*(t).$$

Thus, the function $\tilde{u}(t)$ is piecewise continuous and its only values are the vertices of the polytope $F_0(\tilde{x}(t))$.

Now, let us consider the optimization of nonautonomous delay-differential inclusions of the form

$$\begin{aligned} &\text{infimum} && J(x(\cdot)) = \int_0^1 g(x(t), t)dt + \varphi_0(x(1)), \\ &\text{subject to} && \dot{x}(t) \in F(x(t), x(t-h), t), \quad \text{a.e } t \in [0, 1], \\ &&& x(t) = \xi(t), \quad t \in [-h, 0], \quad x(1) \in M_1. \end{aligned} \tag{4.67}$$

For simplicity we assume that $F(\cdot, \cdot, t)$ is convex, g, φ_0 are continuous and convex on x, M_1 is a convex set, and $\xi(t)$, $t \in [-h, 0]$ is an absolutely continuous initial function. It is required to find a trajectory $x(t)$, $t \in [-h, 1]$ satisfying almost everywhere the delay-differential inclusion in Eq. (4.67) and that minimizes the cost functional $J(x(\cdot))$.

Theorem 4.15. Let $F(\cdot, \cdot, t)$ be a convex, closed, and bounded mapping, and moreover let the mapping $t \to \operatorname{gph} F(\cdot, \cdot, t)$ be lower semicontinuous. Then the necessary condition for the trajectory $\tilde{x}(t)$, $t \in [-h, 1]$ of the problem in Eq. (4.67) to satisfy $(\tilde{x}(t), \tilde{x}(t-h)) \in \operatorname{int} \operatorname{dom} F(\cdot, \cdot, t)$ with the initial condition to be optimal is that there exist a number $\lambda_0 \geq 0$, vector x_e^*, and a pair of Lipschitzian functions $x^*(\cdot)\eta^*(\cdot)$, not all zero, such that

1. $x_e^* - x^*(1) \in \lambda_0 \partial\varphi(\tilde{x}(1), 1)$, $\quad x_e^* \in K_M^*$, $\ \eta^*(1) = 0$,

2. $(-\dot{x}^*(t), \eta^*(t)) \in F^*(x^*(t); \quad (\tilde{x}(t), \tilde{x}(t-h), \dot{\tilde{x}}(t)), t)$
$+ \{\eta^*(t+h) - \lambda_0 \partial g(\tilde{x}(t), t)\} \times \{0\}, \quad \text{a.e. } t \in [0, 1-h), \quad \partial g(\tilde{x}(0), 0) = 0,$

3. $(-\dot{x}^*(t), \eta^*(t)) \in F^*(x^*(t); \quad (\tilde{x}(t), \tilde{x}(t-h), \dot{\tilde{x}}(t)), t)$
$+ \{-\lambda_0 \partial g(\tilde{x}(t), t)\} \times \{0\}, \quad \text{a.e. } t \in [1-h, 1],$

4. $\dot{\tilde{x}}(t) \in F(\tilde{x}(t), \tilde{x}(t-h), x^*(t), t), \quad \text{a.e. } t \in [0, 1].$

In addition, if $\lambda_0 > 0$, then these conditions are sufficient for optimality of $\tilde{x}(\cdot)$.

□ The proof of necessity is analogous to the nonautonomous problem (Theorem 4.3). The only difference is that in the process of discretization of a continuous problem (Eq. (4.67)), we use Theorem 4.1. To avoid long calculations, we omit it. Let us prove sufficiency, if $\lambda_0 > 0$. Without loss of generality, we set $\lambda_0 = 1$. By Theorem 2.1, we have

$$F^*(y^*; (x, x_1, y, t)) = \partial_{(x,x_1)} H(x, x_1, y^*, t), \quad y \in F(x, x_1; y^*, t).$$

Using this formula, condition $(\tilde{x}(t), \tilde{x}(t-h)) \in \operatorname{int} \operatorname{dom} F(\cdot, \cdot, t)$, and Theorem 1.28, it is easy to see that conditions (b) and (c) are equivalent to the following relations, respectively:

$$\begin{aligned} &H(x(t), x(t-h), x^*(t), t) - H(\tilde{x}(t), \tilde{x}(t-h), x^*(t), t) - g(x(t), t) + g(\tilde{x}(t), t) \\ &\leq \langle -\dot{x}^*(t) - \eta^*(t+h), x(t) - \tilde{x}(t) \rangle + \langle \eta^*(t), x(t-h) - \tilde{x}(t-h) \rangle, \\ &\quad \text{a.e. } t \in [0, 1-h). \end{aligned} \tag{4.68}$$

$$\begin{aligned} &H(x(t), x(t-h), x^*(t), t) - H(\tilde{x}(t), \tilde{x}(t-h), x^*(t), t) - g(x(t), t) + g(\tilde{x}(t), t) \\ &\leq \langle -\dot{x}^*(t), x(t) - \tilde{x}(t) \rangle + \langle \eta^*(t), x(t-h) - \tilde{x}(t-h) \rangle, \quad \text{a.e. } t \in [1-h, 1]. \end{aligned} \tag{4.69}$$

Then, since $H(x(t), x(t-h), x^*(t), t) \geq \langle \dot{x}(t), x^*(t) \rangle, \quad \text{a.e. } t \in [0, 1]$ for an arbitrary feasible solution $x(\cdot)$, by condition (d) of theorem we obtain from Eq. (4.68)

$$\begin{aligned} &\langle \dot{x}(t), x^*(t) \rangle \leq \langle \dot{\tilde{x}}(t), x^*(t) \rangle + g(x(t), t) - g(\tilde{x}(t), t) \\ &+ \langle -\dot{x}^*(t) - \eta^*(t+h), x(t) - \tilde{x}(t) \rangle + \langle \eta^*(t), x(t-h) - \tilde{x}(t-h) \rangle, \\ &\quad \text{a.e. } t \in [0, 1-h). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt} \langle x(t), x^*(t) \rangle \leq g(x(t), t) - g(\tilde{x}(t), t) \\ &+ \langle -\dot{x}^*(t) - \eta^*(t+h), x(t) - \tilde{x}(t) \rangle + \langle \eta^*(t), x(t-h) - \tilde{x}(t-h) \rangle, \\ &\quad \text{a.e. } t \in [0, 1-h) \end{aligned}$$

or

$$\begin{aligned}&\frac{d}{dt}\langle x(t), x^*(t)\rangle \leq g(x(t),t) - g(\tilde{x}(t),t) + \frac{d}{dt}\langle \tilde{x}(t), x^*(t)\rangle\\ &- \langle \eta^*(t+h), x(t) - \tilde{x}(t)\rangle + \langle \eta^*(t), x(t-h) - \tilde{x}(t-h)\rangle, \quad \text{a.e. } t \in [0, 1-h).\end{aligned}$$

Integrating this inequality over interval [0, 1, − h], we obtain

$$\begin{aligned}\langle x(1-h), x^*(1-h)\rangle - \langle x(0), x^*(0)\rangle &\leq \int_0^{1-h}[g(x(t),t) - g(\tilde{x}(t),t)]dt\\ &\quad + \langle \tilde{x}(1-h), x^*(1-h)\rangle\\ - \langle \tilde{x}(0), x^*(0)\rangle - \int_0^{1-h}\langle \eta^*(t+h), x(t) - \tilde{x}(t)\rangle dt &+ \int_0^{1-h}\langle \eta^*(t), x(t-h)\\ - \tilde{x}(t-h)\rangle dt\end{aligned} \tag{4.70}$$

Analogously, from Eq. (4.69), it follows that

$$\begin{aligned}\langle x(1), x^*(1)\rangle - \langle x(1-h), x^*(1-h)\rangle &\leq \int_{1-h}^1[g(x(t),t) - g(\tilde{x}(t),t)]dt\\ + \langle \tilde{x}(1), x^*(1)\rangle - \langle \tilde{x}(1-h), x^*(1-h)\rangle &+ \int_{1-h}^1\langle \eta^*(t), x(t-h) - \tilde{x}(t-h)\rangle dt\end{aligned} \tag{4.71}$$

On the other hand, because $\tilde{x}(t) = x(t) = \xi(t), \;\; t \in [-h, 0]$, we get

$$\int_0^h \langle \eta^*(t), x(t-h) - \tilde{x}(t-h)\rangle dt = \int_{-h}^0 \langle \eta^*(t+h), x(t) - \tilde{x}(t)\rangle dt = 0.$$

Hence, summing the inequalities in Eqs. (4.70) and (4.71), we deduce that

$$\langle x(1), x^*(1)\rangle \leq \int_0^1 [g(x(t),t) - g(\tilde{x}(t),t)]dt + \langle \tilde{x}(1), x^*(1)\rangle \tag{4.72}$$

Moreover, condition (a) implies that the inequalities

$$\begin{aligned}\varphi_0(x(1)) - \varphi_0(\tilde{x}(1)) &\geq \langle x(1) - \tilde{x}(1), x_e^* - x^*(1)\rangle,\\ \langle x(1) - \tilde{x}(1), x_e^*\rangle &\geq 0\end{aligned} \tag{4.73}$$

hold. Thus, the inequalities in Eqs. (4.72) and (4.73) give us

$$\begin{aligned}&\int_{1-h}^1 [g(x(t),t) - g(\tilde{x}(t),t)]dt + \varphi_0(x(1)) - \varphi_0(\tilde{x}(1))\\ &\geq \langle x(1) - \tilde{x}(1), x^*(1)\rangle + \langle x(1) - \tilde{x}(1), x_e^* - x^*(1)\rangle = \langle x(1) - \tilde{x}(1), x_e^*\rangle \geq 0,\end{aligned}$$

which completes the proof of the theorem. ■

At the end of this section, note that for nonautonomous polyhedral differential inclusions (4.67) without delay effect ($h = 0$), where $F(x, t) = \{y \in \mathbb{R}^n : A(t)x - B(t)y \leq d(t)\}$,

$A(t)$, $B(t)$ are $(n-1)$ order continuously differentiable $m \times n$ matrices, and $d(t)$ is a continuous column vector, by analogy we can prove theorems like Theorems 4.12 and 4.14, which can be characterized as theorems on the finiteness of the number of switchings. But it should be pointed out that in this case the condition for generality of position is absolutely different. Indeed, let $w_{ki}(t)$, $i \in J_K$ be a rib connected with the each vertex of number k. Indeed, take

$$G_{ki}^0(t) = w_{ki}(t), \quad G_{ki}^1(t) = \frac{dG_{ki}^0(t)}{dt} - C_k(t)G_{ki}^0(t), \quad G_{ki}^l(t)\frac{dG_{ki}^{l-1}(t)}{dt} - C_k(t)G_{ki}^{l-1}(t)$$
$$l = 1, 2, \ldots, n-1; \quad C_k(t) = B_{J_k}^{-1}(t)A_{J_k}(t).$$

Then we assume satisfaction of the following condition for generality of position: let for all $t \in [t_0, t_1]$ the set of vectors

$$G_{ki}^0(t), \ G_{ki}^1(t), \ldots, \ G_{ki}^{n-1}(t), \quad i \in J_k, \quad k = 1, \ldots, q$$

be linearly independent for each vertex of number k and rib w_{ki}, $i \in J_k$. Here, q is the number of vertices of the polyhedron. It can easily be seen that the generality of position in Eq. (4.61) is obtained from linear independence of the above vectors, if we take $A(t) \equiv A$, $B(t) \equiv B$, $d(t) \equiv d$.

4.5 Bolza Problems for Differential Inclusions with State Constraints

In this section, a Bolza problem of optimal control, whose dynamic constraint are given by some class of nonconvex differential inclusions, is considered. Sufficient conditions are proved in terms of the Euler–Lagrange inclusion, Hamiltonian inclusion, and Weierstrass–Pontryagin maximum condition. Here, a new concept of LAM, defined by means of the Hamiltonian function, is introduced. Of course, a jump condition connected with state constraints and formulated t_1-transversality conditions are essential. Our problem with a varying time interval labeled by (P) consists of the following:

$$\begin{array}{lll} & \text{infimum} & J[x(\cdot), t_1] = \int_{t_0}^{t_1} g(x(t), t)dt + \varphi_0(x(t_1), t_1), \\ (P) & \text{subject to} & \dot{x}(t) \in F(x(t), t), \quad \text{a.e } t \in [t_0, t_1], \\ & & x(t_0) = x_0, \quad x(t_1) \in M_1, \\ & & x(t) \in \Phi(t), \quad t \in [t_0, t_1], \end{array}$$

where, as usual, $F(\cdot, t)$, $t \in [t_0, t_1]$ is a nonautonomous multivalued function, g, φ: $\mathbb{R}^n \times [t_0, t_1] \to \mathbb{R} \cup \{+\infty\}$; M_1 is a set of target points $x(t_1)$; and the initial time moment t_0 is fixed, whereas the instant t_1 is nonfixed. The integral in the cost functional J is understood as that of Lebesgue. A feasible solution $x(t)$ of problem (P)

is absolutely continuous. The state constraint is satisfied everywhere; i.e., $x(t) \in \Phi(t)$ $\forall t \in [t_0, t_1]$. Below we show that an adjoint trajectory has jumps, which are typical for control systems with state constraints, and among sufficient conditions there appears a condition of jumps (see also Ref. [221]), where the number of jump points may be countable.

Definition 4.3. The multivalued function defined by the formula

$$F^*(v^*; (x^0, v^0), t) = \{x^* : H(x, v^*) - H(x^0, v^*) \leq \langle x^*, x - x^0 \rangle \quad \forall x \in \mathbb{R}^n\},$$
$$v^0 \in F(x^0; v^0)$$

is called the LAM to the nonconvex mapping F at the point $(x^0, y^0) \in \text{gph } F$. Note that for smooth functions $H(\cdot, y^*)$, the inequality in this definition can be given by the Weierstrass function [111], which plays an important role in classical variational calculus problems. Obviously, for a convex multivalued function $F(\cdot, t)$, this formula can be expressed by the subdifferential of the Hamiltonian function contained in Theorem 2.1. On the other hand, it can easily be checked that F^* is a convex closed set at a given point.

Proposition 4.1. For a nonconvex multivalued function $F(\cdot, t)$, the value of the LAM at a point $(x, v) \in \text{gph } F(\cdot, t)$ is a convex closed set.
☐ Indeed it follows from Definition 2.2 that if $x_1^*, x_2^* \in F^*(v^*(x, v), t)$, then $\lambda_1 x_1^* + \lambda_2 x_2^* \in F^*(v^*, (x, v), t)$, i.e., the LAM is convex valued. On the other hand, for a sequence $x_n^* \in F^*(v^*, (x, v), t)$ such that $x_n^* \to x_0^*$ as $n \to \infty$, by Definition 2.2

$$M(x_1, v^*, t) - M(x, v^*, t) \leq \langle x_0^*, x_1 - x \rangle, \quad \text{i.e., } x_0^* \in F^*(v^*, (x, v), t)$$

and the value of the LAM is a closed set. The proof is completed. ■

Let $\tilde{x}(t)$, $t \in [t_0, t_1]$ be any feasible solution of problem (P). We construct the adjoint differential inclusion of the adjoint trajectory $x^*(t)$ as follows:

a. $-\dot{x}^*(t) \in F^*(x^*(t); \quad (\tilde{x}(t), \dot{\tilde{x}}(t)), t)$,
b. $\tilde{x}(t) \in F(\tilde{x}(t); \quad x^*(t), t)$, a.e. $t \in [t_0, t_1]$,

which would be fulfilled for all $x \in \Phi(t)$. Assume that the solution $x^*(t)$, $t \in [t_0, t_1]$ satisfies the adjoint differential inclusion a almost everywhere and may be represented as a sum of an absolutely continuous function and a function of jumps. Note that an inclusion of the form (a) usually is called an Euler–Lagrange inclusion. By the definition of the Argmaximum set, it is easy to see that (b) implies the Weierstrass–Pontryagin maximum principle.

Let us denote the points of jumps and the values of jumps of $x^*(t)$ by

$$\tau_i, (i = 1, 2, \ldots), t_0 < \tau_i < t_1; \quad x_i^* = x^*(\tau_i + 0) - x^*(\tau_i - 0), (i = 1, 2, \ldots),$$

respectively.

Definition 4.4. Let $W_{M_1 \cap \Phi(t)}$ be a support function to the set $M_1 \cap \Phi(t)$. Then we say that for a feasible solution $\tilde{x}(t)$ of problem (P), the t_1-transversality condition is satisfied on the set M_1 if the inequality

$$-\langle x^*(t), \tilde{x}(t)\rangle > W_{M_1 \cap \Phi(t)}(-x^*(t)), \quad t_0 \leq t < t_1$$

holds strictly. Clearly, $\tilde{x}(t) \notin M_1 \cap \Phi(t)$ and so $\tilde{x}(t) \notin M_1$ for all $t \in [t_0, t_1)$. In other words, the t_1-transversality condition guarantees that the point $\tilde{x}(t)$ belongs to the set M_1 only at the instant $t = t_1$.

Definition 4.5. If the inequality

$$J[x(\cdot), \theta'] < J[x(\cdot), \theta'']$$

holds for all $\theta', \theta'' \in [t_0, t_1]$, where $\theta' < \theta''$ and for all feasible trajectories of the problem (P), then the cost functional $J[x(\cdot), t]$ is said to be monotonically increasing with respect to t.

Remark 4.8. The monotonicity condition of $J[x(\cdot), t]$ in t for all feasible solutions $x(t)$ is not very restrictive or nonverifiable. For example, it is fulfilled for time optimal control problems, $(\varphi \equiv 0,\ g \equiv 1)$ for problems with quadratic cost functional, for Lagrange $[\varphi(x, t) \equiv 0]$ problems with nonnegative integrand g and so on.

Theorem 4.16. Let $\tilde{x}(\cdot)$ be any feasible arc of a nonconvex problem (P) and let there exists an adjoint feasible function $x^*(\cdot)$ satisfying almost everywhere conditions (a) and (b) and which may be represented as the sum of an absolutely continuous function and a function of jumps. Moreover, assume that the Bolza cost functional $J[x(\cdot), t]$ is monotonically increasing with respect to t for any feasible arc $x(\cdot)$ of the problem (P) and that the following conditions are satisfied:

i. $g(x,t) - g(\tilde{x}(t), t) \geq \langle -x^*(t), x - \tilde{x}(t)\rangle$ a.e. $t \in [t_0, t_1]$ and $\forall x \in \Phi(t)$,
ii. $\varphi(x, t_1) - \varphi(\tilde{x}(t_1)) \geq \langle -x^*(t_1), x - \tilde{x}(t_1)\rangle \quad \forall x \in \Phi(t_1) \cap M_1$,
$\varphi(x, t_1) - \varphi(\tilde{x}(t_1)) \geq \langle -x^*(t_1), x - \tilde{x}(t_1)\rangle \quad \forall x \in \Phi(t_1) \cap M_1$,
iii. The jumps x_i^* satisfy the jump conditions $\langle \tilde{x}(\tau_i), x_i^*\rangle = W_{\Phi(\tau_i)}(x_i^*)\ (x_i^*),\ i = 1, 2, \ldots$.
iv. For a feasible solution $\tilde{x}(\cdot)$ of problem (P), the t_1-transversality condition on the set M_1 is satisfied.

Then the arc $\tilde{x}(\cdot)$ is optimal.

□ Let $x(\cdot)$ be an arbitrary feasible solution of the problem (P), which transfers the state from the fixed point x_0 to the set M_1 in the time interval $[t_0, \theta]$. We shall show that

$$J[x(\cdot), \theta] \geq J[\tilde{x}(\cdot), t_1].$$

By Definition 4.3 for almost all $t \in [t_0, t_1]$, we can rewrite the inclusion (a) as follows:

$$H(x(t), x^*(t), t) - H(\tilde{x}(t), x^*(t), t) \leq \langle -\dot{x}^*(t), x(t) - \tilde{x}(t)\rangle.$$

or in view of condition (b),

$$H(x(t), x^*(t), t) - \langle \tilde{x}(t), x^*(t) \rangle \le \langle -\dot{x}^*(t), x(t) - \tilde{x}(t) \rangle. \tag{4.74}$$

Since by definition of Hamiltonian functions, $H(x(t), x^*(t), t) \ge \langle x(t), x^*(t) \rangle$ for all feasible trajectories, from Eq. (4.74), we have

$$\frac{d\psi(t)}{dt} \ge 0 \text{ for almost all } t \in [t_0, t_1], \text{ where } \psi(t) = \langle -x^*(t), x(t) - \tilde{x}(t) \rangle.$$

Then, integrating the last inequality over the interval $[t_0, t_1]$ and taking into account that $x(t) = \tilde{x}(t) = x_0$, we find that

$$\int_{t_0}^{t_1} \dot{\psi}(t)dt = \langle -x^*(t_1), x(t_1) - \tilde{x}(t_1) \rangle \ge 0 \tag{4.75}$$

Note that t_1 is free and the last inequality is true for any $t_1 = \theta$. Since $x(\cdot)$, $\tilde{x}(\cdot)$ are absolutely continuous, the function $\psi(\cdot)$ can be represented as a sum of an absolutely continuous function and a function of jumps (see, for instance, Ref. [139])

$$\psi(\theta) = \psi(t_0) + \int_{t_0}^{\theta} \dot{\psi}(t)dt + \sum_{i \in I(t)} [\psi(\tau_i + 0) - \psi(\tau_i - 0)], \quad I(t) = \{i : \tau_i \in [t_0, t]\} \tag{4.76}$$

Further, by condition (iii) we derive that the values of the jumps of ψ at points τ_i, $i = 1, 2, \ldots$ are the following quantities

$$\psi(\tau_i + 0) - \psi(\tau_i - 0) = \langle x(\tau_i) - \tilde{x}(\tau_i), -x_i^* \rangle = -\langle x(\tau_i), x_i^* \rangle + W_{\Phi(\tau_i)}(x_i^*).$$

Now, since $x(\tau_i) \in \Phi(\tau_i)$, it is evident that

$$\psi(\tau_i + 0) - \psi(\tau_i - 0) \ge 0 \quad \forall \tau_i \in [t_0, \theta]$$

or

$$\sum_{i \in I(\theta)} [\psi(\tau_i + 0) - \psi(\tau_i - 0)] \ge 0.$$

Therefore, from Eq. (4.76), we find that $\psi(\theta) \ge \psi(t_0)$, or

$$\langle x(\theta) - \tilde{x}(\theta), -x^*(\theta) \rangle \ge \langle x(t_0) - \tilde{x}(t_0), -x^*(t_0) \rangle.$$

Since $x(t) = \tilde{x}(t) = x_0$, the right-hand side of the latter inequality is zero, so

$$\langle x(\theta), -x^*(\theta) \rangle \ge \langle \tilde{x}(\theta), -x^*(\theta) \rangle \tag{4.77}$$

Thus, from the t_1-transversality condition on the set M_1 of condition (iv) and from Eq. (4.77), we observe that

$$\langle x(\theta), -x^*(\theta)\rangle > W_{M_1\cap\Phi(t)}(-x^*(\theta)) \tag{4.78}$$

Let $\Delta J = J[x(\cdot),\theta] - J[\tilde{x}(\cdot),t_1]$ be an increment of the Bolza cost functional J, obtained by the transition from arc $\tilde{x}(\cdot)$ to the arc $x(\cdot)$. Then (see Refs. [155,157]) we have

$$\begin{aligned}\Delta J &= \varphi(x(\theta),\theta) + \int_{t_0}^{\theta} g(x(t),t)dt - \varphi(\tilde{x}(t_1),t_1) - \int_{t_0}^{t_1} g(\tilde{x}(t),t)dt \\ &= \varphi(x(\theta),\theta) + \int_{t_0}^{\theta} g(x(t),t)dt - \varphi(x(t_1),t_1) - \int_{t_0}^{t_1} g(x(t),t)dt + \varphi(x(t_1),t_1) \\ &\quad + \int_{t_0}^{t_1} g(x(t),t)dt - \varphi(\tilde{x}(t_1),t_1) - \int_{t_0}^{t_1} g(\tilde{x}(t),t)dt\end{aligned} \tag{4.79}$$

On the other hand, conditions (ii) and (iii) of the theorem imply that

$$\begin{aligned}&\varphi(x(t_1),t_1) - \varphi(\tilde{x}(t_1),t_1) + \int_{t_0}^{t_1}[g(x(t),t) - g(\tilde{x}(t),t)]dt \\ &\geq -\langle x^*(t_1), \ x(t_1) - \tilde{x}(t_1)\rangle - \int_{t_0}^{t_1}\langle x^*(t), \ x(t) - \tilde{x}(t)\rangle dt.\end{aligned}$$

Then, since this inequality holds for any $t\in[t_0, t_1]$, the relation in Eq. (4.79) gives us

$$\Delta J \geq \varphi(x(\theta),\theta) + \int_{t_0}^{\theta} g(x(t),t)dt - \varphi(x(t_1),t_1) - \int_{t_0}^{t_1} g(x(t),t)dt. \tag{4.80}$$

To prove the optimality of $\tilde{x}(\cdot)$, let us assume the contrary; i.e., for all feasible arc $x(t)$,

$t\in[t_0, \theta]$, $x(\theta)\in M_1$ the increment $\Delta J < 0$, i.e., $J[x(.),\theta] < J[\tilde{x}(\cdot),t_1]$.

Because of Eq. (4.80), $J[x(\cdot), \theta] < J[x(\cdot), t_1]$. Therefore, from the monotonicity of J we have $\theta < t_1$. Thus, by the t_1-transversality condition, it follows from Eq. (4.78) that $x(\theta)\notin M_1\cap\Phi(\theta)$ and so $x(\theta)\notin M_1$; i.e., the arc $x(\cdot)$ cannot realize the transition from the interval $[t_0, \theta]$ to the set M_1. This is a contradiction. ■

Corollary 4.5. If in problem (P), $F(\cdot, t)$, $t\in[t_0, t_1]$ is a closed convex multivalued function, then conditions (a) and (b) of Theorem 4.16 can be rewritten in Hamiltonian form

$$\begin{aligned}&-\dot{x}^*(t) \in \partial_x H(\tilde{x}(t), x^*(t), t), \\ &\dot{\tilde{x}}(t) \in \partial_{y^*} H(\tilde{x}(t), x^*(t), t).\end{aligned}$$

□ The proof is elementary (see also Corollary 4.1). By Definition 4.2, for convex multivalued functions F, the Hamiltonian function is concave in the first argument and so F^* is the subdifferential of a Hamiltonian function. ■

Remark 4.9. If $\theta = t_1$ is fixed, then it follows from Eq. (4.80) that $\Delta J \geq 0$; i.e., $\tilde{x}(\cdot)$ is optimal. Therefore, the monotonicity of $J[x(\cdot), t]$ on t is superfluous.

Example 4.3. Consider the following time optimal problem described by the equation

$$\begin{cases} \dot{x}^1 = x^2, \\ \dot{x}^2 = u, \end{cases} \tag{4.81}$$

where u is the control parameter $|u| \leq 1$ and $|x^2| \leq 1$, $t_0 = 0$, $x(0) = x_0$, $x(t_1) = 0$.

Here, x_0 is an arbitrary initial point. We have the state constraints

$$\Phi(t) \equiv \{x = (x^1, x^2) : -1 \leq x^2 \leq 1\}. \tag{4.82}$$

According to the classical theory of optimal control, if $u(t) = 0$, then the solution runs along the line $x^2 = k$ (k is constant). Moreover, if $u(t) = +1$, then the solution $x(t)$ runs along the parabola.

$x^1 - 1/2(x^2)^2 = k$. Similarly, $x(t)$ runs along the parabola $x^1 + 1/2(x^2)^2 = k$, if $u(t) = -1$. Take an arbitrary point x_0, for example, in the third orthant of $\mathbb{R}^2 (x_0 = (x_0^1, x_0^2) \leq 0)$ (for all other points the description will be similar). Thus, the control $u(t)$ has a form

$$\tilde{u}(t) = \begin{cases} +1, & \text{if } t \in [0, \tau_1), \\ 0, & \text{if } t \in [\tau_1, \tau_2], \\ -1, & \text{if } t \in (\tau_2, t_1]. \end{cases} \tag{4.83}$$

By applying Theorem 4.16, we shall show that the given control $\tilde{u}(t)$ and the corresponding solution $\tilde{x}(t)$ are optimal. We have the multivalued function

$$F(x,t) = \{y = (y^1, y)^2 \in \mathbb{R}^2 : y^1 = x^2, \ |y^2| \leq 1\}.$$

The Hamiltonian function has the form

$$H(x, y^*) = x^2 y^{1*} + |y^{2*}| \tag{4.84}$$

Then, by definition of the LAM, we get

$$\begin{aligned} F^*(x^*(t); (\tilde{x}(t), \dot{\tilde{x}}(t)), t) = \{ -\dot{x}^*(t) : H(x, x^*(t), t) - H(\tilde{x}(t), x^*(t), t) \\ \leq \langle -\dot{x}^*(t), x - \tilde{x}(t) \rangle \} \end{aligned}$$

for all $x \in \mathbb{R}^2$ and $\dot{\tilde{x}}(t) \in F(\tilde{x}(t); x^*(t), t)$. In other words,

$$(x^1 - \tilde{x}^1(t))\dot{x}^{1*} + (x^2 - \tilde{x}^2(t))(\dot{x}^{2*} + x^{1*}) \leq 0 \tag{4.85}$$

must hold for all $x \in \mathbb{R}^2$. As we see from Eq. (4.85), F^* depends on $\tilde{x}(t)$, whether it is the boundary or interior point of $\Phi(t)$ (see Eq. (4.82)):

$$F^*(x^*; (\tilde{x}(t), \dot{\tilde{x}}(t)), t) = \begin{cases} [x^* = (x^{1^*}, x^{2^*}) : \dot{x}^{1^*} = 0, \quad \dot{x}^{2^*} + x^{1^*} \le 0], & \text{if } \tilde{x}^2(t) = -1, \\ [x^* = (x^{1^*}, x^{2^*}) : \dot{x}^{1^*} = 0, \quad \dot{x}^{2^*} + x^{1^*} = 0], & \text{if } |\tilde{x}^2(t)| < 1, \\ [x^* = (x^{1^*}, x^{2^*}) : \dot{x}^{1^*} = 0, \quad \dot{x}^{2^*} + x^{1^*} \ge 0], & \text{if } \tilde{x}^2(t) = +1. \end{cases} \tag{4.86}$$

Using Eq. (4.84), the condition (b) has the form

$$\tilde{u}(t) x^{2*}(t) = |x^{2*}(t)|$$

and so for those instances $t \in [0, t_1]$ where $x^{2^*}(t) \neq 0$, we get the formula $\tilde{u}(t) = \text{sign } x^{2^*}(t)$ and clearly $|\tilde{u}(t)| \le 1$, if $x^{2^*}(t) = 0$. Moreover, the t_1-transversality condition on the set M_1 (condition (iv)) of Theorem 4.16 in our problem consists of the following:

$$\langle \tilde{x}(t), x^*(t) \rangle < 0 \quad \forall t \in [0, t_1) \tag{4.87}$$

Now we shall construct the adjoint function $x^*(t)$, which is even absolutely continuous. Then the jump condition (iii) of Theorem 4.16 will hold for $x^*(t)$, automatically. In this case we set for definiteness $x^{1^*}(t) = 1$. Furthermore, from the form of $\tilde{u}(t)$ given by Eq. (4.83), we get

$$x^{2^*}(t) = \begin{cases} > 0, & \text{if } t \in [0, \tau_1), \\ = 0, & \text{if } t \in [\tau_1, \tau_2], \\ < 0, & \text{if } t \in (\tau_2, t_1]. \end{cases}$$

The function $x^{1^*}(t)$ must be a solution of the differential equations and inequalities contained in Eq. (4.86) on the intervals $[0, \tau_1)$, $[\tau_1, \tau_2]$, $(\tau_2, t_1]$, respectively.

If we add to this condition the condition of absolute continuity, then we get a unique function, $x^{1^*}(t)$, having the form

$$x^{2^*}(t) = \begin{cases} \tau_1 - t, & \text{if } t \in [0, \tau_1), \\ 0, & \text{if } t \in [\tau_1, \tau_2], \\ \tau_2 - t, & \text{if } t \in (\tau_2, t_1]. \end{cases}$$

We need to check only the t_1-transversality condition in Eq. (4.87) on the set M_1. At once we observe that since $\tilde{x}^1(t) < 0, \ \tilde{x}^2(t) \ge 0, \ t \in [\tau_1, t_1)$, Eq. (4.87) is satisfied on $[\tau_1, t_1)$. It remains to check this condition on the interval $[0, \tau_1)$. Setting $\tilde{u}(t) = +1$ in Eq. (4.81), we find that

$$\tilde{x}^1(t) = \frac{1}{2} t^2 + x_0^2 t + x_0^1; \quad \tilde{x}^2(t) = t + x_0^2; \quad x_0 = (x_0^1, x_0^2).$$

Using the condition $\tilde{x}^2(\tau_1)=1$, we define $\tau_1=1-x_0^2$. Inserting this into Eq. (4.87), we reduce it to the form

$$-\frac{1}{2}(\tau_1-t)^2+\left(x_0^1-\frac{1}{2}(x_0^2)^2+\frac{1}{2}\right)<0, \quad t\in[0,\tau_1).$$

Therefore, $(\tilde{x}(t),\tilde{u}(t))$ satisfies all the hypotheses of the theorem and so is optimal.

4.6 Optimal Control of Hereditary Functional-Differential Inclusions with Varying Time Interval and State Constraints

In this section, Bolza problems of optimal control theory with a varying time interval given by convex and nonconvex hereditary differential inclusions (P_N), (P_V) are considered. We derive sufficient optimality conditions for neutral functional-differential inclusions. The existence of constraint conditions implies jump conditions for the conjugate variable. The sufficient conditions under the t_1-transversality condition are proved incorporating Euler–Lagrange and Hamiltonian-type inclusions. The basic concept of obtaining optimality conditions is that of the LAM and some specially proved equivalence theorems. Then the application of these results is illustrated by solving some typical examples.

At first we consider a problem with neutral discrete inclusions:

$$\begin{aligned}
(P_D)\quad &\text{infimum} \quad \sum_{t=1}^{T} g(x_t,t),\\
&\text{subject to} \quad x_{t+1}\in F_t(x_t,x_{t-h},x_{t-h-1}), \quad t=0,\ldots,T-1,\\
&\qquad x_t=\xi_t,\ \ t=-h,\ -h+1,\ldots,0,\\
&\qquad x_t\in\Phi_t,\ \ t=1,\ldots,T, \quad x_T\in Q,
\end{aligned}$$

where $g(\cdot,t):\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is a function taking values on the extended line for all $t=1,\ \ldots,\ T$, $F_t:\mathbb{R}^{3n}\to P(\mathbb{R}^n)$ is a multivalued function, ξ_t are fixed vectors $(t=-h,\ \ldots,\ 0)$, Q and Φ_t are sets in $\mathbb{R}^n$, and T, h are fixed natural numbers. It is required to find a feasible trajectory $\{x_t\}_{t=-h}^{T}$ minimizing $\sum_{t=1}^{T} g(x_t,t)$. The term "neutral discrete inclusions" will become clearer later on.

Our second problem is a Bolza problem with state constraints for neutral-type functional-differential inclusions: choose an arc, which is an absolutely continuous function on $[t_0-h,\ t_0)$ and $[t_0,\ t_1]$ ($t=t_0$ could be a point of discontinuity) and then

$$\begin{aligned}
&\text{infimum} \quad J\ [x(\cdot),t_1)] := \varphi(x(t_1),t_1)+\int_{t_0}^{t_1} g(x(t),t)dt,\\
(P_N)\quad &\text{subject to} \quad \dot{x}(t)\in F\ (x(t),\ \ x(t-h),\ \dot{x}(t-h),t) \quad \text{a.e. } t\in[t_0,t_1],\\
&\qquad x(t)=\xi(t) \quad \forall t\in[t_0-h,t_0), \quad h>0,\\
&\qquad x(t)\in\Phi(t) \quad \forall t\in[t_0,t_1],\\
&\qquad x(t_1)\in Q,
\end{aligned}$$

where $F(\cdot, t):\mathbb{R}^{3n} \to P(\mathbb{R}^n)$ is a multivalued function for all fixed $t \in [t_0, t_1]$, the target set $Q \subset \mathbb{R}^n$ is a set of points $x(t_1)$; $\Phi:[t_0, t_1] \to P(\mathbb{R}^n)$ is a multivalued function; g, $\varphi:\mathbb{R}^n \times [t_0, t_1] \to \mathbb{R} \cup \{+\infty\}$, $\xi(\cdot)$ is an absolutely continuous function on $[t_0 - h, t_0)$, t_0 is fixed, t_1 is generally free, and $h > 0$ is a constant delay. The integral in the cost functional J is understood in the sense of Lebesgue.

It can be seen that such problems contain time delays not only in the state variables but also in the velocity variables. The next problem considered is an optimal control problem (P_N) with functional-differential inclusions that are linear in velocities and involve a neutral-type operator given in the Hale form [99,204,208]:

$$\frac{d}{dt}[x(t) - Ax(t-h)] \in F(x(t),\ \ x(t-h),\ \ t).$$

The third problem is the same problem as (P_N), but with a variable delay-differential inclusion:

$$\begin{aligned}
&\text{infimum} && J[(x(.), t_1]: = \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} g\ \ (x(t), t)dt,\\
(P_V)\quad &\text{subject to} && \dot{x}(t) \in F(x(t),\ \ x(t-h(t)), t)\quad \text{a.e. } t \in [t_0, t_1],\\
& && x(t) = \xi(t),\ \ t_0 - h(t_0) \le t \le t_0,\\
& && x(t) \in \Phi(t)\quad \forall t \in [t_0, t_1],\\
& && x(t_1) \in Q,
\end{aligned}$$

where $F(\cdot, t):\mathbb{R}^{2n} \to P(\mathbb{R}^n)$, $h(\cdot)$ is a differentiable function satisfying the inequalities $h(t) > 0$, $h'(t) < 1$ and $\xi()$ is absolutely continuous on $t_0 - h(t_0) \le t \le t_0$. We minimize the cost functional J over an absolutely continuous functions $x:[t_0 - h(t_0), t_1] \to \mathbb{R}^n$ satisfying the indicated delay-differential inclusion (P_V) with initial condition $x(t) = \xi(t)$, $t_0 - h(t_0) \le t \le t_0$ and state constraint on $[t_0, t_1]$.

Section 4.5 deals with a complicated problem with differential inclusion (P_V), which involves a variable delay. For such nonconvex problems, a sufficient condition is formulated. The simple examples considered show that the results obtained coincide with the results of classical optimal control theory [94,187,221].

Hypothesis (H1). Suppose that in problem (P_D), the multivalued functions are such that the cones of tangent directions $K_{\operatorname{gph} F_t}(\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t-h+1}, \tilde{x}_{t+1})$, $K_{\Phi_t}(\tilde{x}_t)$, $K_Q(\tilde{x}_T)$ are local tents, where $\tilde{x}_t$ are the points of the optimal trajectory $\{\tilde{x}_t\}_{t=-h}^{T}$ of problem (P_D). Furthermore, the functions $g(\cdot, t)$ admit a CUA $h(\overline{x}, \tilde{x}_t)$ of the points $\tilde{x}_t$ that is continuous with respect to $\overline{x}$ and consequently $\partial g(\tilde{x}_t, t) := \partial h_t(0, \tilde{x}_t)$ is defined.

Hypothesis (H2). Let the problem (P_D) be convex (i.e., it involves functions, multivalued functions, and sets that are convex), and $\{x_t^0\}_{t=-h}^{T}$ is a feasible trajectory. Then suppose either of the two following conditions:

a. $$\begin{aligned}&(x_t^0, x_{t-h}^0, x_{t-h+1}^0, x_{t+1}^0) \in \operatorname{ri} \operatorname{gph}\ F_t,\quad t = 0, \ldots, T-1,\\ &x_t^0 \in \operatorname{ri} \Phi_t \cap\ \operatorname{dom}\ g(., t),\quad t = 1, \ldots, T,\ \ x_T^0 \in \operatorname{ri} Q,\end{aligned}$$

b. $(x_t^0, x_{t-h}^0, x_{t-h+1}^0, x_{t+1}^0) \in \operatorname{int} \operatorname{gph}\ F_t$, $t = 0, \ldots, T-1$, $x_t^0 \in \operatorname{int} \Phi_t$, $t = 1, T$, $x_T^0 \in Q$ and $g(\cdot, t)$ are continuous at a point x_t^0.

Optimization of Neutral Discrete Inclusions (P_D)

Theorem 4.17. Assume hypothesis H1 for a neutral-type nonconvex problem (P_D). Then, for the optimality of the trajectory $\{\tilde{x}_t\}_{t=-h}^{T}$, it is necessary that there exist a number $\lambda \in \{0, 1\}$ and vectors $x^*,\ x_t^*,\ \eta_t^*,\ \varphi_t^*,\ t = 0, 1, \ldots, T$, not all equal to zero, such that

$$\text{i.}\quad \begin{aligned} &(x_t^* - \eta_{t+h}^* - \varphi_{t+h-1}^*,\ \eta_t^*,\ \varphi_t^*) \in F_t^*(x_{t+1}^* : (\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{tt-h+1}, \tilde{x}_{t+1})) \\ &\quad - \{\lambda\ \partial_x g(\tilde{x}_t, t) - K_{\Phi_t}^*(\tilde{x}_t)\} \times \{0\} \times \{0\}, \quad t = 0, \ldots, T-1-h, \\ &\lambda \partial_x(\tilde{x}_0, 0) = 0, \quad K_{\Phi_0}^*(\tilde{x}_0) = 0, \end{aligned}$$

$$\text{ii.}\quad \begin{aligned} &(x_t^*,\ \eta_t^*,\ \varphi_t^*) \in F_t^*(x_{t+1}^* : (\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{tt-h+1}, \tilde{x}_{t+1}) \\ &- \{\lambda\ \partial_x g(\tilde{x}_t, t) - K_{\Phi_t}^*(\tilde{x}_t)\} \times \{0\} \times \{0\}, \quad t = T-h, \ldots, T-1, \end{aligned}$$

$$\text{iii.}\quad x^* - x_T^* \in \lambda \partial_x g\ (\tilde{x}_T, T) - K_{\Phi_T}^*(\tilde{x}_T),\ \ x^* \in K_Q^*(\tilde{x}_T),\ \ \eta_T^* = 0,\ \ \varphi_T^* = 0.$$

□ Let us introduce the vector $w = (x_{-h}, x_{-h+1}, \ldots, x_0, \ldots, x_T) \in \mathbb{R}^{n(h+1+T)}$ and define in the space $\mathbb{R}^{n(h+1+T)}$ the following sets:

$$\begin{gathered} M_t = \{w : (x_t, x_{t-h}, x_{t-h+1}, x_{t+1}) \in \text{gph}\ \ F_t\}, \quad t = 0, \ldots, \quad T-1, \\ N = \{w : w_t = \xi_t, t = -h, \quad -h+1, \ldots, 0\}. \\ P_t = \{w : x_t \in \Phi_t,\ \ t = 1, \ldots, T\}, \quad M_T = \{w : x_T \in Q\}. \end{gathered}$$

Obviously, the problem (P_D) is equivalent to the following minimization problem:

$$\begin{aligned} &\text{infimum} \qquad g(w) = \sum_{t=1}^{T} g\ \ (x_t, t), \\ &\text{subject to} \quad w \in N \cap \left(\bigcap_{t=1}^{T} M_t\right) \cap \left(\bigcap_{t=1}^{T} P_t\right). \end{aligned} \tag{4.88}$$

In order to formulate a necessary condition (Theorem 3.4) for the minimization problem in Eq. (4.88), we use the dual cones

$$\begin{gathered} K_{M_t}^*(w) = \{w^* : (x_t^*, x_{t-h}^*, x_{t-h+1}^*, x_{t+1}^*) \in K_{\text{gph}F_t}^*(x_t, x_{t-h}, x_{t-h+1}, x_{t+1}) \\ x_k^* = 0,\ \ k \neq t,\ \ t-h,\ \ t-h+1,\ \ t+1\}, \quad w^* = (x_{-h}^*, \ldots, x_0^*, \ldots, x_T^*) \in \mathbb{R}^{n(h+1+T)}, \\ K_{M_T}^*(w) = \{w^* : x_T^*, \in K_Q^*(x_T),\ \ x_t^* = 0,\ \ t < T\}, \\ K_{P_t}^*(w) = \{w^* : x_t^* \in K_{\Phi_t}^*(x_t),\ \ x_k^* = 0,\ \ k \neq t\}, \quad t = 1, \ldots, T, \\ K_N^*(w) = \{w^* : x_t^* = 0,\ \ t = -h, \quad -h+1, \ldots, 0\}. \end{gathered}$$

Then by the minimization theorems mentioned above, there are vectors

$$\begin{gathered} w^*(t) \in K_{M_t}^*(\tilde{w}),\ \ t = 0, \ldots, T,\ \ w_h^* \in K_N^*(\tilde{w}), \quad \overline{w}^*(t) \in K_{P_t}^*(\tilde{w}),\ \ t = 1, \ldots,\ \ T, \\ \tilde{w} = (\tilde{x}_{-h}, \tilde{x}_{-h+1}, \ldots, \tilde{x}_T), \quad \tilde{x}_t = \xi_t,\ \ t = -h, \ldots,\ \ 0 \end{gathered}$$

and a number $\lambda \in \{0, 1\}$, such that

$$\lambda w_0^* = w_b^* + \sum_{t=0}^{T} w^*(t) + \sum_{t=1}^{T} \overline{w}^*(t), \quad w_b^* \in \partial_w g(\tilde{w}) \tag{4.89}$$

Now, by rewriting Eq. (4.89), component-wise we have

$$\lambda x_{t0}^* = x_t^*(t) + x_t^*(t-1) + x_t^*(t+h) + x_t^*(t+h-1) + \overline{x}_t^*(t), \quad t = 0, \ldots, \ T-1-h, \tag{4.90}$$

$$\lambda x_{t0}^* = x_t^*(t) + x_t^*(t-1) + \overline{x}_t^*(t), \quad t = T-h, \ldots, \ T-1, \tag{4.91}$$

$$\lambda x_{t0}^* = x_T^*(T-1) + x_T^*(T) + \overline{x}_T^*(T), \tag{4.92}$$

$$x_{t-h}^*(t) = 0, \quad x_{t-h+1}^*(t) = 0, \ t = T, \quad x_{t0}^* \in \partial_x g(\tilde{x}_t, t), \ t = 1, \ldots, T. \tag{4.93}$$

Then by definition of LAM in the sense of Section 3.6, it follows from the relations in Eqs. (4.90) and (4.91) that

$$\begin{aligned}&(\lambda x_{t0}^* - x_t^*(t+h) - x_t^*(t+h-1) - x_t^*(t-1) - \overline{x}_t^*(t), \ x_{t-h}^*(t), \ x_{t-h+1}^*(t)) \\ &\in F_t^*(-x_{t+1}^*(t); (\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t-h+1}, \tilde{x}_{t+1})), \quad t = 0, \ldots, \ T-1-h,\end{aligned} \tag{4.94}$$

$$\begin{aligned}&(\lambda x_{t0}^* - x_t^*(t-1) - \overline{x}_t^*(t), x_{t-h}^*(t), x_{t-h+1}^*(t)) \\ &\in F_t^*(-x_{t+1}^*(t); (\tilde{x}_t, \tilde{x}_{t-h}, \tilde{x}_{t-h+1}, \tilde{x}_{t+1})), \quad t = T-h, \ldots, \ T-1.\end{aligned} \tag{4.95}$$

Finally, denoting $x_t^* \equiv -x_t^*(t-1)$, $\eta_t^* \equiv x_{t-h}^*(t)$, $\varphi_t^* \equiv x_{t-h+1}^*(t)$, $x^* = x_T^*(T)$ and taking into account Eqs. (4.92–4.95), we obtain conditions (i)–(iii). ■

Remark 4.10. For a convex problem (P_D) under hypothesis H2, the conditions (i)–(iii) are also sufficient for the optimality of the trajectory $\{\tilde{x}_t\}_{t=-h}^T$. This is because the representation (4.5) holds with the parameter $\lambda = 1$.

Let L be any natural number, t_1 a fixed number, and $\delta = h/(L)$. Then we define the points $t_0 + k\delta$, $k = -L, \ldots, T$, $t_0 + (T+1)\delta = t_1$, where the natural T satisfies the inequality $t_0 + T\,\delta \le t_1 < t_0 + (T+1)\delta$ with respect to the problem (P_N), and we associate the discrete-approximating problem:

$$(P_{DA}) \quad \begin{aligned}&\text{infimum} \quad J_\delta\ (x(t)) := \varphi(x(t_1), t_1) + \textstyle\sum_{t=t_0, t_0+\delta, \ldots, t_1-\delta} \delta g(x(t), t) \\ &\text{subject to} \quad \Delta x(t) \in F(x(t), \ \ x(t-h), \ \ \Delta x(t-h), t), \quad t = t_0, \\ &\qquad t_0 + \delta, \ldots, t_1 - \delta, \\ &\qquad x(t) = \xi(t), \ \ t = t_0 - h, \ \ t_0 - h + \delta, \ldots, t_0 - \delta, \\ &\qquad x(t) \in \Phi(t), \ \ t = t_0, \ \ t_0 + \delta, \ldots, t_1, \\ &\qquad x(t_1) \in Q,\end{aligned} \tag{4.96}$$

where $\Delta x(t) = x(t+\delta) - x(t)/(\delta)$.

In order to formulate a sufficient condition for the continuous problem (P_N), it is required to define the form of the adjoint inclusion for it. With this aim we use the discrete inclusion in Eq. (4.96) and Theorem 4.17. Let us write Eq. (4.97) in the so-called neutral discrete form

$$x(t+\delta) \in Q(x(t), x(t-h), x(t-h+\delta), t), \tag{4.97}$$

$$Q(x_1, x_2, x_3, t) = x_1 + \delta F\left(x_1, x_2, \frac{x_3 - x_2}{\delta}, t\right). \tag{4.98}$$

Then it follows from Theorem 4.17 that the adjoint inclusion for the discrete-approximating problem with discrete inclusion (Eq. (4.97)) must be expressed in terms of LAM Q^*. That is why the main problem is the connection between the LAMs Q^* and F^*. This connection is of principal importance for the proposal method in our problem.

Remark 4.11. Comparing the discrete-approximation inclusions in Eq. (4.97) with the discrete inclusions in the problem (P_D) we see that the name "neutral" is justified.

Theorem 4.18. Let $Q(\cdot, t)$ be defined by Eq. (4.98) and $K_{\text{gph } Q(\cdot,t)}(x_1, x_2, x_3, y)$, $(x_1, x_2, x_3, y) \in \text{gph } Q(\cdot, t)$ be a local tent. Then

$$K_{\text{gph } F(\cdot,t)}\left(x_1, x_2, \frac{x_3 - x_2}{\delta}, \frac{y - x_1}{\delta}\right)$$

is a local tent to gph $F(\cdot, t)$ and the following inclusions are equivalent:

1. $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}) \in K_{\text{gph } Q(\cdot,t)}(x_1, x_2, x_3, y)$,
2. $\left(\bar{x}_1, \bar{x}_2, (\bar{x}_3 - \bar{x}_2)/\delta, (\bar{y} - \bar{x}_1)/\delta\right) \in K_{\text{gph } F(\cdot,t)}\left(x_1, x_2, (x_3 - x_2)/\delta, (y - x_1)/\delta\right)$.

□ It follows from the definition of local tent that for functions ri$(\bar{z})$, $i = 0, 1, 2, 3$. $\bar{z} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y})$ possessing the property $r_i(\bar{z})\|\bar{z}\|^{-1} \to 0$, $\bar{z} \to 0$, we have

$$\begin{aligned} &y + \bar{y} + r_0(\bar{z}) \in x_1 + \bar{x}_1 + r_1(\bar{z}) \\ &+ \delta F\left(x_1 + \bar{x}_1 + r_1(\bar{z}), x_2 + x_2 + r_2(\bar{z}), \frac{x_3 + \bar{x}_3 + r_3(\bar{z}) - x_2 - \bar{x}_2 - r_2(\bar{z})}{\delta}, t\right) \end{aligned}$$

for small $\bar{z} \in K \subseteq \text{ri } K_{\text{gph } Q(\cdot,t)}$. Transforming this inclusion, we have

$$\begin{aligned} &\frac{y - x_1}{\delta} + \frac{\bar{y} - \bar{x}_1}{\delta} + \frac{r_0(\bar{z}) - r_1(\bar{z})}{\delta} \\ &\in F\left(x_1 + \bar{x}_1 + r_1(\bar{z}), x_2 + x_2 + r_2(\bar{z}),\ \frac{x_3 + x_3}{\delta} + \frac{\bar{x}_3 - \bar{x}_2}{\delta} + \frac{r_3(\bar{z}) - r_2(\bar{z})}{\delta}, t\right) \end{aligned}$$

Then condition (2) is clear. By going in the reverse direction, we obtain condition (1). ■

Theorem 4.19. Let $K_{\text{gph } Q(\cdot,\, t)}$ be a local tent for the multivalued function $Q_{(\cdot,\, t)}$. Then the following inclusions are equivalent:

1. $(x_1^*, x_2^*, x_3^*) \in Q^*(y^*, (x_1, x_2, x_3, y), t)$,

2. $((x_1^* - y^*)/\delta, (x_2^* + x_3^*)/\delta, x_3^*) \in F^*(y^*; (x_1, x_2, (x_3 - x_2)/\delta,\ (y - x_1)/\delta, t))$.

□ Let us prove (1)⇒(2). By the definition of LAM, the relation (1) means that

$$\langle \bar{x}_1,\ x_1^* \rangle + \langle \bar{x}_2,\ x_2^* \rangle + \langle \bar{x}_3,\ x_3^* \rangle - \langle \bar{y}, y* \rangle \geq 0,$$

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}) \in K_{\text{gph } Q(\cdot,\, t)}(x_1, x_2, x_3, y). \tag{4.99}$$

Because of Theorem 4.18, the inequality in Eq. (4.99) can be rewritten in the form

$$\langle \bar{x}_1, \psi_1^* \rangle + \langle \bar{x}_2, \psi_2^* \rangle + \left\langle \frac{\bar{x}_3 - \bar{x}_2}{\delta}, \psi_3^* \right\rangle - \left\langle \frac{\bar{y} - \bar{x}_1}{\delta}, \psi^* \right\rangle \geq 0 \tag{4.100}$$

where ψ_1^*, ψ_2^*, ψ^* are to be determined. By comparing Eq. (4.99) with Eq. (4.100), it is not hard to see that

$$\psi^* = y^*, \quad \psi_1^* = \frac{x_1^* - y^*}{\delta}, \quad \psi_2^* = \frac{x_2^* + x_3^*}{\delta}.$$

By analogy, the inverse implication (2)⇒(1) can be shown. Now, let us return to the discrete-approximation problem (P_{DA}) with inclusion (4.97). The hypotheses (i)–(iii) of Theorem 4.17 take the following form:

$$\begin{gathered}(x^*(t) - \eta^*(t+h) - \varphi^*(t+h-\delta),\ \ \eta^*(t),\ \ \varphi^*(t)) \\ \in Q^*(x^*(t+\delta),\ \ (\tilde{x}(t),\ \ \tilde{x}(t-h),\ \ \tilde{x}(t-h+\delta),\ \ \tilde{x}(t+\delta)), t), \\ -\{\lambda_\delta\, \partial_x g(\tilde{x}\ \ (t), t) - K^*_{\Phi(t)}(\tilde{x}(t))\} \times \{0\} \times \{0\}, \quad \lambda = \lambda_\delta \in \{0,\ 1\}\end{gathered} \tag{4.101}$$

$$\begin{gathered}\partial_x g(\tilde{x}(t_0),\ \ t_0) = 0,\ \ K^*_{\Phi(t_0)}(\tilde{x}(t_0)) = 0,\ \ t = t_0,\ \ t_0 + \delta, \ldots, t_1 - h - \delta, \\ (x^*(t), \eta^*(t), \varphi^*(t)) \in Q^*(x^*(t+\delta);\ \ (\tilde{x}(t))\tilde{x}(t-h),\ \ \tilde{x}(t-h+\delta),\ \ \tilde{x}(t+\delta)), t) \\ -\{\lambda_\delta\, \partial_x g(\tilde{x}(t), t) - K^*_{\Phi(t)}(\tilde{x}(t))\} \times \{0\} \times \{0\}, \quad t = t_1 - h, \ldots, t_1 - \delta,\end{gathered} \tag{4.102}$$

$$\begin{gathered}x^* - x^*(t_1) \in \lambda_\delta\, \partial_x \varphi(\tilde{x}(t_1) - K^*_{\Phi(t_1)}(\tilde{x}(t_1)), x^* \in K^*_Q(\tilde{x}\ (t_1)), \\ \eta^*(t_1) = 0,\ \ \varphi^*(t_1) = 0\end{gathered} \tag{4.103}$$

Then by using Theorem 4.19, it is easy to see that the conditions in Eqs. (4.101) and (4.102) can be expressed as follows:

$$\left(-\Delta x^*(t)-\frac{\eta^*(t+h)-\varphi^*(t+h)}{\delta}+\Delta\varphi^*(t+h-\delta),\frac{\eta^*(t)-\varphi^*(t)}{\delta},\varphi^*(t)\right)$$
$$\in F(x^*(t+\delta),\tilde{x}(t),\tilde{x}(t-h),\Delta\tilde{x}(t-h),\Delta\tilde{x}(t)),t)$$
$$-\{\lambda_\delta\partial_x g(\tilde{x}(t),t)-K^*_{\Phi(t)}(\tilde{x}(t))\}\times\{0\}\times\{0\},\quad t=t_0,\ \ t_0+\delta,\ldots,t_1-\delta-h, \tag{4.104}$$

$$\left(-\Delta x^*(t),\ \ \frac{\eta^*(t)+\varphi^*(t)}{\delta},\ \ \varphi^*(t)\right)\in F^*(x^*(t+\delta),(\tilde{x}(t),\tilde{x}\,(t-h),\Delta\tilde{x}\,(t-h),$$
$$\Delta\tilde{x}\,(t)),\ \ t)$$
$$-\{\lambda_\delta\partial_x\,g\,(\tilde{x}(t),t)-K^*_{\Phi(t)}(\tilde{x}\ \ (t)\}\times\{0\}\times\{0\},\quad t=t_1-h,\ldots,\ \ t_1-\delta. \tag{4.105}$$

Let us summarize the obtained result. ■

Corollary 4.5. Let assumptions (H1) for the difference neutral-type nonconvex problem (P_{DA}) be satisfied. Then for the optimality of the trajectory $\{\tilde{x}(t)\}_{t=-h}^{t_1}$, it is necessary that there exist a number $\lambda_\delta\in\{0,\ 1\}$ and grid functions $x^*(t)$, $\eta^*(t)$, $\varphi^*(t)$, $t=t_0$, $t_0+\delta$, $\ldots$, t_1, not all equal to zero, satisfying the conditions in Eqs. (4.101)–(4.103). Moreover, let us assume that $\varphi(\cdot)$ and $g(\cdot,t)$ are proper convex functions and continuous at the points of some feasible grid trajectory $\{x^0(t)\}_{t=-h}^{t_1}$. Then under hypothesis H2, the conditions in Eqs. (4.103)–4.105 are also sufficient for optimality in the convex problem (P_{DA}).

It should be noted that setting $\lambda_\delta=1$, denoting the expression $\eta^*(t)+\varphi^*(t)/(\delta)$ by $\zeta^*(t)$ and passing to the formal limit in Eqs. (4.104) and (4.105) when $L\to\infty$ and consequently $\delta\to 0$, we have

$$\begin{aligned}&(-\dot{x}^*(t)-\zeta^*(t+h)+\dot{\varphi}^*(t+h),\ \ \zeta^*(t),\ \ \varphi^*(t))\\ &\in\ F^*(x^*(t),\ \ (\tilde{x}(t),\ \ \tilde{x}(t-h),\ \ \dot{\tilde{x}}(t-h),\ \ \dot{\tilde{x}}(t)),t)\end{aligned} \tag{4.106}$$

$$\begin{aligned}&-\{\partial_x g\,(\tilde{x}(t),t)-K^*_{\Phi(t)}\ \ (\tilde{x}(t)\}\times\{0\}\times\{0\},\quad t\in[t_0,t_1-h),\\ &(-\dot{x}^*(t),\ \zeta^*(t),\ \ \varphi^*(t))\in F^*(x^*(t),(\tilde{x}(t),\tilde{x}(t-h),\ \ \dot{\tilde{x}}(t-h),\ \ \dot{\tilde{x}}(t)),t)\\ &-\{\partial_x g(\tilde{x}(t),t)-K^*_{\Phi(t)}(\tilde{x}(t)\}\times\{0\}\times\{0\},\quad t\in[t_1-h,t_1].\end{aligned} \tag{4.107}$$

Below we will verify that the differential inclusions in Eqs. (4.106) and (4.107) are just the needed adjoint inclusions for convex problem (P_N) with fixed time interval $[t_0,\,t_1]$.

Sufficient Condition for the Optimality of Neutral Differential Inclusions

Let $\tilde{x}(t),\ t\in[t_0-h,t_1],\ \tilde{x}(t)=\xi(t),\ \ t\in[t_0-h,t_0]$ be some feasible solution of the nonconvex problem (P_N). First, let us construct the adjoint differential inclusion for the adjoint variables $\{x^*(\cdot),\zeta^*(\cdot),\varphi^*(\cdot)\}$. Of course, for this we use the approximating method demonstrated above in Section 4.6. Thus, the adjoint inclusions expressed in terms of the LAM consist of the following:

a. $(-\dot{x}^*(t)-\zeta^*(t+h)+\dot{\varphi}^*(t+h),$
$\zeta^*(t),\ \ \varphi^*(t))\in F^*(x^*(t),(\tilde{x}(t),\tilde{x}(t-h),\dot{\tilde{x}}(t-h),\dot{\tilde{x}}(t)),\ t)\quad$ a.e. $t\in[t_0,\ t_1-h]$,

b. $(-\dot{x}^*(t),\ \ \zeta^*(t),\ \ \varphi^*(t))\ \in F^*(x^*(t),\ \ (\tilde{x}(t),\ \ \tilde{x}(t-h),\ \ \dot{\tilde{x}}(t-h),\ \ \dot{\tilde{x}}(t)),\ \ t)$
a.e. $t\in[t_1-h,\ t_1]$, $\varphi^*(t_1)=0$,

c. $\dot{\tilde{x}}(t)\in F\ \ (\tilde{x}(t),\ \ \tilde{x}(t-h),\ \ \dot{\tilde{x}}(t-h),\ \ x^*(t),t)\quad$ a.e. $t\in[t_0,t_1]$.

which should be fulfilled for all $x\in\Phi(t)$.

Here, a feasible solution is understood to be a triplet $\{x^*(\cdot),\zeta^*(\cdot),\varphi^*(\cdot)\}$ satisfying conditions (a) and (b) almost everywhere, such that $\zeta^*(\cdot),\ \varphi^*(\cdot)$ are absolutely continuous and $x^*(\cdot)$ may be represented as a sum of an absolutely continuous function and a function of jumps. Let us denote the points of the jumps and the values of the jumps of $x^*(\cdot)$ by $\tau_i,\ \ t_0<\tau_i<t_1$ and $x_i^*=x^*(\tau_i+0)-x^*(\tau_i-0),\quad i=1,2,\ldots$, respectively.

Let the feasible solution $\tilde{x}(t)$ be t_1-transversal on the set Q; i.e., the condition

$$-\langle x^*(t),\ \ \tilde{x}(t)\rangle\ >W_{Q\cap\Phi(t)}(-x^*(t)),\quad t_0\ \le\ t<t_1.$$

holds for every $t\in[t_0,t_1]$, where $W_{Q\cap\Phi(t)}$ is a support function for $Q\cap\Phi(\mathrm{t})$.

Furthermore, assume that in problem (P_N), the Bolza cost functional $J[x(\cdot),t]$ is monotonically increasing with respect to t (see Section 4.5).

Finally, we can formulate sufficient conditions for optimality in the form of Theorem 4.20.

Theorem 4.20. Let $\tilde{x}(\cdot)$ be any feasible arc of the nonconvex problem (P_N) and let there exist a triplet of feasible solutions $\{x^*(\cdot),\zeta^*(\cdot),\varphi^*(\cdot)\}$ almost everywhere, satisfying the adjoint inclusions (a) and (b) and the condition (c). Moreover, assume that the Bolza cost functional $J[x(t),t]$ is monotonically increasing with respect to t for any feasible arc $x(\cdot)$ of the problem (P_N) and that the following conditions are satisfied:

i. $g(x,t)-g(\tilde{x}(t),t)\ge\langle-x^*(t),x-\tilde{x}(t)\rangle\quad$ a.e. $t\in[t_0,\ t_1]$ and $\forall x\in\Phi(t)$,

ii. $\varphi(x,t_1)-\varphi(\tilde{x}(t_1))\ge\ \langle-x^*(t_1),x-\tilde{x}(t_1)\rangle\quad\forall x\in\Phi(t_1)\cap Q$,

iii. The jumps x_i^* satisfy the jump conditions $\langle\tilde{x}(\tau_i),x_i^*\rangle=W_{\Phi(\tau_i)}(x_i^*),\quad i=1,2,\ldots$

iv. $\tilde{x}(\cdot)$ satisfies the t_1-transversality condition on the set Q.

Then the arc $\tilde{x}(\cdot)$ is optimal.

□ Let $x(t) \in \Phi(t)$, $t \in [t_0, t_1]$ be an arbitrary feasible arc realizing the transition from the interval $[t_0, \theta]$ to the set of target points Q. Let us prove that $J[x(\cdot), \theta] \geq J[\tilde{x}(\cdot), t_1]$. Using the definition of LAM, we can rewrite the inclusions (a) and (b) as follows:

$$\begin{aligned} &H(x(t),\ x(t-h),\ \dot{x}(t-h),\ x^*(t), t) - H(\tilde{x}(t),\ \tilde{x}(t-h),\ \dot{\tilde{x}}(t-h),\ x^*(t), t) \\ &\leq \langle -\dot{x}^*(t) - \zeta^*(t+h) + \dot{\varphi}^*(t+h), x(t) - \tilde{x}(t)\rangle + \langle \zeta^*(t), x(t-h) \\ &- \tilde{x}(t-h)\rangle + \langle \varphi^*(t),\ \dot{x}(t-h) - \dot{\tilde{x}}(t-h)\rangle, \quad t \in [t_0,\ t_1 - h), \end{aligned} \tag{4.108}$$

$$\begin{aligned} &H(x(t),\ x(t-h),\ \dot{x}(t-h),\ x^*(t), t) - H(\tilde{x}(t),\ \tilde{x}(t-h),\ \dot{\tilde{x}}(t-h),\ x^*(t), t) \\ &\leq \langle -\dot{x}^*(t),\ x(t) - \tilde{x}(t)\rangle + \langle \zeta^*(t), x(t-h) - \tilde{x}(t-h)\rangle \\ &+ \langle \varphi^*(t),\ \dot{x}(t-h) - \dot{\tilde{x}}(t-h)\rangle, \quad t \in [t_1 - h,\ t_1], \end{aligned} \tag{4.109}$$

respectively.

On the other hand, by condition (c) and the definition of the Hamiltonian function H, we have

$$H(\tilde{x}(t),\ \tilde{x}(t-h),\ \dot{\tilde{x}}(t-h),\ x^*(t), t) = \langle \dot{\tilde{x}}(t),\ x^*(t)\rangle, \tag{4.110}$$

$$H(x(t),\ x(t-h),\ \dot{x}(t-h),\ x^*(t),\ t) \geq \langle \dot{x}(t),\ x^*(t)\rangle, \quad t \in [t_0, t_1].$$

Let us denote $\Psi(t) = \langle x(t) - \tilde{x}(t),\ -x^*(t)\rangle$. Taking into account Eq. (4.108) to (4.110), we obtain

$$\begin{aligned} \frac{d\Psi(t)}{dt} &\geq \langle x(t) - \tilde{x}(t),\ \zeta^*(t+h)\rangle + \langle x(t) - \tilde{x}(t),\ -\dot{\varphi}^*(t+h)\rangle \\ &+ \langle x(t-h) - \tilde{x}(t-h), -\zeta^*(t)\rangle + \langle \dot{x}(t-h) - \dot{\tilde{x}}(t-h), -\varphi^*(t)\rangle, \quad t \in [t_0, t_1 - h), \end{aligned} \tag{4.111}$$

$$\frac{d\Psi(t)}{dt} \geq \langle x(t-h) - \tilde{x}(t-h), -\zeta^*(t)\rangle + \langle \dot{x}(t-h) - \dot{\tilde{x}}(t-h), -\varphi^*(t)\rangle, \quad t \in [t_1 - h, t_1] \tag{4.112}$$

Now, integrating both sides of Eqs. (4.111) and (4.112) on the intervals $[t_0, t_1 - h)$ and $[t_1 - h, t_1]$, respectively, and adding them we have

$$\begin{aligned} \int_{t_0}^{t_1} \dot{\Psi}(t)dt &\geq \int_{t_0}^{t_1} \langle x(t-h) - \tilde{x}(t-h),\ -\zeta^*(t)\rangle dt + \int_{t_0}^{t_1} \langle \dot{x}(t-h) - \dot{\tilde{x}}(t-h),\ -\varphi^*(t) > dt \\ &+ \int_{t_0}^{t_1 - h} \langle x(t) - \tilde{x}(t),\ -\dot{\varphi}^*(t+h)\rangle dt + \int_{t_0}^{t_1 - h} \langle x(t) - \tilde{x}(t),\ \zeta^*(t+h)\rangle dt. \end{aligned} \tag{4.113}$$

Since

$$\int_{t_0}^{t_1-h} \langle x(t) - \tilde{x}(t), \ \ \zeta^*(t+h)\rangle dt = \int_{t_0+h}^{t_1} \langle x(t-h) - \tilde{x}(t-h), \ \ \zeta^*(t)\rangle dt$$

and

$$\int_{t_0}^{t_0+h} \langle x(t-h) - \tilde{x}(t-h), \ \ \zeta^*(t)\rangle dt = 0 \ \ (x(t) = \tilde{x}(t) = \xi(t) \quad \forall t \in [t_0 - h, t_0)$$

it follows from Eq. (4.113) that

$$\begin{aligned}
\int_{t_0}^{t_1} \dot{\Psi}(t)dt &\geq \int_{t_0}^{t_1} \langle \dot{x}(t-h) - \dot{\tilde{x}}(t-h), \ \ -\varphi^*(t)\rangle dt + \int_{t_0}^{t_1-h} \langle x(t) - \tilde{x}(t), \ \ -\dot{\varphi}^*(t+h)\rangle dt \\
&= \int_{t_0}^{t_1} \langle \dot{x}(t-h) - \dot{\tilde{x}}(t-h), \ \ -\varphi^*(t)\rangle dt + \int_{t_0}^{t_1} \langle x(t-h) - \tilde{x}(t-h), \ \ -\dot{\varphi}^*(t)\rangle dt \\
&\quad + \int_{t_0}^{t_1} \langle x(t-h) - \tilde{x}(t-h), \ \ \dot{\varphi}^*(t)\rangle dt + \int_{t_0+h}^{t_1} \langle x(t-h) - \tilde{x}(t-h), \ \ -\dot{\varphi}^*(t)\rangle dt \\
&= \int_{t_0}^{t_1} d\langle x(t-h) - \tilde{x}(t-h), \ \ -\varphi^*(t)\rangle + \int_{t_0}^{t_0+h} \langle x(t-h) - \tilde{x}(t-h), \ \ \dot{\varphi}^*(t)\rangle dt
\end{aligned}$$

Note that the arcs $x(\cdot)$, $\tilde{x}(\cdot)$ are feasible; i.e., $x(t) = \tilde{x}(t), \ \ t \in [t_0 - h, t_0)$ and $\varphi^*(t_1) = 0$. Therefore,

$$\begin{aligned}
\int_{t_0}^{t_1} \dot{\Psi}(t)dt &\geq \langle x(t_1 - h) - \tilde{x}(t_1 - h), -\varphi^*(t_1)\rangle + \langle x(t_0 - h) - \tilde{x}(t_0 - h), \varphi^*(t_0)\rangle \\
&\quad + \int_{t_0}^{t_0+h} \langle x(t-h) - \tilde{x}(t-h), \ \ \dot{\varphi}^*(t)\rangle dt = 0.
\end{aligned}$$

Consequently,

$$\int_{t_0}^{t_1} \dot{\psi}(t)dt = \langle x(t_1) - \tilde{x}(t_1), \ -x^*(t_1)\rangle \geq 0. \tag{4.114}$$

Moreover, $x(\cdot)$, $\tilde{x}(\cdot)$ are absolutely continuous functions, so $\Psi(\cdot)$ can be represented as a sum of an absolutely continuous function and a jump function (see Section 4.5):

$$\Psi(\theta) = \Psi(t_0) + \int_{t_0}^{\theta} \dot{\Psi}(t)dt + \sum_{i \in N(\theta)} [\Psi(\tau_i + 0) - \Psi(\tau_i - 0)], \quad N(t) = \{i : \tau_i \in [t_0, t]\}. \tag{4.115}$$

Using the jump condition (iii), we can calculate the values of the jumps of $\Psi(t)$ at the points τ_i, $i = 1, 2, \ldots$

$$\Psi(\tau_i + 0) - \Psi(\tau_i - 0) = \langle x(\tau_i) - \tilde{x}(\tau_i), \; -x_i^* \rangle = -\langle x(\tau_i), \quad x_i^* \rangle + W_{\Phi(\tau_i)}(x_i^*).$$

Then from $x(\tau_i) \in \Phi(\tau_i)$, it is obvious that $\langle x(\tau_i), x_i^* \rangle \leq W_{\Phi(\tau_i)}(x_i^*)$, so $\Psi(\tau_i + 0) - \Psi(\tau_i - 0) \geq 0$ for any $\tau_i \in [t_0, \theta]$. Hence, the sum of the jumps in Eq. (4.115) is nonnegative. Then, since the inequality in Eq. (4.114) is correct for any point $t_1 = \theta$, we find that $\Psi(\theta) \geq \Psi(t_0)$; i.e.,

$$\langle x(\theta) - \tilde{x}(\theta), \; -x^*(\theta) \rangle \geq \langle x(t_0) - \tilde{x}(t_0), \; -x^*(t_0) \rangle = 0.$$

Then the t_1-transversality condition implies that

$$\langle x(\theta), \quad -x^*(\theta) \rangle \geq \langle \tilde{x}(\theta), \quad -x^*(\theta) \rangle > W_{Q \cap \Phi(\theta)}(-x^*(\theta)). \tag{4.116}$$

On the other hand, in Section 4.5, we proved that

$$\Delta J \geq \varphi(x(\theta), \theta) + \int_{t_0}^{\theta} g(x(t), t)dt - \varphi(x(t_1), t_1) - \int_{t_0}^{t_1} g(x(t), t)dt, \tag{4.117}$$

where $\Delta J = J[x(\cdot), \theta] - J[\tilde{x}(\cdot), t_1]$ is the increment of the Bolza cost functional J, obtained by the transition from arc $\tilde{x}(\cdot)$ to arc $x(\cdot)$

To prove the optimality of $\tilde{x}(\cdot)$, let us assume the contrary—i.e., for all feasible arcs $x(t)$, $t \in [t_0 - h, \theta]$, $x(t) = \xi(t)$, $t \in [t_0 - h, t_0)$, $x(\theta) \in Q$, the increment $\Delta J < 0$; i.e., $J\,[x(\cdot), \theta] < J\,[\tilde{x}(\cdot), \; t_1]$. Then because of the inequality in Eq. (4.117), $J[x(\cdot), \theta] < J[x(\cdot), t_1]$.

Therefore, from the monotonicity of J, we have $\theta < t_1$. Thus, by the t_1-transversality condition, it follows from Eq. (4.116) that $x(\theta) \notin Q \cap \Phi(\theta)$, so $x(\theta) \notin Q$; i.e., the arc $x(\cdot)$ cannot realize the transition from the interval $[t_0, \theta]$ to the set Q. This means that $\tilde{x}(\cdot)$ is an optimal arc.

Corollary 4.6. Let $F(\cdot, t)$ be convex for all fixed t, and suppose that $F(x, t)$ is a closed set. Then conditions (a) and (b) of Theorem 3.1 can be rewritten in Hamiltonian form:

$$(a_H) \quad (-\dot{x}^*(t) - \zeta^*(t+h) + \dot{\varphi}^*(t+h), \zeta^*(t), \varphi^*(t)) \\ \in \partial_x H(\tilde{x}(t), \tilde{x}(t-h), \dot{\tilde{x}}(t-h), x^*(t), \ t) \quad \text{a.e. } t \in [t_0, \ t_1 - h),$$

$$(b_H) \quad (-\dot{x}^*(t), \zeta^*(t), \varphi^*(t)) \in \partial_x H(\tilde{x}(t), \tilde{x}(t-h), \dot{\tilde{x}}(t-h), x^*(t), t) \\ \text{a.e. } t \in [t_1 - h, t_1], \quad \varphi^*(t_1) = 0,$$

$$\dot{\tilde{x}}(t) \in \partial_{y*} H(\tilde{x}(t), \tilde{x}(t-h), \dot{\tilde{x}}(t-h), x^*(t), t) \quad \text{a.e. } t \in [t_0, t_1].$$

☐ The proof is elementary (see the proof of Corollary 4.5). ■

The following theorem involves another sufficient condition for the optimality of problem (P_N) with a convex structure and a fixed time interval $[t_0, t_1]$.

Theorem 4.21. Let $\tilde{x}(t)$, $t \in [t_0 - h, t_1]$, $\tilde{x}(t) = \xi(t)$, $t \in [t_0 - h, \ t_0)$ be an admissible arc for the convex problem (P_N) with a fixed time interval. Furthermore, assume that the triplet $\{x^*(\cdot), \zeta^*(\cdot), \varphi^*(\cdot)\}$ satisfies the adjoint differential inclusions in Eqs. (4.106) and (4.107), where

$$x^* - x^*(t_1) \in \partial\varphi(\tilde{x}(t_1)) - K^*_{\Phi(t_1)}(\tilde{x}(t_1)), \quad x^* \in K^*_Q(\tilde{x}(t_1)), \quad \varphi^*(t_1) = 0.$$

Then the arc $\tilde{x}(\cdot)$ is optimal.

☐ Using the definition of LAM and Eq. (4.110), it is easy to see that

$$\begin{aligned} &\langle \dot{x}(t) - \dot{\tilde{x}}(t), x^*(t) \rangle - g(x(t), t) + g(\tilde{x}(t), t) \\ &\le \langle -\dot{x}^*(t) - \zeta^*(t+h) + \dot{\varphi}^*(t+h), x(t) - \tilde{x}(t) \rangle + \langle \zeta^*(t), x(t-h) - \tilde{x}(t-h) \rangle \\ &\quad - \langle \overline{x}^*(t), x(t) - \tilde{x}(t) \rangle + \langle \varphi^*(t), \dot{x}(t-h) - \dot{\tilde{x}}(t-h) \rangle, \quad t \in [t_0, t_1 - h), \end{aligned}$$

$$\begin{aligned} &\langle \dot{x}(t) - \dot{\tilde{x}}(t), x^*(t) \rangle - g(x(t), t) + g(\tilde{x}(t), t) \le \langle -\dot{x}^*(t), x(t) - \tilde{x}(t) \rangle \\ &+ \langle \zeta^*(t), x(t-h) - \tilde{x}(t-h) \rangle - \langle \overline{x}^*(t), x(t) - \tilde{x}(t) \rangle + \langle \varphi^*(t), \dot{x}(t-h) - \dot{\tilde{x}}(t-h) \rangle, \\ &\qquad t \in [t_1 - h, t_1], \\ &\overline{x}^*(t) \in K^*_{\Phi(t)}(x^*(t)), \quad t \in [t_0, t_1]. \end{aligned}$$

Now as a result of simple calculations and the definition of the adjoint cone from these inequalities, we have

$$\begin{aligned} &\frac{d}{dt} \langle x(t) - \tilde{x}(t), x^*(t) \rangle + \langle \zeta^*(t+h) - \dot{\varphi}^*(t+h), x(t) - \tilde{x}(t) \rangle \\ &- \langle \zeta^*(t), x(t-h) - \tilde{x}(t-h) \rangle - \langle \varphi^*(t), \dot{x}(t-h) - \dot{\tilde{x}}(t-h) \rangle \\ &\quad \le g(x(t), t) - g(\tilde{x}(t), t), \quad t \in [t_0, \ t_1 - h), \end{aligned} \tag{4.118}$$

$$\frac{d}{dt}\langle x(t)-\tilde{x}(t),x^*(t)\rangle-\langle\zeta^*(t),x(t-h)-\tilde{x}(t-h)\rangle$$
$$-\langle\varphi^*(t),\dot{x}(t-h)-\dot{\tilde{x}}(t-h)\rangle\leq g\,(x(t),t)-g\,\,(\tilde{x}\,(t),t),\quad t\in[t_1-h,t_1].$$

By integrating both sides of Eq. (4.118) on $[t_0, t_1-h)$ and $[t_1-h, t_1]$, respectively, it can be shown as in the inequality in Eq. (4.113) that

$$\langle x^*(t_1),\;\; x(t_1)-\tilde{x}(t_1)\rangle\leq\int_{t_0}^{t_1}[g(x(t),t)-g(\tilde{x}(t),t)]dt. \tag{4.119}$$

On the other hand, the hypothesis of the theorem at $t=t_1$ implies that

$$\langle x^*-x^*(t_1)+\overline{x}^*(t_1),x(t_1)-\tilde{x}(t_1)\rangle\leq\varphi(x(t_1))-\varphi(\tilde{x}(t_1)). \tag{4.120}$$

Adding the inequalities in Eqs. (4.119) and (4.120), we see that $J(x(\cdot))\geq J(\tilde{x}(\cdot))$ for all feasible arcs $x(\cdot)$. The proof is complete. ■

Remark 4.12. Let one of the variables, for example, the second variable of the multivalued function $F(\cdot,\cdot,\cdot,t)$ be missing; i.e., $F(x_1, x_2, x_3, t)\equiv F(x_1, x_3, t)$. Then it can be shown from the definition of LAM that $x_2^*\equiv 0$. This means that in the adjoint differential inclusions (a) and (b) of Theorem 4.20 and in Eqs. (4.106) and (4.107), $\zeta^*(t)\equiv 0$, $t\in[t_0, t_1]$ identically.

Now let us investigate the optimality conditions for an important type of differential inclusions in the Hale form

$$\frac{d}{dt}[x(t)-Ax(t-h)]\in F(x(t),x(t-h),t). \tag{4.121}$$

First, we prove Lemma 4.8.

Lemma 4.8. Let $P(x_1, x_2, x_3, t)=F(x_1, x_2, t)+Ax_3$, where A is an $n\times n$ matrix and the cone of tangent directions $K_{\text{gph }P(\cdot,t)}(x_1, x_2, x_3, y)$, $(x_1, x_2, x_3, y)\in\text{gph }P(\cdot,t)$ is a local tent. Then

$$K_{\text{gph }F(\cdot,t)}(x_1,x_2,y-Ax_3,t)$$

is a local tent to gph $F(\cdot,t)$ and the following inclusions are equivalent

1. $(\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{y})\in K_{\text{gph }P(\cdot,t)}(x_1,x_2,x_3,y)$,
2. $(\overline{x}_1,\overline{x}_2,\overline{y}-A\overline{x}_3)\in K_{\text{gph }F(\cdot,t)}(x_1,x_2,y-Ax_3,t)$.

□ Let us prove (1)⇒(2). By the definition of a local tent, (1) means that there exist functions

$$r_i(\overline{z}),\quad i=0,1,2,3,\quad \overline{z}=(\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{y})\text{ such that }r_i(\overline{z})\left\|\overline{z}\right\|^{-1}\to 0\ \text{ as }\overline{z}\to 0,$$

and

$$y+\overline{y}+r_0(\overline{z}) \in P(x_1+\overline{x}_1+r_1(\overline{z}),\ \ x_2+\overline{x}_2+r_2(\overline{z}),\ \ x_3+\overline{x}_3+r_3(\overline{z}),\ t)$$

or

$$y+\overline{y}+r_0(\overline{z}) \in F(x_1+\overline{x}_1+r_1(\overline{z}),\ \ x_2+\overline{x}_2+r_2(\overline{z}),t)+A(x_3+\overline{x}_3+r_3(\overline{z})) \tag{4.122}$$

or sufficiently small $\overline{z} \in K \subseteq \text{ri}\, K_{\text{gph}\ P(\cdot,\ t)}(x_1,x_2,x_3,y)$. On the other hand,

$$y+\overline{y}+r_0(\overline{z})-A(x_3+\overline{x}_3+r_3(\overline{z}))=y-Ax_3+\overline{y}-A\overline{x}_3+r_0(\overline{z})-Ar_3(\overline{z}) \tag{4.123}$$

and

$$[r_0(\overline{z})-Ar_3(\overline{z})]\left\|\overline{z}\right\|^{-1}\to 0 \quad \text{whenever } \overline{z}\to 0.$$

Then because of Eq. (4.123), the validity of inclusion (2) is justified through Eq. (4.122). In just the same way, the reverse direction can be proved; i.e., (2)$\Rightarrow$(1). This ends the proof of the lemma. ■

In Theorem 4.22, A^* is the adjoint (transposed) matrix of A.

Theorem 4.22. Let $K_{\text{gph}\ P}(\cdot,t)$ be a local tent for the multivalued function $P_{(\cdot,\ t)}$. Then for the LAMs P^* and F^*, the following inclusions are equivalent:

1. $(x_1^*,x_2^*,x_3^*) \in P^*\ \ (y^*;(x_1,x_2,x_3,y),t)$,
2. $(x_1^*,x_2^*) \in F^*(y^*;(x_1,x_2,y-Ax_3),t)$,

where $Ay^*=x_3^*$.

□ Let (1) be given and let us prove that (1)$\Rightarrow$(2). By the definition of LAM, (1) implies the inequality in Eq. (4.99) for $(\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{y}) \in K_{\text{gph}\ P(\cdot,\ t)}\ (x_1,x_2,x_3,y)$.

Then using Lemma 4.8, we rewrite the equality in Eq. (4.99) in the form

$$\langle\overline{x}_1,\alpha_1\rangle+\langle\overline{x}_2,\alpha_2\rangle-\langle\overline{y}-Ax_3,\alpha\rangle\geq 0,$$

$$(\overline{x}_1,\overline{x}_2,\overline{y}-A\overline{x}_3) \in K_{\text{gph}\ F(\cdot,\ t)}(x_1,x_2,y-Ax_3) \tag{4.124}$$

where α_1, α_2, α are to be determined. Finally from Eq. (4.99), represented for $K_{\text{gph}\ P(\cdot,\ t)}$, and Eq. (4.124) it follows that $x_i^*=\alpha_i(i=1,2)$, $\ y^*=\alpha$, and $x_3^*=A^*y^*$.

By analogy it is easy to conclude that (2)$\Rightarrow$(1). The proof is completed. ■

Suppose now that the differential inclusion in the problem (P_N) has the Hale form (Eq. (4.121)). Clearly, this inclusion is equivalent to the inclusion

$$\dot{x}(t) \in P(x(t),x(t-h),\dot{x}(t-h),t) \tag{4.125}$$

where $P(x_1, x_2, x_3, t)$ is defined as in Lemma 4.8.

Let us return to conditions (a) and (b) of Theorem 4.20. If conditions (a) and (b) of Theorem 4.20 rewritten in terms of the LAM P^* are satisfied, then the transition to the LAM F^* is required. In turn, we use Theorem 4.22, in which the condition $Ay^* = x_3^*$ means that $A^*x^*(t) = \varphi^*(t)$, $t \in [t_0, t_1]$. Consequently, conditions (a) and (b) of Theorem 4.20 are transformed as follows:

$$\begin{aligned}&(\mathrm{a}') \quad (-\dot{x}^*(t) - \zeta^*(t+h) + A^*\dot{x}^*(t+h), \;\; \zeta^*(t)) \\ &\in F^*(x^*(t), (\tilde{x}(t), \tilde{x}(t-h)), \dot{\tilde{x}}(t) - A\dot{\tilde{x}}(t-h), t), \quad \text{a.e. } t \in [t_0, t_1 - h),\end{aligned}$$

$$\begin{aligned}&(\mathrm{b}') \quad (-\dot{x}^*(t), \zeta^*(t)) \in F^*(x^*(t), (\tilde{x}(t), \tilde{x}(t-h), \dot{\tilde{x}}(t) - A\dot{\tilde{x}}(t-h), t), \\ &\qquad \text{a.e. } t \in [t_1 - h, t_1].\end{aligned}$$

As for condition (c), we notice the following. Let us denote the Hamiltonian function of multivalued functions F and P by H_F and H_P, respectively. It is easy to see that

$$H_P(x_1, x_2, x_3; y^*, t) = \langle x_3, A^*y^* \rangle + H_F(x_1, x_2; y^*, t)$$

so the Argmaximum sets coincide:

$$\begin{aligned}P(x_1, x_2, x_3, y^*, t) &= \{y \in P(x_1, x_2, x_3, t) : \langle y, y^* \rangle = H_P(x_1, x_2, x_3, y^*, t)\} \\ &= \{y_1 \in F(x_1, x_2, t) : \langle y_1, y^* \rangle = H_F(x_1, x_2, y^*, t)\} = F(x_1, x_2, y^*, t), y_1 = y - Ax_3.\end{aligned} \tag{4.126}$$

Using Eq. (4.126), we ensure that condition (c) of Theorem 4.20 is reduced to

$$(\mathrm{c}') \quad \frac{d}{dt}[\tilde{x}(t) - A\tilde{x}(t-h) \in F(x(t), \;\; x(t-h), x^*(t), t), \quad \text{a.e. } t \in [t_0, t_1].$$

Note that (a′)–(c′) should be fulfilled for all $x \in \Phi(t)$.

Corollary 4.7. Let $\tilde{x}(\cdot)$ be any feasible trajectory of the problem (P_N) with differential inclusion of the Hale form (Eq. (4.121)), and let there exist a pair of feasible solutions $\{x^*(\cdot), \zeta^*(\cdot)\}$ of the adjoint inclusion, satisfying the conditions (a′)–(c′). Moreover, let the remaining conditions of Theorem 4.20 be fulfilled. Then $\tilde{x}(\cdot)$ is optimal.

Sufficient Condition for Optimality in Problem (P_V)

Let us return to the problem (P_V) and formulate a theorem for the sufficiency of the optimality of an arc $\tilde{x}(\cdot)$. Moreover, let $t = r(\tau)$ be a solution of the equation $\tau = t - h(t)$ or the inverse function of τ.

Theorem 4.23. Assume $\tilde{x}(\cdot)$ to be some feasible solution of the problem (P_V) and the pair of functions $\{x^*(\cdot),\zeta^*(\cdot)\}$ that satisfy the following conditions almost everywhere

a. $(-\dot{x}^*(t)-\dot{r}(t)\zeta^*(r(t)),\ \zeta^*(t)) \in F^*(x^*(t);(\tilde{x}(t),\ \tilde{x}(t-h(t),\ \dot{\tilde{x}}(t)),\ t)$ a.e. $t\in[t_0,t_1-h(t_1)]$,

b. $(-\dot{x}^*(t),\zeta^*(t)) \in F^*(x^*(t);(\tilde{x}(t),\tilde{x}(t-h(t)),\dot{\tilde{x}}(t),t)$ a.e. $t\in[t_1-h(t_1),t_1]$,

c. $\dot{\tilde{x}}(t)\in F(\tilde{x}(t),\tilde{x}(t-h(t)),x^*(t),t)$ a.e. $t\in[t_0,t_1]$

where $\zeta^*(\cdot)$ is absolutely continuous and $x^*(\cdot)$ is a sum of an absolutely continuous function and jump function. Then under conditions (i)–(iv) of Theorem 4.20, the arc $\tilde{x}(\cdot)$ is optimal in problem (P_V).

□ We recall that one of the distinctive places in the proof of Theorem 4.20 is the inequalities in Eqs. (4.111) and (4.112). As mentioned in Remark 4.12, in the present case $F(x_1,x_2,x_3,t)\equiv F(x_1,x_2,t)$, so $x_3^*\equiv 0$. Therefore, $\varphi^*(t)\equiv 0$, $t\in[t_0,t_1]$. Thus, by analogy to Eqs. (4.111) and (4.112), the following inequalities hold:

$$\frac{d\Psi(t)}{dt}\geq\langle x(t)-\tilde{x}(t),\dot{r}(t)\zeta^*(r(t))\rangle-\langle x(t-h(t))-\tilde{x}(t-h(t)),\zeta^*(t)\rangle,\ \ t\in[t_0,t_1-h(t_1)), \tag{4.127}$$

$$\frac{d\Psi(t)}{dt}\geq-\langle x(t-h(t))-\tilde{x}(t-h(t)),\zeta^*(t)\rangle,\ \ t\in[t_1-h(t_1),t_1]. \tag{4.128}$$

In addition to these, it is not hard to see that

$$\int_{t_0}^{t_1}\langle x(t-h(t))-\tilde{x}(t-h(t)),\zeta^*(t)\rangle dt=\int_{t_0}^{t_1-h(t_1)}\langle x(t)-\tilde{x}(t),\zeta^*(r(t))\rangle\dot{r}(t)dt \tag{4.129}$$

Adding the integrated inequalities in Eqs. (4.127) and (4.128) and taking into account Eq. (4.129), we have the same inequality (4.114). Moreover, based on the inequality in Eq. (4.114) in a similar way, the validity of the theorem can be shown. ■

Now, let us illustrate the suggested method in some simple examples.

Example 4.4. Let us consider the neutral-type problem (P_N), where

$$\dot{x}(t)=Ax(t)+A_1x(t-h)+A_2\dot{x}(t-h)+Bu(t),\quad u(t)\in U. \tag{4.130}$$

Here A, A_1, A_2 are $n\times n$ matrices, B is an $n\times r$ matrix, and $U\subset\mathbb{R}^r$ is a convex set. Obviously, Eq. (4.130) can be reduced to problem (P_N) with differential inclusion:

$$\dot{x}(t)\in F(x(t),x(t-h),\dot{x}(t-h)), \tag{4.131}$$

$$F(x_1, x_2, x_3) = Ax_1 + Ax_2 + Ax_3 + BU.$$

It is easy to see that

$$H(x_1, x_2, x_3, y^*) = \langle x_1, A^* y^* \rangle + \langle x_2, A_1^* y^* \rangle + \langle x_3, A_2^* y^* \rangle + \sup_{u \in U} \langle Bu, y^* \rangle$$

so

$$F^*(y^*; (x_1, x_2, x_3, y)) = \begin{cases} (A^* y^*, A_1^* y^*, A_2^* y^*), & \text{if } -B^* y^* \in K_U^*(\tilde{u}), \\ \varnothing, & \text{if } -B^* y^* \notin K_U^*(\tilde{u}), \end{cases}$$

where A^* is the adjoint (transposed) matrix of A.

Then by the adjoint differential inclusions (a) and (b) of Theorem 4.20,

$$\begin{gathered} -\dot{x}^*(t) - \zeta^*(t+h) + \dot{\varphi}^*(t+h) = A^* x^*(t), \\ \zeta^*(t) = A_1^* x^*(t), \quad \varphi^*(t) = A_2^* x^*(t), \quad t \in [t_0, \ t_1 - h), \\ -\dot{x}^*(t) = A^* x^*(t), \quad t \in [t_1 - h, t_1], \end{gathered}$$

so

$$\dot{x}^*(t) = -A^* x^*(t) - A_1^* x^*(t+h) + A_2^* \dot{x}^*(t+h), \quad t \in [t_0, t_1 - h),$$

$$\dot{x}^*(t) = -A^* x^*(t), \quad t \in [t_1 - h, t_1]. \tag{4.132}$$

Besides, taking into account the conditions $-B^* y^* \in K_U^*(\tilde{u})$ and (c) of Theorem 4.20, we have

$$\langle x^*(t), B\tilde{u}(t) \rangle = \sup_{u \in U} \langle x^*(t), Bu(t) \rangle \tag{4.133}$$

where $\tilde{u}(\cdot)$ is a controlling parameter corresponding to $\tilde{x}(\cdot)$. Thus, the conditions (a)–(c) of Theorem 4.20 for problem (P_N) with Eqs. (4.130) and (4.131) consist of the conditions in Eqs. (4.132) and (4.133). Obviously, Eq. (4.130) or (4.131) can be rewritten in the form

$$\frac{d}{dt}[x(t) - A_2 x(t-h)] \in F_1(x(t), x(t-h)) \quad \text{a.e. } t \in [t_0, t_1] \tag{4.134}$$

where $F_1(x_1, x_2) = Ax_1 + A_1 x_2 + BU$. Let us write out the conditions (a′) − (c′) of Corollary 4.7. First of all, it is perfectly clear that the problem (P_N) with Eqs. (4.131) and (4.134) has the same sufficient conditions. In fact, one can easily check that

$$F_1^*(y^*; (x_1, x_2, y - A_2 x_3)) = \begin{cases} (A^* y^*, A_1^* y^*), & \text{if } -B^* y^* \in K_U^*(\tilde{u}), \\ \varnothing, & \text{if } -B^* y^* \notin K_U^*(\tilde{u}). \end{cases}$$

Then conditions (a′) − (c′) of Corollary 4.7 have the form

$$-\dot{x}^*(t)-\zeta^*(t+h)+A_2^*\dot{x}^*(t+h)=A^*x^*(t),\quad \zeta^*(t)=A_1^*x^*(t)\ \text{a.e.}\ t\in[t_0,t_1-h),$$

$$\dot{x}^*(t)=-A^*x^*(t),\quad \zeta^*(t)=A_1^*x^*(t)\ \text{a.e.}\ t\in[t_1-h,t_1]$$

or

$$\begin{aligned}&\frac{d}{dt}\left[x^*(t)-A_2^*x^*(t+h)\right]=-A^*x^*(t)-A_1^*x^*(t+h)\quad \text{a.e.}\ t\in[t_0,t_1-h)\\&\dot{x}^*(t)=-A^*x^*(t)\quad \text{a.e.}\ t\in[t_1-h,t_1]\end{aligned}\tag{4.135}$$

which is nothing but Eq. (4.132) and in the present case the condition (c′) simply consists of Eq. (4.133). Note that the adjoint inclusion in Eq. (4.135) and the adjoint inclusion of Theorem 6.1 of Ref. [205] coincide.

Example 4.5. Now let us consider a problem (P_V) with a differential equation and variable delay:

$$\dot{x}(t)=Ax(t)+A_1x(t-h(t))+Bu(t),\quad u(t)\in V.\tag{4.136}$$

A, A_1 are $n\times n$ matrices, B is an $n\times r$ matrix, and $\mathrm{V}\subset\mathbb{R}^r$ is a convex set. Let us replace Eq. (4.10) with the differential inclusion:

$$\dot{x}(t)\in F(x(t),x(t-h(t)),\tag{4.137}$$

$$F(x_1,x_2)=Ax_1+A_1x_2+BV.$$

By a similar calculation to Example 4.4, we can conclude that the conditions (a) and (b) of Theorem 4.23 are transformed in

$$\begin{gathered}-\dot{x}^*(t)-\zeta^*(r(t))\dot{r}(t)=A^*x^*(t),\quad t\in[t_0,t_1-h(t_1)),\\ \zeta^*(t)=A_1^*x^*(t),\\ -\dot{x}^*(t)=A^*x^*(t),\quad t\in[t_1-h(t_1),t_1],\end{gathered}\tag{4.138}$$

respectively.

Finally, Eq. (4.138) can be rewritten as follows:

$$\dot{x}^*(t)=-A^*x^*(t)-A_1^*x^*(r(t))\dot{r}\ \ (t),\quad t\in[t_0,t_1-h(t_1)],\tag{4.139}$$

$$\dot{x}^*(t)=-A^*x^*(t),\quad t\in[t_1-h(t_1),t_1].\tag{4.140}$$

That is, conditions (a), (b), and (c) of Theorem 4.23 for this problem are the conditions in Eqs. (4.139), (4.140), and (4.133), respectively. Thus, Examples 4.4 and 4.5 show that for concrete problems, the conditions of the proved theorems and well-known results of classical optimal control theory coincide. Similarly, consult Mordukhovich [188] or Gabasov and Kirillova [94] for more detailed information.

4.7 Optimal Control of HODI of Bolza Type with Varying Time Interval

In this section, a sufficient condition is formulated for problems with HODI, under the t_1-transversality condition. In particular, the time optimal control problem is studied. LAMs are used for both the convex and the nonconvex cases. Furthermore, the application of these results is demonstrated by solving one example.

Agarwal and O'Regan [1] presented new fixed-point theorems for weakly sequentially upper semicontinuous maps. These results were then used to establish existence principles for second-order differential equations and inclusions.

Auslender and Mechler [23] give necessary and sufficient conditions for the existence of a solution to second-order differential inclusions with state constraints. Furthermore, they introduced and studied second-order interior tangent sets to obtain such conditions.

Benchohra and Ntouyas [31,34], Benchohra et al. [32,33] investigated the existence of solutions for initial and boundary value problems for second-order impulsive functional-differential inclusions in Banach spaces.

In the next section, we consider the following optimization problem for HODI on the nonfixed time interval $[t_0, t_1]$:

$$\text{infimum } J[x(\cdot),t_1] := \int_{t_0}^{t_1} g(x(t),x'(t),\ldots,x^{(s-1)}(t),t)dt + \varphi(x(t_1),x'(t_1),\ldots,x^{(s-1)}(t_1),t_1) \tag{4.141}$$

$$(P_H) \quad \text{subject to } L_s x(t) \in F(x(t),t), \quad t \in [t_0,t_1], \tag{4.142}$$

and

$$x(t_0) = x_0, x'(t_0) = x_1, \ldots, x^{(s-1)}(t_0) = x_{s-1}, \tag{4.143}$$

$$x(t_1) \in Q_0, x'(t_1) \in Q_1, \ldots, x^{(s-1)}(t_1) \in Q_{s-1}, \tag{4.144}$$

where $F : \mathbb{R}^n \times [t_0, t_1] \to P(R^n)$ is a multivalued function and L_s is an sth-order differential expression:

$$L_s x = p_0 \frac{d^s x}{dt^s} + p_1 \frac{d^{s-1} x}{dt^{s-1}} + \cdots + p_{s-1} \frac{dx}{dt} + p_s x$$

with p_i, $i = 0, 1, \ldots, s$ some real constants. The initial time moment t_0 is fixed. We label this problem as (P_H). The problem is to find a solution $\tilde{x}(\cdot)$ satisfying almost everywhere the HODI in Eq. (4.142) on the time interval $[t_0, t_1]$ and with the initial and final state conditions in Eqs. (4.143) and (4.144), which minimizes $J[x(\cdot), t_1]$. Here, a feasible solution $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[t_0, t_1]$ together with the higher order derivatives up until $s-1$, for which $x^{(s)}(\cdot) \in L_1^n$. Observe that such a class of functions $W_{1,s}^n([t_0, t_1])$ is a Banach space, endowed with different equivalent norms. For instance,

$$\|x(\cdot)\| = \sum_{k=0}^{s-1} |x^{(k)}(t_0)| + \|x^{(s)}(\cdot)\|_1 \quad \text{or} \quad \|x(\cdot)\| = \sum_{k=0}^{s} \|x^{(k)}(\cdot)\|_1$$

$$\text{where } \|x^{(k)}(\cdot)\|_p = \left(\int_{t_0}^{t_1} |x^{(k)}(t)|^p \right)^{\frac{1}{p}},$$

$1 \le p < \infty$ and $|x|$ is a Euclidean norm in $\mathbb{R}^n$.

Note that there are clear relations

$$x^{(k-1)}(t_1) = \int_{t_0}^{t_1} x^{(k)}(t)dt + x^{(k-1)}(t_0), \quad k = 1, 2, \ldots, s$$

and if $\tilde{x}(\cdot)$ is another feasible solution of problem (P_H), then

$$L_{k-1}(x(t_1) - \tilde{x}(t_1)) = \int_{t_0}^{t_1} L_k(x(t) - \tilde{x}(t))dt, \quad k = 1, 2, \ldots, s.$$

We say that the solution $x(\cdot)$ of the HODI in Eq. (4.142) transfers the state from the initial point $(x_0, x_1, \ldots, x_{s-1})$ to the final set $Q = Q_0 \times \cdots \times Q_{s-1}$ on the time interval $[t_0, t_1]$ if the boundary condition in Eqs. (4.143) and (4.144) holds.

Let $W_A(\cdot)$ be a support function for a set $A \in P(R^n)$ as before; i.e., $W_A(x^*) = \sup_{x \in A} \langle x^*, x \rangle$.

Proposition 4.2. Let Q be the Cartesian product of the sets $Q_1 \in P(R^n)$, $i = 0, 1, \ldots, s-1$; i.e., $Q = Q_0 \times Q_1 \times \cdots \times Q_{s-1} \subset (R^n)^s$.

Then the equality $W_Q(x^*) = \sum_{i=0}^{s-1} W_{Q_i}(x_i^*)$, $x^* = (x_0^*, \ldots, x_{s-1}^*) \in (R^n)^s$ holds.

☐ Indeed we have

$$W_Q(x^*) = \sup_{x \in Q} \langle x^*, x \rangle = \sup_{x_i \in Q_i} \left\{ \sum_{i=0}^{s-1} \langle x_i^*, x_i \rangle \right\} = \sum_{i=0}^{s-1} W_{Q_i}(x_i^*). \blacksquare$$

Let $\tilde{x}(\cdot)$ be a feasible solution of the problem (P_H). For a given adjoint trajectory $x^*(\cdot)$ on $[t_0, t_1]$, let us construct the following higher-order adjoint differential inclusion (HOADI):

i. $L_s^* x^*(t) \in F^*(x^*(t), (\tilde{x}(t), L_s\tilde{x}(t)), t), \quad t \in [t_0, t_1]$,

ii. $L_s\tilde{x}(t) \in F(\tilde{x}(t), x^*(t), t), \quad t \in [t_0, t_1]$,

where L_s^* is the adjoint differential expression; i.e.,

$$L_s^* x^* = (-1)^s \frac{d^s(p_0 x^*)}{dt^s} + (-1)^{-1} \frac{d^{s-1}(p_1 x^*)}{dt^{s-1}} + \cdots + p_s x^*. \tag{4.145}$$

Here, an adjoint trajectory $x^*(\cdot)$ satisfies the HOADI almost everywhere on $[t_0, t_1]$.

Definition 4.6. Let $\tilde{x}(\cdot)$ be a feasible solution of the problem (P_H) and suppose that $\tilde{x}(\cdot)$ is an adjoint trajectory satisfying HOADI. We say that there is a t_1-transversality condition on the final state set $Q = Q_0 \times Q_1 \times \cdots \times Q_{s-1}$ if the inequality

$$-\sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(t), \tilde{x}^{(k)}(t)\rangle > \sum_{k=0}^{s-1} W_{Q_k}(-L_{s-k-1}^* x^*(t))$$

holds for $t_0 \le t < t_1$.

Let us introduce the following s-dimensional vector-function and differential expression:

$$z(t) = (x(t), x'(t), \ldots, x^{(s-1)}(t)),$$

$$L^* x^*(t) = (L_0^* x^*(t), L_1^* x^*(t), \ldots, L_{s-1}^* x^*(t)), \quad L_0^* x^*(t) = p_0 x^*(t),$$

respectively.

Then by Proposition 4.2, it follows from t_1-transversality that the inequality

$$-\langle L^* x^*(t), \tilde{z}(t)\rangle > W_Q(-L^* x^*(t)), \quad t_0 \le t < t_1$$

holds. In other words, the t_1-transversality condition guarantees that the point $\tilde{z}(t_1)$ belongs to the set Q only at the moment $t = t_1$ or there is at least one k such that $\tilde{x}^{(k)}(t) \notin Q_k$ for all $t \in [t_0, t_1)$.

Furthermore, assume that the Bolza cost functional $J[x(\cdot), t]$ is monotone with respect to the argument t; that is to say, the inequality $J[x(\cdot), \theta_1] < J[x(\cdot), \theta_2]$ holds for any $\theta_1, \theta_2 \in [t_0, t_1]$ $(\theta_1 < \theta_2)$ and for all feasible arcs $x(\cdot)$ of the problem (P_H).

Theorem 4.24. Let $\tilde{x}(\cdot)$ be a feasible arc of the HODI, which transfers the state from the point $(x_0, x_1, \ldots, x_{s-1})$ to the set Q on the time interval $[t_0, t_1]$. Assume that there exists an absolutely continuous adjoint function $x^*(\cdot)$ on the time interval $[t_0, t_1]$ of the HOADI (i) and (ii), together with its higher-order derivatives up until

$s-1$, for which $x^{*(s)}(\cdot) \in L_1^n$. Furthermore, suppose that $J[x(\cdot), t]$ is monotonic increasing with respect to t for any feasible arc of the HODI in Eq. (4.142) and that

a. $g(x, y_1, \ldots, y_{s-1}, t) - g(\tilde{x}(t), \tilde{x}'(t), \ldots, \tilde{x}^{(s-1)}, t)$
$\geq \langle -L_{s-1}^* x^*(t), x - \tilde{x}(t)\rangle + \langle -L_{s-2}^* x^*(t), y_1 - \tilde{x}'(t)\rangle + \cdots + \langle -p_0 x^*(t), y_{s-1} - \tilde{x}^{(s-1)}(t)\rangle$
$\forall x, y_i \in R^n, \cdots i = 1, \ldots, s-1,$ a.e. $t \in [t_0, t_1]$,

b. $\varphi(x, y_1, \ldots, y_{s-1}, t_1) - \varphi(\tilde{x}(t_1), \tilde{x}'(t_1), \ldots, \tilde{x}^{s-1}(t_1), t_1)$
$\geq \langle -L_{s-1}^* x^*(t_1), x - \tilde{x}(t_1)\rangle + \langle -L_{s-2}^* x^*(t_1), y_1 - \tilde{x}'(t_1)\rangle + \cdots + \langle -p_0 x^*(t_1), y_{s-1}$
$- \tilde{x}^{(s-1)}(t_1)\rangle, x \in Q_0, y_i \in Q_i, \quad i = 1, \ldots, s-1$

c. t_1-transversality condition on $Q = Q_0 \times \cdots \times Q_{s-1}$.

Then the arc $\tilde{x}(\cdot)$is optimal.

☐ Let $x(\cdot)$ be a feasible arc realizing the transition from the interval $[t_0, t^*]$ to the set $Q = Q_0 \times \cdots \times Q_{s-1}$. Let us show that $J[x(\cdot), t^*] \geq J[\tilde{x}(\cdot), t_1]$. We can rewrite condition (i) as follows:

$$H(x(t), x^*(t), t) - H(\tilde{x}(t), x^*(t), t) \leq \langle L_s^* x^*(t), x(t) - \tilde{x}(t)\rangle \tag{4.146}$$

By condition (ii),

$$H(\tilde{x}(t), x^*(t), t) = \langle L_s \tilde{x}(t), x^*(t)\rangle \tag{4.147}$$

and by the definition of H,

$$H(x(t), x^*(t), t) \geq \langle L_s x(t), x^*(t)\rangle \tag{4.148}$$

Taking into account Eqs. (4.147) and (4.148) in (4.146), we have

$$\langle L_s(x(t) - \tilde{x}(t)), x^*(t)\rangle \leq \langle L_s^* x^*(t), x(t) - \tilde{x}(t)\rangle. \tag{4.149}$$

Now it is easy to see that

$$\begin{aligned} &\left\langle \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, x^*(t)\right\rangle - \left\langle (-1)^k \frac{d^k x^*(t)}{dt^k}, x(t) - \tilde{x}(t)\right\rangle \\ &= \frac{d}{dt}\left\langle \frac{d^{k-1}(x(t) - \tilde{x}(t))}{dt^{k-1}}, x^*(t)\right\rangle - \frac{d}{dt}\left\langle \frac{d^{k-2}(x(t) - \tilde{x}(t))}{dt^{k-2}}, \frac{dx^*(t)}{dt}\right\rangle \\ &+ \cdots + (-1)^{k-1}\frac{d}{dt}\left\langle x(t) - \tilde{x}(t), \frac{d^{k-1}x^*(t)}{dt^{k-1}}\right\rangle, \quad k = 1, 2, \ldots, s, \end{aligned} \tag{4.150}$$

$$\langle p_s(x(t) - \tilde{x}(t)), x^*(t)\rangle - \langle x(t) - \tilde{x}(t), p_s x^*(t)\rangle = 0. \tag{4.151}$$

Multiplying both sides of Eq. (4.150) by p_{s-k} and summing on k and taking into account Eq. (4.151), we have

$$\langle L_s(x(t)-\tilde{x}(t)), x^*(t)\rangle - \langle L_s^* x^*(t), x(t)-\tilde{x}(t)\rangle = \frac{d}{dt}\langle L_{s-1}(x(t)-\tilde{x}(t)), x^*(t)\rangle$$
$$-\frac{d}{dt}\left\langle L_{s-2}(x(t)-\tilde{x}(t)), \frac{dx^*(t)}{dt}\right\rangle + \cdots + (-1)^{s-1}\frac{d}{dt}\left\langle L_0(x(t)-\tilde{x}(t)), \frac{d^{s-1}x^*(t)}{dt^{s-1}}\right\rangle \tag{4.152}$$

where $L_0(x(t)-\tilde{x}(t)) = p_0(x(t)-\tilde{x}(t))$.

Recall that $x(\cdot), \tilde{x}(\cdot)$ are feasible arcs and $x(t_0)=\tilde{x}(t_0)$. Therefore, integrating the two sides of Eq. (4.152) on the time interval $[t_0, t_1]$ we obtain

$$\int_{t_0}^{t_1}[\langle L_s(x(t)-\tilde{x}(t)), x^*(t)\rangle - \langle L_s^* x^*(t), x(t)-\tilde{x}(t)\rangle]dt = \langle L_{s-1}(x(t_1)-\tilde{x}(t_1)), x^*(t_1)\rangle$$
$$-\left\langle L_{s-2}(x(t_1)-\tilde{x}(t_1)), \frac{dx^*(t_1)}{dt}\right\rangle + \cdots + (-1)^{s-1}\left\langle p_0(x(t_1)\tilde{x}(t_1)), \frac{d^{s-1}x^*(t_1)}{dt^{s-1}}\right\rangle \tag{4.153}$$

Direct verification shows that Eq. (4.153) can be rewritten as follows:

$$\sum_{k=0}^{s-1}(-1)^k\langle L_{s-k-1}(x(t_1)-\tilde{x}(t_1)), x^{*(k)}(t_1)\rangle = \sum_{k=0}^{s-1}\langle L_{s-k-1}^* x^*(t_1), x^{(k)}(t_1)-\tilde{x}^{(k)}(t_1)\rangle \tag{4.154}$$

Consequently, it follows from Eqs. (4.149) and (4.154) that

$$\begin{aligned}&\int_{t_0}^{t_1}[\langle L_s(x(t)-\tilde{x}(t)), x^*(t)\rangle - \langle L_s^* x^*(t), x(t)-\tilde{x}(t)\rangle]dt \\ &\quad= \sum_{k=0}^{s-1}\langle L_{s-k-1}^* x^*(t_1), x^{(k)}(t_1)-\tilde{x}^{(k)}(t_1)\rangle \le 0\end{aligned} \tag{4.155}$$

Let $\Delta J = J[x(\cdot), t^*] - J[x(\cdot), t_1]$ be the increment of the Bolza functional J obtained by the transition from arc $\tilde{x}(\cdot)$ to the arc $x(\cdot)$. Then we have

$$\Delta J = \varphi(x(t^*), x'(t^*), \ldots, x^{(s-1)}(t^*), t^*) + \int_{t_0}^{t*} g(x(t), x'(t), \ldots, x^{(s-1)}(t), t)dt$$
$$-\varphi(\tilde{x}(t_1), \tilde{x'}(t_1), \ldots, \tilde{x}^{(s-1)}(t_1), t_1) - \int_{t_0}^{t_1} g(\tilde{x}(t), \tilde{x'}(t), \ldots, \tilde{x}^{(s-1)}(t), t)dt$$

Then from hypotheses (a) and (b) and from Eq. (4.155), we obtain

$$
\begin{aligned}
\Delta J \ge{}& \varphi(x(t^*), x'(t^*), \ldots, x^{(s-1)}(t^*), t^*) + \int_{t_0}^{t_1^*} g(x(t), x'(t), \ldots, x^{(s-1)}(t), t)dt \\
&- \varphi(x(t_1), x'(t_1), \ldots, x^{(s-1)}(t_1), t_1) - \int_{t_0}^{t_1} g(x(t), x'(t), \ldots, x^{(s-1)}(t), t)dt \\
&- \sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(t_1), x^*(t_1), x^{(k)}(t_1) - \tilde{x}^{(k)}(t_1)\rangle - \sum_{k=0}^{s-1} \int_{t_0}^{t_1} \langle L_{s-k-1}^* x^*(t), x^{(k)}(t) - \tilde{x}^{(k)}(t)\rangle dt \\
&- \sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(t_1), x^*(t_1), x^{(k)}(t_1) - \tilde{x}^{(k)}(t_1)\rangle - \sum_{k=0}^{s-1} \int_{t_0}^{t_1} \langle L_{s-k-1}^* x^*(t), x^{(k)}(t) - \tilde{x}^{(k)}(t)\rangle dt
\end{aligned}
\tag{4.156}
$$

$$
\begin{aligned}
&\ge \varphi(x(t^*), x'(t^*), \ldots, x^{(s-1)}(t^*), t^*) + \int_{t_0}^{t_1^*} g(x(t), x'(t), \ldots, x^{(s-1)}(t), t)dt \\
&- \varphi(x(t_1), x'(t_1), \ldots, x^{(s-1)}(t_1), t_1) - \int_{t_0}^{t_1} g(x(t), x'(t), \ldots, x^{(s-1)}(t), t)dt.
\end{aligned}
$$

To prove the optimality of $\tilde{x}(\cdot)$, let us assume the contrary—for any feasible arc $x(\cdot)$, $t \in [t_0, t^*]$, $x(t_0) = x_0$, $x'(t_0) = x_1$, $\ldots$, $x^{(s-1)}(t_0) = x_{s-1}$, $x^k(t^*) \in Q_k$, $k = 0, 1, \ldots, s-1$, let the increment be negative; i.e., $\Delta J < 0$ or $J[x(\cdot), t^*] < J[\tilde{x}(\cdot), t_1]$. Then by Eq. (4.156), this means that $J[x(\cdot), t^*] < J[x(\cdot), t_1]$. Since $J[x(\cdot), t]$ is monotone on t, we conclude that $t^* < t_1$.

On the other hand, by Eq. (4.155), it is clear that for $t = t^*$

$$
\sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(t^*), \tilde{x}^{(k)}(t^*)\rangle \ge \sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(t^*), x^{(k)}(t^*)\rangle.
$$

Finally, by the $t = t^*$ transversality property on Q, we get

$$
-\sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(t^*), x^{(k)}(t^*)\rangle > \sum_{k=0}^{s-1} W_{Q_K}(-L_{s-k-1}^* x^*(t^*)) = W_Q(-L^* x^*(t^*)).
\tag{4.157}
$$

Thus, remembering that $t^* < t_1$, the inequality in Eq. (4.157) implies that there is at least one k for which $x^{(k)}(t^*) \notin Q_k$; i.e., the arc $x(\cdot)$ cannot realize the transition from the interval $[t_0, t^*]$ to the set Q. This means that $\tilde{x}(\cdot)$ is the optimal arc. ■

Remark 4.13. It is obvious that the Bolza-type problem (P_H) with fixed time interval $t^* = t_1$ and the inequality in Eq. (4.156) implies that $\Delta J \geq 0$, and consequently $\tilde{x}(\cdot)$, is optimal. So the t_1-transversality condition in the considered case is superfluous. Moreover, the monotonicity condition on $J[x(\cdot), t]$ in t is unnecessary.

Remark 4.14. Suppose that at the moment $t = t_1$ there is only one restriction; e.g., $x(t_1) \in Q_0$. This means that $Q_k = \mathbb{R}^n$, $k = 1, 2, \ldots, s-1$. Then by Proposition 4.2,

$$W_{Q_k}(x_k^*) = \begin{cases} 0, & x_k^* = 0, \\ +\infty, & x_k^* \neq 0, \end{cases}$$

and in the hypotheses (a) and (b) of Theorem 4.24, $L_{s-k}^* x^*(t) \equiv 0$, $k = 1, \ldots, s-1$, $t \in [t_0, t_1]$ and the t_1-transversality condition consists of the following:

$$-\langle L_{s-1}^* x^*(t), \tilde{x}(t) \rangle > W_{Q_0}(-L_{s-1}^* x^*(t)), \quad t_0 \leq t < t_1.$$

Remark 4.15. Note that in the theory of linear differential operators, a formula similar to that in Eq. (4.153) is called Lagrange's formula [139].

For the sake of simplicity, we now consider the time optimal control problems in Eqs. (4.142)–(4.144). This problem is to find a feasible solution $x(\cdot)$ of the differential inclusion in Eq. (4.142), which transfers the state to the set $Q = Q_0 \times \cdots \times Q_{s-1}$ in the least time.

Theorem 4.25. Let $\tilde{x}(\cdot)$ be a feasible arc of the differential inclusion in Eq. (4.142), which transfers the state to the set Q on the interval $[t_0, t_1]$. Assume that there exists an adjoint function $x^*(\cdot)$ of the HOADI (i) and (ii) of Theorem 4.24, which is an absolutely continuous function together with its higher-order derivatives up until order $s-1$. Further assume that the t_1-transversality condition on the set $Q = Q_0 \times \cdots \times Q_{s-1}$ holds.

Then the solution $\tilde{x}(\cdot)$ is optimal.

☐ Suppose that the solution $\tilde{x}(\cdot)$ is not an optimal one. Then there exists a feasible solution $x(\cdot)$ of the HODI (i) that does a transfer from the initial state $[x(t_0), x'(t_0), \ldots, x^{(s-1)}(t_0)]$ to the final set Q on the time interval $[t_0, \theta]$ and $\theta < t_1$. Then considering the feasible functions $x(\cdot), \tilde{x}(\cdot)$ on the time interval $[t_0, \theta]$ by analogy to the proof of Theorem 4.24, we have the same inequality (Eq. (4.155)) rewritten for point $t_1 = \theta$:

$$\sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(\theta), x^{(k)}(\theta) - \tilde{x}^{(k)}(\theta) \rangle \leq 0$$

or

$$-\sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(\theta), x^{(k)}(\theta) \rangle \geq -\sum_{k=0}^{s-1} \langle L_{s-k-1}^* x^*(\theta), \tilde{x}^{(k)}(\theta) \rangle \tag{4.158}$$

Since for $t_0 \le \theta < t_1$ the t_1-transversality condition is assumed to hold, we have from the inequality in Eq. (4.158)

$$-\sum_{k=0}^{s-1} \langle L^*_{s-k-1} x^*(\theta), x^{(k)}(\theta)\rangle > \sum_{k=0}^{s-1} W_{Q_k}(-L^*_{s-k-1} x^*(\theta)), \quad t_0 \le \theta < t_1. \tag{4.159}$$

But Eq. (4.159) contradicts the inclusions $x(\theta) \in Q_0$, $x'(\theta) \in Q_1, \ldots, x^{(s-1)}(\theta) \in Q_{s-1}$.

This contradiction proves our theorem. ■

Now in the cost function in Eq. (4.141), let $g(x, y_1, \ldots, y_{s-1}, t) \equiv g(x, t)$, $\varphi(x, y_1, \ldots, y_{s-1}, t) \equiv \varphi(x, t)$ and suppose that $g(\cdot, t)$ and $\varphi(\cdot, t_1)$ are convex functions; i.e.,

$$J[x(\cdot), t_1] = \int_{t_0}^{t_1} g(x(t), t)dt + \varphi(x(t_1), t_1).$$

Theorem 4.26. Suppose that $g(\cdot, t)$, $\varphi(\cdot, t_1)$ and the multivalued function $F(\cdot, t)$ are convex and that Q_i, $i = 0, 1, \ldots, s-1$ are convex sets in Eq. (4.141)–(4.144).

Let $\tilde{x}(\cdot)$ be any feasible arc of this convex problem and assume that there exists an absolutely continuous function $x^*(\cdot)$ on $[t_0, t_1]$ together with its higher-order derivatives up until $s-1$, which satisfies the HOADI

$$L^*_s x^*(t) \in F^*(x^*(t), (\tilde{x}(t), L_s\tilde{x}(t)), t) - \partial g(\tilde{x}(t), t), \quad t \in [t_0, t_1].$$

Further assume that $J[x(\cdot), t]$ is monotonically increasing with respect to t for any feasible arc $x(\cdot)$ of the HODI in Eq. (4.142) and that the following conditions are satisfied:

1. $L^*_{s-k-1} x^*(t_1) \in K^*_{Q_k}(\tilde{x}^{(k)}(t_1)), \quad k = 1, 2, \ldots, s-1,$
2. $-L^*_{s-1} x^*(t_1) \in \partial\varphi(\tilde{x}(t_1), t_1),$
3. t_1-transversality condition on $Q_1 \times \cdots \times Q_{s-1}$.

Then the arc $\tilde{x}(\cdot)$ is optimal.

□ It is not hard to see that in the considered case, the inequality in Eq. (4.146) from the proof of Theorem 4.24 has a form:

$$H(x(t), x^*(t), t) - H(\tilde{x}(t), x^*(t), t) - g(x(t), t) + g(\tilde{x}(t), t) \le \langle L^*_s x^*(t), x(t) - \tilde{x}(t)\rangle.$$

so (see Eq. (4.149))

$$\langle L_s(x(t) - \tilde{x}(t)), x^*(t)\rangle - \langle L^*_s x^*(t), x(t) - \tilde{x}(t)\rangle \le g(x(t), t) - g(\tilde{x}(t), t) \tag{4.160}$$

By hypothesis 2,

$$\langle -L^*_{s-1}x^*(t_1), x(t_1) - \tilde{x}(t_1)\rangle \le \varphi(x(t_1), t_1) - \varphi(\tilde{x}(t_1), t_1) \tag{4.161}$$

Taking into account Eq. (4.155) and integrating the inequalities in Eqs. (4.160) and (4.161) on the interval $[t_0, t_1]$ and adding them, we have

$$J[x(.), t_1] - J[\tilde{x}(.), t_1] \ge \sum_{k=1}^{s-1} \langle L^*_{s-k-1}x^*(t_1), x^{(k)}(t_1) - \tilde{x}^{(k)}(t_1)\rangle dt.$$

By hypothesis 1, the right-hand side of the last inequality is nonnegative; i.e.,

$$J[x(\cdot), t_1] - J[\tilde{x}(\cdot), t_1] \ge 0. \tag{4.162}$$

Now using the inequality in Eq. (4.162) and the same method as in the proof of Theorem 4.24, we can show that (see Eq. (4.156))

$$\Delta J \ge \varphi(x(t^*), t^*) + \int_{t_0}^{t*} g(x(t), t)dt - \varphi(x(t_1), t_1) - \int_{t_0}^{t_1} g(x(t), t)dt$$

and $t^* < t_1$. Then by analogy with Theorem 4.24 using the t_1-transversality condition, it can be shown that $\tilde{x}(\cdot)$ is optimal. ■

Remark 4.16. At once the inequality in Eq. (4.162) implies that if t_1 is fixed, then conditions (1) and (2) are sufficient for the optimality of $\tilde{x}(\cdot)$.

Let us consider the following familiar time optimal control problem:

$$\text{infimum } t_1 = \int_{t_0}^{t_1} 1 \cdot dt \quad \text{subject to} \quad \frac{d^2x}{dt^2} = u, u \in [-1, 1], \tag{4.163}$$

$$x(0) = x_0, \quad x'(0) = x_1, \quad x(t_1) = x'(t_1) = 0.$$

It is required to find the optimal control $\tilde{u}(\cdot)$ such that the corresponding arc $\tilde{x}(\cdot)$ minimizes t_1. In the present case [see (P_H)]

$$L_s x = \frac{d^2x}{dt^2}, s = 2, F(x, t) = \{u : |u| \le 1\}, \varphi \equiv 0, g \equiv 1, Q_0 = Q_1 = \{0\}.$$

So the time optimal control problem in Eq. (4.163) can be written as follows:

$$\text{infimum } t_1 \text{ subject to } L_2 x \in F(x, t), \tag{4.164}$$

$$x(0) = x_0, \quad x'(0) = x_1, \quad x(t_1) = x'(t_1) = 0.$$

Clearly,

$$H(x, v_2^*, t) = \max_u \{uv_2^* : |u| \leq 1\} = |v_2^*| \tag{4.165}$$

so

$$F^*(v_2^*, (x, v_2), t) = \partial_x H(x, v_2^*, t) \equiv 0, \quad v_2 \in F(x, v_2^*, t) = \{-1, 1\}. \tag{4.166}$$

Then taking into account $L_2^* x^* = d^2x^*/(dt^2)$, we have (see Theorem 4.25)

$$\frac{d^2x^*}{dt^2} = 0, \quad t \in [0, t_1],$$

for which the solution is a linear function of the form $x^*(t) = c_1 t + c_2$, where c_1, c_2 are arbitrary constants.

Then Eq. (4.165) implies that $\tilde{u}(t)x^*(t) = |x^*(t)|$, or

$$\tilde{u}(t) = \begin{cases} \text{sign}\, x^*(t), & \text{if } x^*(t) \neq 0, \\ \forall u_0 \in [-1, 1], & \text{if } x^*(t) = 0. \end{cases} \tag{4.167}$$

Furthermore, $x^*(t)$, $t \in [0, t_1]$ as a linear function vanishes, which is not more than one time on $[0, t_1]$. Therefore, for optimal control $\tilde{u}(\cdot)$ by Eq. (4.167), there are four possibilities:

$$\tilde{u}(t) \equiv 1, \quad x^*(t) \neq 0, \ \ t \in [0, t_1], \tag{4.168}$$

$$\tilde{u}(t) \equiv -1, \quad x^*(t) \neq 0, \ \ t \in [0, t_1], \tag{4.169}$$

$$\tilde{u}(t) = \begin{cases} 1, & 0 \leq t < \tau, \\ -1, & \tau < t \leq t_1, \end{cases} \tag{4.170}$$

$$\tilde{u}(t) = \begin{cases} -1, & 0 \leq t < \tau, \\ 1, & \tau < t \leq t_1. \end{cases} \tag{4.171}$$

Observe that $x^*(\tau) = 0$ and the values of the control functions $u(\cdot)$ at a point of discontinuity τ are unessential. Using Eqs. (4.168)–(4.171), by solving the Cauchy problem,

$$\frac{d^2x(t)}{dt^2} = \tilde{u}(t), \quad x(t_1) = x'(t_1) = 0 \tag{4.172}$$

we have a unique solution with the initial point $x(0) = x_0$, $x'(0) = x_1$.

Thus, substituting Eq. (4.168) into (4.172), we have

$$x(t_1) = \frac{t_1^2}{2} + x_1 t + x_0$$

and the conditions $x(t_1) = 0$, $x'(t_1) = t_1 + x_1$ give us

$$t_1 = -x_1 + \sqrt{x_1^2 - 2x_0}, \quad \text{and} \quad x_1^2 - 2x_0 = 0 \quad \text{or} \quad x_0 = \frac{x_1^2}{2}, \quad (x_1 < 0).$$

Consequently, in the case $\tilde{u}(t) = +1$, the Cauchy problem in Eq. (4.172) corresponds to the initial points x_0, x_1 for which $x_0 = (x_1^2/2)$, $x_1 < 0$. By analogy, if $\tilde{u}(t) = -1$ we have from Eq. (4.172) initial points satisfying $x_0 = (-x_1^2/2)$, $x_1 > 0$. The family of solution-arcs AB and CD of parabolas are illustrated in Figure 4.1.

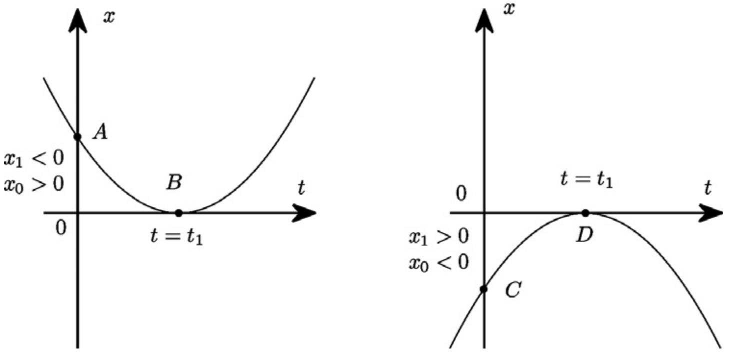

Figure 4.1 Family of solutions consisting of one parabola.

In a similar way, in Figure 4.2, for every optimal control in Eqs. (4.170) and (4.171), taking the values ± 1 and having not more than two intervals of constancy which corresponds to the initial points satisfying $x_0 = -x_1|x_1|/2$:

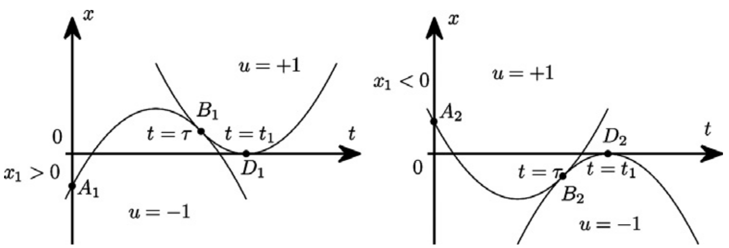

Figure 4.2 Family of solutions consisting of two parabolas.

As is shown in Figure 4.2, the trajectories ($A_1B_1D_1$ and $A_2B_2D_2$) consist of two pieces of parabolas and at points $t = \tau$ (see Figure 4.2(a)) the value of u switches its value and becomes equal to $+1$ and -1 (see Figure 4.2(b)), respectively.

5 On Duality of Ordinary Discrete and Differential Inclusions with Convex Structures

5.1 Introduction

In this chapter, we construct the dual problems to the primary convex problems for ordinary discrete (DSI) and differential (DFI) inclusions. Duality theory, by virtue of the importance of its applications, is one of the central directions in optimal control theory. For example, in mathematics and economics, duality theory is interpreted in the form of prices; in mechanics, potential energy and complementary energy are in a dual relationship—the displacement field and the stress field are solutions to the primary and dual problems, respectively. Besides these applications, duality often makes it possible to simplify the computational procedure and to construct a generalized solution to variational problems that do not have classical solutions. The duality theorems allow you to conclude that a sufficient condition for an extremum is an *extremal relation* for the primary and dual problems. The latter means that if some pair (x, x^*) satisfies this extremal relation, then x and x^* are solutions to a primary and a dual problem, respectively. We remark that a significant part of the investigations of Ekeland and Temam [75,76] for simple variational problems are connected with such problems and that there are similar results for ordinary DFI in Mahmudov [142,144−149,155,157,158,161], Mahmudov and Değer [172], Mordukhovich [185,186,199], Rockafellar and Wolenski [233,234], and Richard and Yung [227].

In this chapter, the operations of addition and infimal convolution of convex functions is the starting point for the construction of duality theory.

In Section 5.2, we consider an important class of optimization problems of mathematical programs with equilibrium constraints $y \in F(x)$, where the multifunction $F(x)$ appears in various aspects of equilibrium theory related to mechanical and economic modeling. We shall investigate this problem in the abstract form

$$\text{(EC)} \quad \text{infimum } g(x, y) \text{ subject to } y \in F(x) \cap N, \quad x \in M.$$

Note that in Ref. [212], the main attention is paid to the case in which the equilibrium multifunction F is given in the form $F(x) = \{y \in Y : 0 \in f(x, y) + Q(x, y)\}$ with $f : X \times Y \to Z$ and $Q : X \times Y \to P(Z)$, where $P(Z)$ is a set of all nonempty

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subsets of Z. The dual problem for mathematical programs with equilibrium constraints is the following maximization problem:

$$(\mathrm{EC}^*) \qquad \sup_{x^*, x_0^*, y^*, y_0^*} \{-g^*(x_0^*, y_0^*) + M_F(x_0^* - x^*, y^* - y_0^*) - W_N(-y^*) - W_M(-x^*)\}.$$

In what follows, we prove that if v and v^* are the values of optimization problem of mathematical programs with equilibrium constraints (EC) and its dual problem (EC*), respectively, then $v \geq v^*$ for all feasible solutions of primary and dual problems. Moreover, if a certain "regularity condition" (see Definition 5.1) is satisfied, then the existence of a solution to one of these problems implies the existence of a solution to the dual problem, where $v = v^*$, and in the case where $v > -\infty$ the dual problem has a solution.

In particular, under the quasisuperlinearity condition relative to F and sets N, M are formulated necessary and sufficient conditions for the finiteness and attainability of v^*.

Next, in the case of $g(z) \equiv f_0(x)$ in terms of the recession function $f_0^* 0^+$ is formulated one sufficient condition (Theorem 5.5) for the equality of values of primary and dual problems.

One of the main object of discussion of this chapter is an optimization of convex DSI of the form

$$(\mathrm{Pd}) \;\; \inf \sum_{t=0}^{T} g(x_t, t), \text{ subject to } x_{t+1} \in F_t(x_t), \;\; t = 0, \ldots, T-1, \; x_0 \in N_0, \; x_T \in M_T,$$

where $g(\cdot, t)$ and F_t are a convex and a multivalued function, respectively, and N_0 and M_T are convex sets. For this goal, in Section 5.3, we first construct the dual problem for polyhedral DSI considered in Section 4.3; i.e., the problem where $g(\cdot, t)$ is a polyhedral function and F_t is a polyhedral multivalued function given by $F_t(x) \equiv F(x) = \{y : Ax - By \leq d\}$; N_0 and M_T are polyhedral sets. As above, it should be emphasized that we use the operations of addition and infimal convolution of convex functions (Theorem 3.15).

The dual problem for (Pd) is reduced to a problem of the form $\sup\{-\varphi^*(w^*) - \delta_G^*(-w^*)\}$. Calculating polyhedral functions φ^* and δ_G^*, which have concrete forms connected with problem (Pd), we construct the dual problem. Then by using the notion of infimal convolution, we investigate the duality relations for classical linear programming theory. In Section 5.3, on the basis of polyhedral duality results, we establish the dual problem for polyhedral DFI.

In Section 5.4, we construct the dual problem for problem (Pd), which consists of the following:

$$(\mathrm{Pd}^*) \quad \sup_{u_t^*, x_t^*, t=0,\ldots,T} \left\{ -\sum_{t=0}^{T} g^*(u_t^*, t) + \sum_{t=0}^{T-1} M_{F_t}(x_t^* + u_t^*, x_{t+1}^*) - W_{N_0}(x_0^*) - W_{M_T}(-u_T^* - x_T^*) \right\}$$

and prove duality theorems.

In Section 5.5, a Bolza problem of optimal control theory given by convex differential inclusions is considered. For such a problem, the dual problem is constructed and under some conditions the duality relations are established. Then a connection between solutions of dual problems and necessary and sufficient conditions for optimality in primary problem is considered. The constructed dual problem for the discrete-approximation problem allows us to make a bridge between the duality problems for DSI and DFI. Construction of the dual problem to (CP) is realized by passing to the formal limit as the discrete step tends to zero. The results obtained are extended to the delay DSI and delay DFI.

5.2 Duality in Mathematical Programs with Equilibrium Constraints

We consider an important class of optimization problems with equilibrium constraints $y \in F(x)$, where the multivalued function $F(x)$ appears in various aspects of equilibrium theory related to mechanical and economic modeling:

$$\inf g(x, y) \text{ subject to } y \in F(x) \cap N, \quad x \in M, \tag{5.1}$$

where $g : X \times Y \to \mathbb{R} \cup \{+\infty\}$, F, and $M \subseteq X$ are closed proper convex function, convex multivalued function, and subset, respectively. In the more general case discussed in Section 3.7, we established necessary conditions of optimality (Theorem 3.6 and Corollaries 3.8 and 3.9) for the nonconvex problem in Eq. (5.1). Here we construct the dual problem to the primary problem in Eq. (5.1) and establish duality relations for it. To do this we easily reduce the problem in Eq. (5.1) to the convex programming problem with a geometric constraint of the form

$$(G) \qquad \inf g(z) \text{ subject to } z \in A$$

where $A = (\text{gph } F) \cap (X \times N) \cap (M \times Y) \subseteq Z = X \times Y$, $z = (x,y)$. With the use of the infimal convolution of two functions, we have constructed the dual problem for the convex problem considered in Section 3.3 as follows:

$$(G^*) \qquad \sup\{-\mathrm{g}^*(\mathrm{z}^*) - \delta_{\mathrm{A}}^*(-\mathrm{z}^*)\},$$

where $\delta_A(\cdot)$ is the indicator function of A. Here, by Theorem 3.15 of the duality between the operations of addition and infimal convolution of convex functions if there exists a point $z_1 \in A$, where g is continuous, then $(g + \delta_A)^* = g^* \oplus \delta_A^*$, so the values of the primary and dual problems (G) and (G^*) are equal. In addition, if the value of problem (G) is finite, then $0 \in \text{dom}(g + \delta_A)^*$ and the supremum in problem (G^*) is attained. On the other hand, since

$$\delta_A = \delta_{\text{gph } F} + \delta_{X \times N} + \delta_{M \times Y},$$

it is easy to see that

$$\delta_A^*(-z^*) \le \inf\{\delta_{\text{gph F}}^*(z_1^*) + \delta_{X\times N}^*(z_2^*) + \delta_{M\times Y}^*(z_3^*) : z_1^* + z_2^* + z_3^* + z^* = 0\},$$
$$z_i^* = (x_i^*, y_i^*), \quad i = 1, 2, 3. \tag{5.2}$$

By Theorem 1.24, we can write

$$\delta_{\text{gph }F}^*(z_1^*) = \sup\{\langle z_1, z_1^*\rangle : z_1 \in \text{gph } F\},$$
$$\delta_{X\times N}^*(z_2^*) = \sup\{\langle z_2, z_2^*\rangle : z_2 \in X \times N\} = \begin{cases} \sup\{\langle y_2, y_2^*\rangle : y_2 \in N\}, & x_2^* = 0, \\ +\infty, & x_2^* \neq 0 \end{cases}$$
$$\delta_{M\times Y}^*(z_3^*) = \sup\{\langle z_3, z_3^*\rangle : z_3 \in M \times Y\} = \begin{cases} \sup\{\langle y_3, y_3^*\rangle : y_3 \in N\}, & y_3^* = 0, \\ +\infty, & y_3^* \neq 0 \end{cases} \tag{5.3}$$

Definition 5.1. We say that the regularity condition for problem (G) is satisfied, if there is a point $z_0 = (x_0, y_0) \in Z$ such that either (1) $(x_0, y_0) \in$ ri dom $g(x_0, y_0) \in$ ri gph F, $x_0 \in$ ri M, $y_0 \in$ ri N or (2) $(x_0, y_0) \in$ int gph F, $x_0 \in M$, $y_0 \in$ int N and g is continuous at z_0.

Then under the regularity condition in Eq. (5.2), one has the equality sign, and for all z^*, such that $\delta_A^*(-z^*) < +\infty$, the infimum is attained.

Remember that by notation of Eq. (2.5),

$$M_F(x_1^*, y_1^*) = \inf_{(x,y)}\{\langle x, x_1^*\rangle - \langle y, y_1^*\rangle : (x, y) \in \text{gph } F\}.$$

Then in view of the relations in Eqs. (5.2) and (5.3), and Theorem 1.24, we can write

$$\sup\{-g^*(z^*) - \delta_A^*(-z^*)\} \ge \sup\{-g^*(-z^*) + M_F(-x_1^*, y_1^*) - W_N(y_2^*) - W_M(x_3^*)\}, \tag{5.4}$$

where W_C is a support function of C.

Now, by using

$$z^* = (x^*, y^*), \quad z_1^* = (x_1^*, y_1^*), \quad z_2^* = (0, y_2^*), \quad z_3^* = (x_3^*, 0)$$

in the condition $\sum_{i=1}^{3} z_i^* + z^* = 0$ contained in Eq. (5.2), we find that

$$x_1^* + x_3^* + x^* = 0, \quad y_1^* + y_2^* + y^* = 0. \tag{5.5}$$

In what follows, for convenience denoting $x^* = x_0^*$, $y^* = y_0^*$ and then $x_3^* = -x^*$, $y_2^* = -y^*$ in the relation in Eq. (5.4), we observe that the right-hand side of this relation has the form

$$\sup\{-g^*(x_0^*, y_0^*) + M_F(x_0^* - x^*, y^* - y_0^*) - W_N(-y^*) - W_M(-x^*)\} \tag{5.6}$$

Naturally, the constructed problem in Eq. (5.5) is called the dual problem for the primary problem in Eq. (5.1). Thus, comparing all facts in accordance with

Proposition 3.4 and Theorems 3.17 and 3.18, we have obtained the result shown in Theorem 5.1.

Theorem 5.1. If v and v^* are the values of the optimization problems of mathematical programs with equilibrium constraints (Eq. (5.1)) and its dual problem (Eq. (5.5)), respectively, then $v \geq v^*$ for all feasible solutions of primary and dual problems. Moreover, if the regularity condition is satisfied, then the existence of a solution to one of these problems implies the existence of a solution to the other problem, and $v = v^*$ and in the case where $v > -\infty$, the dual problem has a solution.

Assume that we have a problem (Eq. (5.1)), where $g(z) \equiv f_0(x)$. We construct for such a problem its dual problem. At first we calculate the conjugate function g^*.

Proposition 5.1. If $g(z) \equiv f_0(x)$, then we have

$$g^*(z^*) = \begin{cases} f_0^*(x^*), & \text{if } y^* = 0, \\ +\infty, & \text{if } y^* \neq 0. \end{cases}$$

□ In fact, by definition of a conjugate function, we get

$$\begin{aligned} g^*(z^*) &= \sup\{\langle z, z^* \rangle - g(z)\} = \sup\{\langle x, x^* \rangle + \langle y, y^* \rangle - f_0(x)\} \\ &= \sup\{\langle y, y^* \rangle + f_0^*(x^*)\} = \begin{cases} f_0^*(x^*), & \text{if } y^* = 0, \\ +\infty, & \text{if } y^* \neq 0 \end{cases}, \end{aligned}$$

where f_0^* is a closed proper convex function (Theorem 1.21).■

Thus, by Proposition 5.1, the dual problem in Eq. (5.5) to the primary problem below

$$\text{infimum } f_0(x) \text{ subject to } y \in F(x) \cap N, \quad x \in M \tag{5.7}$$

consists of the following:

$$\sup\{-f_0^*(x_0^*) + M_F(x_0^* - x^*, y^*) - W_N(-y^*) - W_M(-x^*)\} \tag{5.8}$$

Similarly, if now $g(z) \equiv f(y)$, then $g^*(z^*) = f^*(y^*)$, $x^* = 0$, so the dual problem is

$$\sup\{-f^*(y^*) + M_F(x_0^*, y_0^* - y^*) - W_N(-y_0^*) - W_M(x_0^*)\}.$$

Theorem 5.2. Let the regularity condition for the problem in Eq. (5.7) be satisfied. Then in order for $\tilde{x}$ be a solution to problem in Eq. (5.7), it is necessary and sufficient that there exist vectors $x^* \in K_M^*(\tilde{x})$, $y^* \in K_N^*(\tilde{y})$, and $x_0^* \in \partial f_0(\tilde{x})$ not all equal to zero, such that

$$x_0^* - x^* \in F^*(y^*; \tilde{z}), \quad \tilde{z} = (\tilde{x}, \tilde{y}), \quad \tilde{y} \in F(\tilde{x}) \cap N.$$

□ The proof is similar to the one for Corollary 3.8. The difference is that under the regularity condition in the convex case, $\lambda = 1$ in Corollary 3.8. Indeed, taking

$$A_0 = \operatorname{dom} f_0 \times Y, \quad A_1 = \operatorname{gph} F, \quad A_2 = X \times N, \quad A_3 = M \times Y,$$

we observe that either

$$(\operatorname{int} A_0) \cap (\operatorname{int} A_1) \cap (\operatorname{int} A_2) \cap A_3 \neq \varnothing \tag{5.9}$$

or

$$(\operatorname{ri} A_0) \cap (\operatorname{ri} A_1) \cap (\operatorname{ri} A_2) \cap (\operatorname{ri} A_3) = \operatorname{ri} \bigcap_{i=0}^{3} A_i \neq \varnothing. \tag{5.10}$$

Therefore, taking into account these conditions, by Theorems 3.3 and 3.4, we get $\lambda = 1$. ■

Theorem 5.3. Let $\tilde{x}$ be a solution to problem (5.7) and let the regularity condition for problem (5.7) be satisfied. Then the triplet of vectors (x_0^*, x^*, y^*) is a solution to the dual problem (5.8) if and only if it satisfies the necessary and sufficient condition of Theorem 5.1.

□ Let (x_0^*, x^*, y^*) be a solution to the dual problem in Eq. (5.8). Since the problem in Eq. (5.7) is equivalent to the problem in Eq. (5.1), where $g(z) \equiv f_0(x)$, it follows by Theorem 3.1 that $0 \in \partial(g(\tilde{z}) + \delta_A(\tilde{z}))$ or by Corollary 1.2 $\tilde{z} \in \partial(g + \delta_A)^*(0)$. In turn, by Theorem 3.15, the latter inclusion yields $\tilde{z} \in \partial(g^* \oplus \delta_A^*)^*(0)$. Moreover, by Proposition 3.2,

$$\begin{aligned} &\partial(g^* \oplus \delta_A^*)^*(0) = \partial g^*(z^*) \cap \partial\delta_A^*(-z^*) \quad \text{and} \\ &\tilde{z} \in \partial g^*(z^*) \cap \partial\delta_A^*(-z^*), \end{aligned} \tag{5.11}$$

where z^* is a solution to the dual problem (G^*). Furthermore, by hypothesis, remember that at points $z^* = (x_0^*, 0)$, $\quad z_1^* = (x^* - x_0^*, y^*)$, $\quad z_2^* = (0, -y^*)$, $\quad z_3^* = (-x^*, 0)$, where $\sum_{i=1}^{3} z_i^* + z^* = 0$, the infimum in Eq. (5.2) is attained; i.e.,

$$\delta_A^*(-z^*) = \delta_{\operatorname{gph} F}^*(z_1^*) + \delta_{X \times N}^*(z_2^*) + \delta_{M \times Y}^*(z_3^*).$$

Then from the formula in Eq. (5.11), we obtain

$$\tilde{z} \in \partial g^*(z^*) \cap \partial\delta_{\operatorname{gph} F}^*(z_1^*) \cap \delta_{X \times N}^*(z_2^*) \cap \delta_{M \times Y}^*(z_3^*)$$

which implies that

$$\tilde{z} \in \partial g^*(z^*), \quad \tilde{z} \in \partial\delta_{\operatorname{gph} F}^*(z_1^*), \quad \tilde{z} \in \partial\delta_{X \times N}^*(z_2^*), \quad \tilde{z} \in \partial\delta_{M \times Y}^*(z_3^*). \tag{5.12}$$

The first inclusion in Eq. (5.12) by Corollary 1.2 implies that $z^* \in \partial g(\tilde{z})$ or $(x_0^*, 0) \in \partial f_0(\tilde{x}) \times \{0\}$; i.e., $x_0^* \in \partial f_0(\tilde{x})$. Similarly, by virtue of the second inclusion in Eq. (5.12), we get $\tilde{z} \in \partial \delta^*_{\text{gph } F}(z_1^*)$ or, equivalently, $z_1^* \in \partial \delta_{\text{gph } F}(\tilde{z})$. Therefore. $(x^* - x_0^*, y^*) \in \partial \delta_{\text{gph } F}(\tilde{z}) = -K^*_{\text{gph } F}(\tilde{z})$ or by the definition of LAM $x_0^* - x^* \in F^*(y^*; \tilde{z})$. Finally, by analogy, the remaining inclusions $\tilde{z} \in \partial \delta^*_{X \times N}(z_2^*)$, $\tilde{z} \in \partial \delta^*_{M \times Y}(z_3^*)$ imply that $y^* \in K_N^*(\tilde{y})$ and $x^* \in K_M^*(\tilde{x})$, respectively. Conversely, we show now that if the vectors x_0^*, x^*, y^* satisfy the necessary and sufficient condition of the theorem, then the triplet (x_0^*, x^*, y^*) is a solution to the dual problem in Eq. (5.8). By Theorem 1.26, $x_0^* \in \partial f_0(\tilde{x})$ if and only if

$$f_0^*(x_0^*) = \langle \tilde{x}, x_0^* \rangle - f_0(\tilde{x}). \tag{5.13}$$

Moreover, by Lemma 2.6, the inclusion $x_0^* - x^* \in F^*(y^*; \tilde{z})$ is equivalent to

$$M_F(x_0^* - x^*, y^*) = \langle \tilde{x}, x_0^* - x^* \rangle - H(\tilde{x}, y^*). \tag{5.14}$$

Also it is clear that the inclusions $x^* \in K_M^*(\tilde{x})$, $y^* \in K_N^*(\tilde{y})$ imply that

$$W_N(-y^*) = \langle \tilde{x}, -y^* \rangle, \quad W_M(-x^*) = \langle \tilde{x}, -x^* \rangle, \tag{5.15}$$

respectively. On the other hand, remember that by Theorem 2.1, $\tilde{y} \in F(\tilde{x}, y^*)$ or, equivalently, $H(\tilde{x}, y^*) = \langle y^*, \tilde{y} \rangle$. Thus, from Eqs. (5.13)–Eq. (5.15), we have

$$\begin{aligned} &-f_0^*(x_0^*) + M_F(x_0^* - x^*, y^*) - W_N(-y^*) - W_M(-x^*) \\ &= -\langle \tilde{x}, x_0^* \rangle + f_0(\tilde{x}) - \langle \tilde{y}, y^* \rangle + \langle x_0^* - x^*, \tilde{x} \rangle - \langle \tilde{y}, -y^* \rangle - \langle \tilde{x}, -x^* \rangle = f_0(\tilde{x}). \end{aligned}$$

Consequently, we get $v^* \geq v$. But by Theorem 5.1, $v \geq v^*$ holds for all feasible solutions to the primary problem in Eq. (5.7) and the dual problem in Eq. (5.8). Keeping these two inequalities in mind, we find that $v = v^*$. ■

Thus, Theorem 5.3 allows us to conclude that a sufficient condition for an extremum is an "extremal relation" between the primary and dual problems. The latter means that if some pair of feasible solutions satisfies this relation, then each of them is a solution to the corresponding (primary and dual) problem [75,76,144–185].

Proposition 5.2. For effective domain of $g^* \oplus \delta_A^*$, we have the inclusion dom $(g^* \oplus \delta_A^*) \supseteq \{(x_1^* + x_3^* + x^*, y_1^* + y_2^*) : M_F(-x_1^*, y_1^*) > -\infty, W_N(y_2^*) < +\infty, W_M(x_3^*) < +\infty, \ x^* \in \text{dom } f_0^*\}$.

In addition, under the regularity condition, there is an equality sign.

□ It is known from Ref. [129] and Theorem 6.5.2, that $\text{dom}(g^* \oplus \delta_A^*) = \text{dom } g^* + \text{dom } \delta_A^*$. Moreover,

$$\text{dom } \delta_A^* \supseteq \text{dom } \delta^*_{\text{gph } F} + \text{dom } \delta^*_{X \times N} + \text{dom } \delta^*_{M \times Y}$$

and under the regularity condition there is an equality sign. Then, taking into account the relations in Eq. (5.3) and $\operatorname{dom} g^* = \{(x^*, 0) : x^* \in \operatorname{dom} f_0^*\}$, we have the desired result. ■

Theorem 5.4. Let a multivalued function F and sets N, M be closed convex, and let the regularity condition be satisfied. Then in order for the value v^* of dual problem in Eq. (5.8) to be finite and attainable, it is necessary that

$$x_0^* \in \operatorname{dom} f_0^*, \quad x_0^* - x^* \in F^*(y^*), \quad -y^* \in (0^+ N)^*, \quad -x^* \in (0^+ M)^*.$$

In addition, under the quasisuperlinearity condition relative to F and sets N, M, these conditions are sufficient for the finiteness and attainability of v^*.

□ Let the value v^* of the dual problem in Eq. (5.8) be finite and attainable. By the same argument as conducted in the proof of Theorem 5.1, $v = v^*$ and $0 \in \operatorname{dom} (g + \delta_A)^* = \operatorname{dom} (g^* \oplus \delta_A^*)$. By Proposition 5.2, this implies that

$$x_1^* + x_3^* + x^* = 0, \quad y_1^* + y_2^* = 0, \quad M_F(-x_1^*, y_1^*) > -\infty, \quad W_N(y_2^*) < +\infty,$$
$$W_M(x_3^*) < +\infty, \quad x^* \in \operatorname{dom} f_0^*$$

or in previous notations $(x_0^* - x^*, y^*) \in \operatorname{dom} M_F$, $-y^* \in W_N$, $-x^* \in \operatorname{dom} W_M$, $x_0^* \in \operatorname{dom} f_0^*$. Then in view of Proposition 2.1, we obtain

$$(x_0^* - x^*, y^*) \in (0^+ \operatorname{gph} F)^*, \quad -y^* \in (0^+ N)^*, \quad -x^* \in (0^+ M)^*$$

By Definition 2.9 of adjoint (not locally adjoint) multivalued function F^*, the first of these inclusions implies that $x_0^* - x^* \in F^*(y^*)$. This ends the proof of the necessity condition. Now in the case of the quasisuperlinearity condition relative to F and sets N, M, we prove sufficiency of these conditions for the finiteness and attainability of v^*. Actually, in view of quasisuperlinearity (Remark 2.1), by Proposition 2.1, we get

$$\operatorname{dom} M_F = (0^+ \operatorname{gph} F)^*, \quad \operatorname{dom} W_N = (0^+ N)^*, \quad \operatorname{dom} W_M = (0^+ M)^*.$$

Therefore, by going in the reverse direction, we have the desired result. ■

In Theorem 5.5, in terms of the recession function $f_0^* 0^+$, we formulate one sufficient condition for the equality of the values of the primary and dual problems. Remember that by Definition 1.25, $\operatorname{epi} (f_0^* 0^+) = 0^+ (\operatorname{epi} f_0^*)$.

Theorem 5.5. Let v and v^* be the values of the primary problem in Eq. (5.7) and the dual problem in Eq. (5.8), respectively. Then in order for $v = v^*$, it is sufficient that there do not exist vectors x^*, y^*, x_0^* such that

$$(f_0^* 0^+)(x_0^*) - M_F(x_0^* - x^*, y^*) + W_N(-y^*) + W_M(-x^*) \leq 0.$$

□ By Lemma 1.1, it may easily be seen that

$$0^+(\text{epi}\, g^*) = \{\bar{z}^*, \upsilon) : \{\bar{x}^*, \upsilon\} \in 0^+(\text{epi}\, f_0^*), \quad \bar{y}^* = 0 \ \ \forall \upsilon \in \mathbb{R}\}.$$

Obviously, the recession function g^*0^+ has the form

$$(g^*0^+)(\bar{z}^*) = (f_0^*0^+)(\bar{x}^*), \quad \bar{y}^* = 0. \tag{5.16}$$

Moreover, from the positive homogeneity of conjugate functions of indicator functions (Theorem 1.24), we get

$$\begin{gathered}(\delta^*_{\text{gph}\, F}0^+)(z_1^*) = \delta^*_{\text{gph}\, F}(z_1^*), \quad (\delta^*_{X\times N}0^+)(z_2^*) = \delta^*_N(y_2^*), \quad x_2^* = 0,\\ (\delta^*_{M\times Y}0^+)(z_3^*) = \delta^*_M(x_3^*), \quad y_3^* = 0\end{gathered} \tag{5.17}$$

From Eqs. (5.16) and (5.17) by virtue of Corollary 16.2.2 of Ref. [228:141], we can assert that if there do not exist vectors $x^*, \ \ y^*, \ \ x_0^*$ such that

$$\begin{gathered}(f_0^*0^+)(\overline{x_0}^*) + \delta^*_{\text{gph}\, F}(z_1^*) + \delta^*_N(y_2^*) + \delta^*_M(x_3^*) \le 0\\ \overline{x_0}^* + x_1^* + x_3^* = 0, \quad y_1^* + y_2^* = 0\end{gathered} \tag{5.18}$$

then ri $(\text{dom}\ g \cap \text{dom}\ \delta_A) \neq \varnothing$. Obviously, taking $A_0 = (\text{dom}\ f_0) \times Y$, $A_1 = \text{gph}\ F$, $A_2 = X \times Y$, $A_3 = M \times Y$, we observe that the latter is equivalent to the relation in Eq. (5.10). On the other hand, the existence of a point $z_0 = (x_0, y_0) \in \text{int gph}\ F$, $x_0 \in M$, $y_0 \in \text{int}\ N$ such that g is continuous at z_0 ensures that the condition in Eq. (5.9) is true. In turn, according to Proposition 4 of Ref. [111:219], if there do not exist vectors $x^*, \ \ y^*, \ \ x_0^*$ such that

$$\begin{gathered}\delta^*_{A_0}(z^*) + \delta^*_{A_1}(z_1^*) + \delta^*_{A_2}(z_2^*) + \delta^*_{A_3}(z_3^*) \le 0\\ z_1^* + z_2^* + z_3^* + z^* = 0,\end{gathered} \tag{5.19}$$

then Eq. (5.9) holds. Furthermore, it is easy to derive (see Theorem 1.22) that

$$\delta^*_{A_0}(z^*) = \begin{cases} \delta^*_{\text{dom}\, f_0}(x^*), & \text{if} \ \ y^* = 0,\\ +\infty, & \text{if} \ \ y^* \neq 0.\end{cases}$$

Also, by Theorem 6.8.5 of Ref. [129:347] $(f_0^*0^+)(\overline{x_0}^*) = \delta^*_{\text{dom}\, f_0}(\overline{x_0}^*)$. It follows that the inequality in Eq. (5.18) is none other than the inequality in Eq. (5.19), and the inconsistency of the inequality in Eq. (5.18) implies that the regularity condition takes place. Hence, in accordance with Theorem 5.1 and the preceding notations $\overline{x_0}^* = x_0^*, \ \ x_3^* = -x^*, \ \ y_2^* = -y^*$, we derive that $v = v^*$. ■

Now, by virtue of Theorem 3.15, we will construct the dual problem to the linear programming problem

$$\text{infimum} \ \ \langle c, x\rangle \ \ \text{subject to} \ \ Bx \le d,$$

where B is an $m\times n$ matrix, $c\in\mathbb{R}^n$, $d\in\mathbb{R}^m$. In this case, $f(x)=\langle c, x\rangle$ is linear and $A=\{x{:}Bx\le d\}$ is a polyhedral set. Hence, for all z^*, such that $\delta_A^*(-z^*)<+\infty$, we have

$$\inf_{x\in A} f(x)=\sup\{-f^*(x^*)-\delta_A^*(-x^*)\}. \tag{5.20}$$

Proposition 5.3. If $A=\{x:Bx\le d\}$, then the indicator function δ_A of A has the form

$$\delta_A(x)=\sup_{y^*\ge 0}\langle y^*, Bx-d\rangle.$$

□ An elementary exercise. ■

Now return to the problem in Eq. (5.20). Clearly,

$$f^*(x^*)=\sup_x\langle x^*-c, x\rangle=\begin{cases}0, & \text{if } x^*=c,\\ +\infty, & \text{if } x^*\ne c.\end{cases}$$

On the other hand, by Proposition 5.3, it is not hard to see that

$$\begin{aligned}\delta_A^*(-x^*)&=\sup_x\{\langle x, -x^*\rangle-\delta_A(x)\}=\sup_x\{\langle x, x^*\rangle-\sup_{y^*\ge 0}\langle y^*, Bx-d\rangle\}\\ &=\sup_x\inf_{y^*\ge 0}\{\langle x, -x^*-B^*y^*\rangle+\langle y^*, d\rangle\}.\end{aligned}$$

Therefore,

$$\delta_A^*(-x^*)=\begin{cases}\inf\limits_{y^*\ge 0}\langle y^*, d\rangle, & \text{if } x^*+B^*y^*=0,\\ +\infty, & \text{if } x^*+B^*y^*\ne 0.\end{cases}$$

Now, since $\operatorname{dom}(f^*\oplus\delta_A^*)=\operatorname{dom} f^*+\operatorname{dom}\delta_A^*$, it follows that if $0\in\operatorname{dom}(f^*\oplus\delta_A^*)$; i.e., $x^*+B^*y^*=0$, then

$$\sup\{-f^*(x^*)-\delta_A^*(-x^*)\}=-\delta_A^*(-c)=\sup_{y^*\ge 0}\{\langle -y^*, d\rangle : c+B^*y^*=0\}.$$

In conclusion, Eq. (5.16) can be rewritten as the following duality relation:

$$\inf\{\langle c, x\rangle : Bx\le d\}=\sup_{y^*\ge 0}\{\langle -y^*, d\rangle : B^*y^*+c=0\}.$$

5.3 Duality in Problems Governed by Polyhedral Maps

In this section, we construct the dual problem for the polyhedral discrete and differential inclusions considered in Section 4.3. Then we will investigate the duality relations and as a special case we will deduce the duality theorem for classical linear programming theory. It should be emphasized that, as above, we use the operations of addition and infimal convolution of convex functions which are dual of each other. In this book, this is the most important case of dual operations as far as applications to extremum problems are concerned. Note that if each function f_i in Theorem 3.15 is polyhedral, then it remains valid if ri (dom f_i) is replaced by dom f_i [228:179]. First, we consider polyhedral optimization for the discrete inclusions considered in Section 4.3 and labeled by (PD):

$$\begin{array}{lll} & \inf & \sum_{t=0}^{T} g(x_t, t), \\ (\mathrm{PD}) & \text{subject to} & x_{t+1} \in F(x_t), \quad t = 0,\ 1, \ldots, T-1, \\ & & x_0 \in N_0, \quad x_T \in M_T, \end{array} \tag{5.21}$$

where

$$g(x_t, t) = \max_{i \in I_t} \{\langle x_t, b_i^t\rangle + \beta_i^t\}, \quad t = 0, \ldots, T,$$
$$F(x) = \{y : Ax - By \le d\}, \qquad N_0 = \{x_0 : Nx_0 \le p\}, \quad M_T = \{x_T : Mx_T \le q\}.$$

Here I_t, $t = 0, \ldots, T$ are finite index sets, $b_i^t \in \mathbb{R}^n$, β_i^t is a real number, $b_i^t \in \mathbb{R}^n$, A, B are $m \times n$ matrices, N, M are rectangular matrices with n columns; and d, p, q are column vectors with corresponding dimensions. It is required to find an optimal trajectory $\{\tilde{x}_t\}_{t=0}^{T}$ of (Eq. (5.21)) that minimizes $\sum_{t=0}^{T} g(x_t, t)$.

Replace the problem (PD) with the following equivalent problem in space $\mathbb{R}^{n(T+1)}$:

$$\inf \ \varphi(w) \ \text{ subject to } \ w \in G = \tilde{N}_0 \cap \left(\bigcap_{t=0}^{T-1} G_t\right) \cap \tilde{M}_T, \tag{5.22}$$

where

$$\begin{gathered} \varphi(w) = \sum_{t=1}^{T} g(x_t, t), \quad w = (x_0, \ldots, x_T) \in \mathbb{R}^{(T+1)n}, \\ G_t = \{w : (x_t, x_{t+1}) \in \operatorname{gph} F\}, \quad t = 0, \ldots, T-1 \\ \tilde{N}_0 = \{w : x_0 \in N_0\}, \quad \tilde{M}_T = \{x_T : x_T \in M_T\}. \end{gathered} \tag{5.23}$$

Now if (dom φ)$\cap$(dom δ_G)$\neq\varnothing$, then on using the operations of addition and infimal convolution of the polyhedral functions (Theorem 3.15 in this book and Theorem 20.1 of Ref. [226]) φ and δ_G, we can write

$$(\varphi + \delta_G)^* = \varphi^* \oplus \delta_G^* \tag{5.24}$$

Thus, by the relation in Eq. (3.33), we derive that

$$\inf\{\varphi(w) : w \in G\} = \sup\{-\varphi^*(w^*) - \delta_G^*(-w^*)\}.$$

The problem

$$\sup\{-\varphi^*(\mathrm{w}^*) - \delta_{\mathrm{G}}^*(-\mathrm{w}^*)\} \tag{5.25}$$

is called the dual problem to the problem (Eq. (5.22)).

Note that if the value v of problem (PD), i.e., the value of problem in Eq. (5.22), is finite, then in the problem in Eq. (5.25), for every w^* the infimum is attained. Let us investigate the conjugate functions φ^* and δ_G^*, which are polyhedral, too (Theorem 19.2 of Ref. [226]). Put $g_i(x_t, t) = \langle x_t, b_i^t \rangle + \beta_i^t$, $t = 0, \ldots, T$, so that $g(x_t, t) = \max_{i \in I_t} g_i(x_t, t)$, $t = 0, \ldots, T$. Remember that, by Definition 1.21, the convex hull of a collection of functions $\{g_i : i \in I_t\}$ on $\mathbb{R}^n$ denoted by $\operatorname{conv}\{g_i : i \in I_t\}$ is the convex hull of the pointwise infimum of the collection:

$$\operatorname{conv}\{g_i : i \in I_t\}(x) = \inf\left\{ x^0 \in \mathbb{R} : (x^0, x) \in \operatorname{conv}\left(\bigcup_{i \in I_t} \operatorname{epi} g_i \right) \right\}.$$

Theorem 5.6. Let $\operatorname{conv}\{g_i : i \in I_t\}$ be the convex hull of a collection of functions $\{g_i : i \in I_t\}$ on $\mathbb{R}^n$. Then we have

$$\varphi^*(w^*) = \sum_{t=0}^{T} \operatorname{conv}\{g_i^*(\cdot, t) : i \in I_t\}(x_t^*).$$

Moreover, for all $w^* = (x_0^*, \ldots, x_T^*) \in \operatorname{dom} \varphi^*$, there are vectors $x_t^{i*} \in \operatorname{dom} g_i^*$, $i \in I_t$ and numbers $\alpha_i^t \geq 0$, $\sum_{i \in I_t} \alpha_i^t = 1$:

$$\varphi^*(w^*) = -\sum_{t=0}^{t} \sum_{i \in I} \alpha_i^t \beta_i^t, \quad x_t^* = \sum_{i \in I} \alpha_i^t b_i^t.$$

☐ It is not hard to see that, on the one hand,

$$\varphi^*(w^*) = \sum_{t=0}^{T} g^*(x_t^*, t), \tag{5.26}$$

and on the other hand, for any $i \in I_t$, $t = 0, \ldots, T$, we have

$$g_i^*(x_t^*, t) = \begin{cases} -\beta_i^t, & \text{if } x_t^* = b_i^t, \\ +\infty, & \text{if } x_t^* \neq b_i^t. \end{cases} \tag{5.27}$$

By using the duality operations between the pointwise maximum of the convex hull of a collection of convex functions (Theorem 2, Section 3.4 of Ref. [111:189]), we can derive that

$$g^*(x_t^*, t) = \operatorname{conv}\{g_i^*(\cdot, t) : i \in I_t\}(x_t^*), \tag{5.28}$$

where for every $w^* \in \operatorname{dom} \varphi^*$ and so every $x_t^{i*} \in \operatorname{dom} g_i^*$, there are vectors $x_t^{i*} \in \operatorname{dom} g_i^*(\cdot, t)$ and numbers $\alpha_i^t \geq 0$, $\sum_{i \in I_t} \alpha_i^t = 1$ such that

$$g^*(x_t^*, t) = \sum_{i \in I} \alpha_i^t g_i^*(x_t^{i*}, t), \quad x_t^* = \sum_{i \in I_t} \alpha_i^t x_t^{i*}, \quad t = 0, \ldots, T. \tag{5.29}$$

But in view of Eq. (5.27), $x_t^{i*} = b_i^t$ is the only point belonging to dom $g_i^*(\cdot, t)$, so the latter relations give us

$$g^*(x_t^*, t) = -\sum_{t=0}^{t}\sum_{i\in I}\alpha_i^t\beta_i^t, \quad x_t^* = \sum_{i\in I}\alpha_i^t b_i^t, \ \ t = 0, \ldots, T.$$

Thus, taking into account Eqs. (5.26), (5.28), and (5.29), we obtain the required result. ■

Let C be given by

$$C = \{w : \tilde{A}w \le \tilde{d}\}, \tag{5.30}$$

where $\tilde{A}, \tilde{d}$ are a rectangular matrix and a column vector, respectively. The most important result about the δ_C^* representation is the following.

Lemma 5.1. Let δ_C be an indicator function of a polyhedral set (Eq. 5.30). Then one has

$$\delta_C^*(w^*) = \begin{cases} \inf\limits_{\lambda\ge 0}\langle\lambda, \tilde{d}\rangle, & \text{if } \ w^* = \tilde{A}^*\lambda, \quad \lambda \ge 0, \\ +\infty, & \text{if } \ w^* \ne \tilde{A}^*\lambda, \quad \lambda \ge 0. \end{cases}$$

□ By Proposition 5.3, we have $\delta_C(w) = \sup\limits_{\lambda\ge 0}\langle\lambda, \tilde{A}w - \tilde{d}\rangle$. Then we can write

$$\begin{aligned}\delta_C^*(w^*) &= \sup_w\{\langle w, w^*\rangle - \sup_{\lambda\ge 0}\langle\lambda, \tilde{A}w - \tilde{d}\rangle\} = \sup_w\{\langle w, w^*\rangle + \inf_{\lambda\ge 0}\langle\lambda, \tilde{d} - \tilde{A}w\rangle\} \\ &= \sup_w \inf_{\lambda\ge 0}\{\langle w, w^* - \tilde{A}^*\lambda\rangle + \inf_{\lambda\ge 0}\langle\lambda, \tilde{d}\rangle\},\end{aligned}$$

whence immediately the desired result. ■

Now we conclude that $\delta_G^*(-w^*)$. Obviously, in accordance with Proposition 5.3,

$$\begin{aligned}\delta_{G_t}(w) &= \sup_{\lambda_t\ge 0}\langle\lambda_t, Ax_t - Bx_{t+1} - d\rangle, \quad t = 0, \ldots, T-1, \\ \delta_{\tilde{N}_0}(w) &= \sup_{\gamma_0\ge 0}\langle\gamma_0, Nx_0 - p\rangle, \quad \delta_{\tilde{M}_T}(w) = \sup_{\gamma_T\ge 0}\langle\gamma_T, Mx_T - q\rangle.\end{aligned} \tag{5.31}$$

Thus, by virtue of Lemma 5.1, it is easy to obtain that

$$\begin{gathered}\delta_{G_t}^*(-w^*(t)) = \begin{cases} \inf\limits_{\lambda_t\ge 0}\langle\lambda_t, d\rangle, & \text{if } x_t^*(t) = -A^*\lambda_t, x_{t+1}^*(t) = B^*\lambda_t \text{ and } x_i^*(t) = 0, \ \ i \ne t, t+1, \\ +\infty, & \text{otherwise,} \end{cases} \\ w^*(t) = (x_0^*(t), \ldots, x_T^*(t)), \quad t = 0, \ldots, T-1.\end{gathered} \tag{5.32}$$

By analogy with the formula in Eq. (5.32), we compute that

$$\begin{gathered}\delta_{\tilde{N}_0}^*(-w^*(t)) = \begin{cases} \inf\limits_{\gamma_0\ge 0}\langle\gamma_0, p\rangle, & \text{if } x_0^*(-1) = -N^*\gamma_0, \quad x_i^*(-1) = 0, \ \ i \ne 0, \\ +\infty, & \text{otherwise,} \end{cases} \\ w^*(-1) = (x_0^*(-1), \ldots, x_T^*(-1));\end{gathered} \tag{5.33}$$

$$\delta^*_{\tilde{M}_T}(-w^*(T)) = \begin{cases} \inf\limits_{\gamma_T \geq 0} \langle \gamma_T, q \rangle, & \text{if } x^*_T(T) = -M^*\gamma_T, \quad x^*_i(T) = 0, \ i \neq T, \\ +\infty, & \text{otherwise,} \end{cases}$$
$$w^*(T) = (x^*_0(T), \ldots, x^*_T(T)).$$

Remember that $\delta_G = \delta_{\tilde{N}_0} + \sum_{t=0}^{T-1} \delta_{G_t} + \delta_{\tilde{M}_T}$ is a polyhedral function, so by Theorem 3.15 in this book or Theorem 20.1 of Ref. [226], we have $\delta^*_G = \delta^*_{\tilde{N}_0} \oplus \sum_{t=0}^{T-1} \delta^*_{G_t} \oplus \delta^*_{\tilde{M}_T}$, or, in more detail,

$$\delta^*_G(-w^*) = \inf\left\{ \sum_{t=0}^{T-1} \delta^*_{G_t}(-w^*(t)) + \delta^*_{\tilde{N}_0}(-w^*(-1)) + \delta^*_{\tilde{M}_T}(-w^*(-T)) : \sum_{i=-1}^{T} w^*(i) = w^* \right\}, \tag{5.34}$$

and if $\delta^*_G(-w^*) < +\infty$, the infimum is attained for all $w^* = (x^*_0, \ldots, x^*_T)$. In view of the formulas in Eqs. (5.32)–(5.34) and the form of the vectors w^*, $w^*(i)$, $i = -1, \ldots,$ T in relation $\sum_{i=-1}^{T} w^*(i) = w^*$, we have

$$\begin{aligned} &\delta^*_G(-x^*_0, \ldots, -x^*_T) = \inf_{\lambda_t \geq 0, \gamma_0 \geq 0, \gamma_T \geq 0} \left\{ \sum_{i=0}^{T-1} \langle \lambda_t, d \rangle + \langle \gamma_0, p \rangle + \langle \gamma_T, q \rangle \right\} \\ &-A^*\lambda_0 - N^*\gamma_0 = x^*_0, \quad -A^*\lambda_t + B^*\lambda_{t-1} = x^*_t, \\ &B^*\lambda_{T-1} - M^*\gamma_T = x^*_T, \quad t = 1, \ldots, T-1. \end{aligned} \tag{5.35}$$

Now taking into account Eq. (5.35) and Theorem 5.6, it is easy to conclude that the dual problem in Eq. (5.25) can be converted into the following problem (PD*) with linear constraints:

$$\begin{aligned} &(\text{PD}^*) \quad \sup_{\alpha^t_i, \lambda_t, \gamma_0, \gamma_T} \left\{ \sum_{i=0}^{T} \sum_{i \in I_t} \alpha^t_i \beta^t_i - \sum_{i=0}^{T-1} \langle \lambda_t, d \rangle - \langle \gamma_0, p \rangle - \langle \gamma_T, q \rangle \right\}, \\ &A^*\lambda_0 + N^*\gamma_0 + \sum_{i \in I_0} \alpha^0_i b^0_i = 0, \ \ A^*\lambda_t - B^*\lambda_{t-1} + \sum_{i \in I_t} \alpha^t_i b^t_i = 0, \quad t = 1, \ldots, T-1, \\ &M^*\gamma_T - B^*\lambda_{T-1} + \sum_{i \in I_T} \alpha^T_i b^T_i = 0; \quad \sum_{i \in I_t} \alpha^t_i = 1, \ \ \alpha^t_i \geq 0, \ \ \lambda_t \geq 0, \ \ \gamma_0 \geq 0, \ \ \gamma_T \geq 0. \end{aligned}$$

We call this problem the dual problem to the primary problem (PD). These results are summarized as follows.

Theorem 5.7. If the effective domains of functions in the primary problem (PD) have a point in common, then the optimal value in (PD) equals the optimal value in (PD*) and both primary (PD) and dual (PD*) problems have solutions.

Remark 5.1. In the case where $T=1$, setting $g(x_1,1)\equiv 0$, $g_0(x)=g(x_0,0)$, $I_0=I$, $b_i^0=b_i$, $\beta_i^0=\beta_i$, the problem (PD) is converted to the following problem:

$$\text{infimum } g_0(x) \text{ subject to } y\in F(x)\cap M_T, \quad x\in N_0 \tag{5.36}$$

for which the dual problem (PD*) has the form

$$\begin{gathered}\sup_{\alpha_i,\lambda_0,\gamma_0,\gamma_1}\left\{\sum_{i\in I}\alpha_i\beta_i-\langle\lambda_0,d\rangle-\langle\gamma_0,p\rangle-\langle\gamma_1,q\rangle\right\},\\ A^*\lambda_0+N^*\gamma_0+\sum_{i\in I}\alpha_i^0b_i^0=0,\quad M^*\gamma_1-B^*\lambda_0=0,\\ \sum_{i\in I}\alpha_i=1,\alpha_i\geq 0,\quad \lambda_0\geq 0,\quad \gamma_0\geq 0,\quad \gamma_T\geq 0.\end{gathered}\tag{5.37}$$

It turns out that if I consists of a single index and B-zero matrix, then taking $M_T=\mathbb{R}^n$, $N=-E$ (E—identity matrix), $d=p=0$, $\beta_i=0$ instead of the problems in Eqs. (5.36) and (5.37), we get the primary problem and the dual problem of linear programming, respectively.

In the next theorem, there is given a connection between duality and necessary and sufficient conditions for optimality of the primary problem.

Theorem 5.8. Let $\{\tilde{x}_t\}_{t=0}^T$ be an optimal solution to the polyhedral optimization problem (PD). Then the collection of dual variables

$$\lambda_t\geq 0,\quad \gamma_0\geq 0,\quad \gamma_T\geq 0,\quad \alpha_i^t\geq 0\ (i\in I_t),\quad \alpha_i^T\geq 0\ (i\in I_T)\ (t=0,\ldots,T-1)$$

is a solution to the dual problem (PD*) if and only if it satisfies the necessary and sufficient conditions (Eqs. (4.27) and (4.28)) of Theorem 4.4.

☐ Let $\tilde{\alpha}_i{}^t\geq 0\ (i\in I_t)$, $\tilde{\lambda}_t\geq 0$, $\tilde{\alpha}_i{}^T\geq 0\ (i\in I_T)$, $\tilde{\gamma}_0\geq 0$, $\tilde{\gamma}_T\geq 0(t=0,\ldots,T-1)$ be a solution to the problem (PD*). We show that it satisfies Eqs. (4.27) and (4.28) of Theorem 4.4. Let us denote

$$\begin{gathered}\tilde{x}_0{}^*=-N^*\gamma_0,\quad \tilde{x}^*=-M^*\tilde{\gamma}_T,\quad \tilde{u}_t{}^*=\textstyle\sum_{i\in I_t}\tilde{\alpha}_i{}^tb_i^t,\ \ t=0,\ldots,T,\\ \tilde{x}_{t+1}{}^*=B^*\tilde{\lambda}_t,\quad \tilde{x}_t{}^*=A^*\tilde{\lambda}_t+\tilde{u}_t{}^*,\ \ t=0,\ldots,T-1\end{gathered}\tag{5.38}$$

From the equivalency of problems (PD) and Eq. (5.22), it is clear that $\delta_{G_t}(\tilde{w})=\delta_{\tilde{N}_0}(\tilde{w})=\delta_{\tilde{M}_T}(\tilde{w})=0$, where $\tilde{w}=\{\tilde{x}_t\}_{t=0}^T$. Moreover, by virtue of Eq. (5.31),

$$\begin{gathered}\langle\tilde{\lambda}_t,A\tilde{x}_t-B\tilde{x}_{t+1}-d\rangle=0,\quad t=0,\ldots,T-1,\\ \langle\tilde{\gamma}_0,N\tilde{x}_0-p\rangle=0,\quad \langle\tilde{\gamma}_T,M\tilde{x}_T-q\rangle=0.\end{gathered}\tag{5.39}$$

Now we shall show that $\tilde{\alpha}_i{}^t = 0$, if $i \notin I_t(\tilde{x}_t)$ (see Eq. (4.28)). In fact, by Theorem 3.16, $\tilde{w}^* \in \partial\varphi(\tilde{w}) \cap K_G^*(\tilde{w})$, where $\tilde{w}$ is a solution to the problem in Eq. (5.22) and $\tilde{w}^* = (u_0^*, \ldots, u_T^*)$ for given $\tilde{\alpha}_i{}^t \geq 0$ $(i \in I_t)$, $\tilde{\lambda}_t \geq 0$, $\tilde{\alpha}_i{}^T \geq 0$ $(i \in I_T)$, $\tilde{\gamma}_0 \geq 0$, $\tilde{\gamma}_T \geq 0$ $(t = 0, \ldots, T-1)$ is a solution (Eq. (5.25)). Then in accordance with Theorem 1.28, it is not hard to see that $\partial\varphi(\tilde{w}) = \partial g(\tilde{x}_0, 0) \times \partial g(\tilde{x}_1, 1) \times \cdots \times \partial g(\tilde{x}_T, T)$ and the inclusion $\tilde{w}^* \in \partial\varphi(\tilde{w})$ implies that $\tilde{u}_t{}^* \in \partial g(\tilde{x}_t, t)$ or $\tilde{u}_t{}^* \in \operatorname{conv}\left\{\cup_{t \in I_t(\tilde{x}_t)} b_i^t\right\}$ $(t = 0, \ldots, T)$. Here with the preceding notations (Eq. (5.38)) from (PD*), we get Eq. (4.27) and the fact that $\tilde{x}^* + \tilde{x}_T{}^* = u_T^*$. Conversely, let us show that if the vectors x^*, x_t^*, $u_t^* = \sum_{i \in I_t(\tilde{x}_t)} \alpha_i^t b_i^t$, $\sum_{i \in I_t(\tilde{x}_t)} \alpha_i^t = 1$ $(t = 0, \ldots, T)$ satisfy the necessary and sufficient conditions (Eqs. (4.27) and (4.28)) of Theorem 4.4, then the corresponding vectors

$$\lambda_t \geq 0, \quad \gamma_0 \geq 0, \quad \gamma_T \geq 0 \quad \text{with } \alpha_i^t \geq 0 \ (i \in I_t), \quad \alpha_i^T \geq 0 \ (i \in I_T) \ (t = 0, \ldots, T-1)$$

form a solution to the dual problem (PD*). At once we observe that these vectors form a feasible solution for (PD*). Indeed, setting $\alpha_i^t = 0$, $i \notin I_t(\tilde{x}_t)$ in Eq. (4.47) of Theorem 4.4, we can rewrite as follows:

$$\begin{aligned} &x_0^* = A^*\lambda_0 + \textstyle\sum_{i \in I_0} \alpha_i^0 b_i^0, \quad x_t^* = A^*\lambda_t + \textstyle\sum_{i \in I_t} \alpha_i^t b_i^t, \\ &\quad x_t^* = B^*\lambda_{t-1}, \ \ x_T^* = B^*\lambda_{T-1}, \quad t = 1, \ldots, T-1. \end{aligned} \tag{5.40}$$

Obviously, the second and third relations of Eq. (5.40) yield the second constraint in problem (PD*). Taking into account the conditions $x_T^* + x^* = u_T^*$, $x_0^* = -N^*\gamma_0$ and $x^* = -M^*\gamma_T$ of Theorem 4.4, the other conditions of problem (PD*) are easily verified. It remains to show that $\lambda_t \geq 0$, $\gamma_0 \geq 0$, $\gamma_T \geq 0$ with $\alpha_i^t \geq 0$ $(i \in I_t)$, $\alpha_i^T \geq 0$ $(i \in I_T)$ $(t = 0, \ldots, T-1)$ maximize the objective function of problem (PD*). Indeed, by the necessary and sufficient condition of Theorem 4.4,

$$\begin{aligned} &\langle \gamma_0, p \rangle = \langle N^*\gamma_0, \tilde{x}_0 \rangle, \ \ \langle \gamma_T, q \rangle = \langle M^*\gamma_T, \tilde{x}_T \rangle, \\ &\langle \lambda_t, d \rangle = \langle \lambda_t, A\tilde{x}_t - B\tilde{x}_{t+1} \rangle, \quad t = 0, \ldots, T-1. \end{aligned}$$

By using the latter inequalities, we can write

$$\begin{aligned} v^* = &\sup_{\alpha_i^t, \lambda_t, \gamma_0, \gamma_T} \left\{ \sum_{i=0}^{T} \sum_{i \in I_t} \alpha_i^t \beta_i^t - \sum_{i=0}^{T-1} \langle \lambda_t, d \rangle - \langle \gamma_0, p \rangle - \langle \gamma_T, q \rangle \right\} \\ \geq &\sum_{i=0}^{T} \sum_{i \in I_t(\tilde{x}_t)} \alpha_i^t \beta_i^t - \sum_{i=0}^{T-1} \langle \lambda_t, A\tilde{x}_t - B\tilde{x}_{t+1} \rangle - \langle N^*\gamma_0, \tilde{x}_0 \rangle - \langle M^*\gamma_T, \tilde{x}_T \rangle. \end{aligned}$$

Thus, by virtue of the constraints in problem (PD*), we have

$$
\begin{aligned}
v^* \quad & \geq \sum_{t=0}^{T}\sum_{i\in I_t}\alpha_i^t\beta_i^t + \sum_{t=1}^{T-1}\left[\left\langle \sum_{i\in I_t}\alpha_i^t\beta_i^t, \tilde{x}_t\right\rangle - \langle B^*\lambda_{t-1}, \tilde{x}_t\rangle + \langle B^*\lambda_t, \tilde{x}_{t+1}\rangle\right] \\
& - \langle N^*\gamma_0, \tilde{x}_0\rangle - \langle M^*\gamma_T, \tilde{x}_T\rangle = \sum_{t=0}^{T}\sum_{i\in I_t(\tilde{x}_t)}\alpha_i^t\beta_i^t + \sum_{t=1}^{T-1}\sum_{i\in I_t(\tilde{x}_t)}\langle \alpha_i^t\beta_i^t, \tilde{x}_t\rangle - \langle B^*\lambda_0, \tilde{x}_1\rangle \\
& + \langle B^*\lambda_{T-1}, \tilde{x}_t\rangle + \langle B^*\lambda_0, \tilde{x}_1\rangle - \langle A^*\lambda_0, \tilde{x}_0\rangle - \langle N^*\gamma_0, \tilde{x}_0\rangle - \langle M^*\gamma_T, \tilde{x}_T\rangle \\
& = \sum_{t=0}^{T}\sum_{i\in I_t(\tilde{x}_t)}\alpha_i^t\beta_i^t + \sum_{t=0}^{T}\sum_{i\in I_t}\langle \alpha_i^t\beta_i^t, \tilde{x}_t\rangle \\
& = \sum_{t=0}^{T}\left\{\sum_{i\in I_t(\tilde{x}_t)}\alpha_i^t(\langle \beta_i^t, \tilde{x}_t\rangle + \beta_i^t)\right\} \\
& = \sum_{t=0}^{T} g(\tilde{x}_t, t).
\end{aligned}
$$

Hence by duality Theorem 5.7, it follows that $v^* = \sum_{t=0}^{T} g(\tilde{x}_t, t)$ and so the collection of $\lambda_t \geq 0$, $\gamma_0 \geq 0$, $\gamma_T \geq 0$ with $\alpha_i^t \geq 0 (i \in I_t)$, $\alpha_i^T \geq 0 \ (i \in I_T) \ (t = 0, \ldots, T-1)$ is a solution to the dual problem (PD*). ■

Remark 5.2. We remark that the first part of the proof of Theorem 5.8 can be developed by calculation of the dual cone $K_G^*(\tilde{w})$ (see Section 4.2).

We now study the optimization of polyhedral differential inclusions of the form of Eqs. (4.31) and (4.32) considered in Section 4.3:

$$
\begin{aligned}
&\text{infimum} \quad && J(x(\cdot)) = \int_0^1 g(x(t), t)\mathrm{d}t + \varphi_0(x(1)), \\
&\text{subject to} \quad && \dot{x}(t) \in F(x(t)), \quad \text{a.e. } t \in [0, 1], \\
& && x(0) \in N_0, \quad x(1) \in M_1,
\end{aligned}
\tag{5.41}
$$

$$
F(x) = \{y : Ax - By \leq d\}, \tag{5.42}
$$

where A, B, are rectangular matrices and column vectors as described in problem (PD). The functions g, φ_0, and the sets N_0, M_1 are the same polyhedral sets defined in (PD). The following problem of determining the supremum

$$
\begin{aligned}
&\sup_{\lambda(t), u^*(t), \gamma_0, \gamma_1}\left\{-\int_0^1 [\langle \lambda(t), d\rangle + g^*(u^*(t), t)]\mathrm{d}t - \varphi_0^*(u^*(1)) - \langle \gamma_0, p\rangle - \langle \gamma_1, q\rangle\right\}, \\
&\qquad B^*\lambda(0) + N^*\gamma_0 = 0, \quad A^*\lambda(t) + B^*\dot{\lambda}(t) + u^*(t) = 0, \\
&\qquad M^*\gamma_1 - B^*\lambda(1) + u^*(1) = 0, \quad \lambda(t) \geq 0, \ \gamma_0 \geq 0, \ \gamma_1 \geq 0
\end{aligned}
\tag{5.43}
$$

we call the dual problem to the primary problem in Eqs. (5.41) and (5.42). Here we assume that $\lambda(\cdot)$ and $u^*(\cdot)$ are absolutely continuous functions defined on the closed interval [0,1]. Let $\sum_{[0,1]}(0, N_0)$ be the family of solutions to the polyhedral differential inclusion in Eq. (5.41) with the initial condition $x(0) \in N_0$ defined on [0,1]. Note that the solutions are defined only in some subinterval of the segment [0,1], and therefore we do not refer to the set $\sum_{[0,1]}(0, N_0)$. Moreover, let $S(\tau,0,N_0)$ be a section of the family of solutions such that

$$S(\tau, 0, N_0) = \left\{ x(\tau) : x(t) \in \sum_{[0,1]}(0, N_0) \right\}, \quad \tau \in [0, 1].$$

In what follows, by aff M we denote the affine hull of a set M.

Theorem 5.9. Let $\tilde{x}(t)$ be an optimal solution to primary problem in Eqs. (5.41) and (5.42) with the polyhedral differential inclusion and let conditions (1) and (2) of Theorem 4.5 be satisfied. Moreover, assume that either

a. $M_1 \cap \operatorname{int} S(1,0,N_0) \neq \varnothing$ or
b. $\operatorname{ri} S(1,0,N_0) \cap \operatorname{ri} M_1 \neq \varnothing$

is fulfilled and there is no hyperplane H containing both aff $S(1,0,N_0)$ and aff M_1. Then a collection $\{\tilde{\lambda}(t),\ \tilde{u}^*(t),\ \tilde{\gamma}_0,\ \tilde{\gamma}_1\}$ is an optimal solution to the dual problem in Eq. (5.43) if and only if conditions (1) and (2) of Theorem 4.5 are satisfied. In addition, the optimal value in Eqs. (5.41) and (5.42) equals the optimal value in Eq. (5.43).

☐ First of all, we establish that for all feasible solutions $x(t)$ and $\{\lambda(t), u^*(t), \gamma_0, \gamma_1\}$ in problems (5.41)–(5.43), respectively

$$\begin{aligned} \int_0^1 g(x(t), t)\mathrm{d}t + \varphi_0(x(1)) \quad &\geq \int_0^1 [\langle \lambda(t), d\rangle + g^*(u^*(t), t)]\mathrm{d}t \\ &-\varphi_0^*(u^*(1)) - \langle \gamma_0, p\rangle - \langle \gamma_1, q\rangle. \end{aligned} \tag{5.44}$$

Indeed for a feasible solution $x(t)$, $t \in [0,1]$

so $\quad Ax(t) - B\dot{x}(t) \leq d, \quad Nx(0) \leq p, \quad Mx(1) \leq q,$

$$\begin{gathered} \langle \lambda(t), d\rangle \geq \langle Ax(t) - B\dot{x}(t),\ \ \lambda(t)\rangle,\ \ \lambda(t) \geq 0, \\ \langle \gamma_0, p\rangle \geq \langle \gamma_0, Nx(0)\rangle,\ \ \gamma_0 \geq 0, \quad \langle \gamma_0, p\rangle \geq \langle \gamma_1, Mx(1)\rangle,\ \ \gamma_1 \geq 0. \end{gathered} \tag{5.45}$$

On using the definition of conjugate function (Section 1.5) and inequalities (5.45), we can write

$$\begin{aligned} &- \int_0^1 [\langle \lambda(t), d\rangle + g^*(u^*(t), t)]\mathrm{d}t - \varphi_0^*(u^*(1)) - \langle \gamma_0, p\rangle - \langle \gamma_1, q\rangle \\ &\leq \int_0^1 [\langle \lambda(t), B\dot{x}(t) - Ax(t)\rangle + g(x(t), t) - \langle x(t), u^*(t)\rangle]\mathrm{d}t + \varphi(x)(1)) \\ &- \langle x(1), u^*(1)\rangle - \langle \gamma_0, Nx(0)\rangle - \langle \gamma_1, Mx(1)\rangle = \int_0^1 g(x(t), t)\mathrm{d}t + \varphi_0(x(1)) \\ &+ \int_0^1 [\langle \lambda(t), B\dot{x}(t) - Ax(t)\rangle - \langle x(t), u^*(t)\rangle]\mathrm{d}t - \langle x(1), u^*(1)\rangle - \langle \gamma_0, Nx(0)\rangle - \langle \gamma_1, Mx(1)\rangle \end{aligned} \tag{5.46}$$

On the other hand, by virtue of the relationships of the problem in Eq. (5.43),

$$\begin{aligned}
&\int_0^1 [\langle \lambda(t), B\dot{x}(t) - Ax(t)\rangle - \langle x(t), u^*(t)\rangle]\mathrm{d}t - \langle x(1), u^*(1)\rangle - \langle \gamma_0, Nx(0)\rangle - \langle \gamma_1, Mx(1)\rangle \\
&\quad = \int_0^1 [\langle \lambda(t), B\dot{x}(t) - Ax(t)\rangle + \langle A^*\lambda(t) + B^*\dot{\lambda}(t), x(t)\rangle]\mathrm{d}t + \langle M^*\gamma_1 - B^*\lambda(1), x(1)\rangle \\
&\quad\ -\langle \gamma_0, Nx(0)\rangle - \langle \gamma_1, Mx(1)\rangle \\
&\quad = \int_0^1 [\langle B^*\lambda(t), \dot{x}(t)\rangle + \langle B^*\dot{\lambda}(t), x(t)\rangle]\mathrm{d}t - \langle B^*\lambda(1), x(1)\rangle - \langle N^*\gamma_0, x(0)\rangle \\
&\quad = \int_0^1 d\langle B^*\lambda(t), x(t)\rangle - \langle B^*\lambda(1), x(1)\rangle - \langle N^*\gamma_0, x(0)\rangle \\
&\quad = \langle B^*\lambda(1), x(1)\rangle - \langle B^*\lambda(0), x(0)\rangle - \langle B^*\lambda(1), x(1)\rangle - \langle N^*\gamma_0, x(0)\rangle = 0.
\end{aligned}$$

Therefore, the validity of the inequality in Eq. (5.44) is immediate from the latter equality and the inequality in Eq. (5.45). Furthermore, conditions (a) and (b) of the theorem implies that in condition (1) of Theorem 4.5, $\lambda_0 = 1$. Indeed, if this is not so, i.e., if $\lambda_0 = 0$, then $\tilde{x}^*(1) + \tilde{x}_e{}^* = 0$, $\tilde{x}_e{}^* = -M^*\tilde{\gamma}_1$, $\tilde{\gamma}_1 \geq 0$, where $\tilde{x}^*(1) = B^*\tilde{\lambda}(1) \in K^*_{S(1,0,N_0)}(\tilde{x}(1))$. Since λ_0, $\tilde{x}^*(1)$, $\tilde{x}_e{}^*$ are not all zero, the cones K_{M_1} and $K_S(1,0,N_0)$ are separated [111,226]. But at the same time it follows from conditions (a) and (b) that these cones have no separation property. This contradiction means that $\lambda_0 = 1$. Thus, in accordance with Theorems 1.26 and 4.5, we conclude that.

$$\varphi_0(\tilde{x}(1)) + \varphi_0^*(\tilde{x}^*(1) - M^*\gamma_1) = \langle \tilde{x}(1), \tilde{x}^*(1) - M^*\gamma_1\rangle;$$

i.e.,

$$\varphi_0^*(\tilde{u}^*(1)) = \langle \tilde{x}(1), \ \tilde{u}^*(1)\rangle - \varphi_0(\tilde{x}(1)).$$

Also it is clear that $g^*(\tilde{u}^*(t), t) = \langle \tilde{x}(t), \ \tilde{u}^*(t)\rangle - g(\tilde{x}(t), t)$ and for $\tilde{x}(t)$, $\tilde{\lambda}(t)$, $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ in the relations in Eq. (5.45), the equality sign holds. In other words, for the trajectory $\tilde{x}(t)$ and collection $\{\tilde{\lambda}(t), \ \tilde{u}^*(t), \ \tilde{\gamma}_0, \tilde{\gamma}_1\}$,

$$\begin{aligned}
\int_0^1 g(\tilde{x}(t), t)\mathrm{d}t + \varphi_0(\tilde{x}(1)) \quad &= \int_0^1 [\langle \tilde{\lambda}(t), d\rangle + g^*(\tilde{u}^*(t), t)]\mathrm{d}t \\
&\quad -\varphi_0^*(\tilde{u}^*(1)) - \langle \tilde{\gamma}_0, p\rangle - \langle \tilde{\gamma}_1, q\rangle.
\end{aligned}$$

The proof is ended. ■

5.4 Duality in Problems Described by Convex Discrete Inclusions

Optimization of nonconvex discrete inclusions in more general conditions was investigated in Section 4.3. In this section, the optimal control problem described by ordinary convex discrete inclusions with state constraints is considered. For such problems, the dual problem is constructed and under some conditions the duality relations are established, and a connection between solutions of dual problem and necessary and sufficient condition for optimality in primary problems is studied. The problem consists of the following:

$$\inf \sum_{t=0}^{T} g(x_t, t) \tag{5.47}$$

(P) subject to

$$\begin{aligned} &x_{t+1} \in F_t(x_t), \quad t = 0, \ldots, T-1, \\ &x_0 \in N, \quad x_T \in M, \\ &x_t \in D_t, \quad t = 1, \ldots, T-1, \end{aligned} \tag{5.48}$$

where $g(\cdot, t)$ are proper convex functions, $F_t : \mathbb{R}^n \to P(\mathbb{R}^n)$ is a multivalued function for all fixed t, and N, M, D_t ($t = 1, \ldots, T-1$) are convex sets in $\mathbb{R}^n$. It is required to find a feasible trajectory $\{x_t\}_{t=0}^{T}$ minimizing Eq. (5.47). We label this problem (P).

In what follows, we say that the regularity condition for the problem in Eqs. (5.47) and (5.48) is satisfied, if for a point x_t^0, $t = 0, \ldots, T$ either

1. $x_t^0 \in \text{ri dom } g(\cdot, t)$, $(x_t^0, x_{t+1}^0) \in \text{ri gph } F_t$, $x_t^0 \in \text{ri } D_t$, $t = 0, \ldots, T$, where $D_0 = N$, $D_T = M$, or
2. $(x_t^0, x_{t+1}^0) \in \text{int gph } F_t$ $(t = 0, \ldots, T)$, $x_t^0 \in \text{int} \in D_t (t = 1, \ldots, T-1)$, $x_0^0 \in \text{int} \in N$, $x_T^0 \in M$ and $g(\cdot, t)$ are continuous at x_t^0.

Theorem 5.10. Let F_t and $g(\cdot, t)$, $t = 1, \ldots, T$ be a convex multivalued mapping and closed proper convex function, respectively. Moreover, let the regularity condition for the problem in Eqs. (5.47) and (5.48) be satisfied. Then in order for $\{\tilde{x}_t\}_{t=0}^{T}$ be an optimal solution to the problem in Eqs. (5.47) and (5.48), it is necessary and sufficient that there exist vectors x^*, x_t^*, $t = 0, \ldots, T$, such that

1. $\begin{aligned} &x_t^* \in F_t^*(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})) + K_{D_t}^*(\tilde{x}_t) - \partial_x g(\tilde{x}_t, t), \\ &K_{D_0}^*(\tilde{x}_0) = \{0\}, \quad t = 0, \ldots, T-1, \end{aligned}$
2. $x^* - x_T^* \in \partial_x g(\tilde{x}_T, T), \quad -x_0^* \in K_N^*(\tilde{x}_0), \quad x^* \in K_M^*(\tilde{x}_T).$

□ As in Section 4.2, setting

$$\begin{aligned} &Q_t = \{w : (x_t, x_{t+1}) \in \text{gph } F_t\}, \quad t = 0, \ldots, T-1, \\ &\tilde{N} = \{w : x_0 \in N\}, \quad \tilde{M} = \{w : x_T \in M\}, \\ &\tilde{D}_t = \{w : w_t \in D_t\}, \quad t = 1, \ldots, T-1 \end{aligned} \tag{5.49}$$

$$w = (x_0, \ldots, x_T) \in \mathbb{R}^{n(1+T)},$$

this problem can be replaced by a convex minimization problem in the space $\mathbb{R}^{n(1+T)}$:

$$\text{infimum } \varphi(w) = \sum_{t=1}^{T} g(x_t, t) \text{ subject to } Q = \tilde{N} \cap \left(\bigcap_{t=0}^{T-1} Q_t \right) \cap \left(\bigcap_{t=1}^{T-1} D_t \right) \cap \tilde{M}. \tag{5.50}$$

In accordance with the regularity condition, it follows from Theorems 1.10 and 1.29 that

$$K_Q^*(\tilde{w}) = K_{\tilde{N}}^*(\tilde{w}) + \sum_{t=0}^{T-1} K_{Q_t}^*(\tilde{w}) + \sum_{t=1}^{T-1} K_{\tilde{D}_t}^*(\tilde{w}) + K_{\tilde{M}}^*(\tilde{w}), \quad \tilde{w} = (\tilde{x}_0, \ldots, \tilde{x}_T).$$

Then the rest of the proof is the same as for Theorem 4.1 (see also Theorem 4.2 and Remark 4.4). ■

We call the following problem, labeled (P^*), the dual problem to the primary problem (P):

$$\sup_{\substack{u_t^*, x_t^*, v_t^* \\ (v_0 = v_T = 0), \\ t=0,\ldots,T}} \left\{ -\sum_{t=0}^{T} g^*(u_t^*, t) + \sum_{t=0}^{T-1} M_{F_t}(x_t^* - v_t^* + u_t^*, x_{t+1}^*) - \sum_{t=1}^{T-1} W_{D_t}(-v_t^*) - W_N(x_0^*) - W_M(-u_T^* - x_T^*) \right\}, \tag{5.51}$$

where W_C is a support function of the set C and

$$M_{F_t}(x^*, y^*) = \inf_{(x,y)} \{\langle x, x^* \rangle - \langle y, y^* \rangle : (x, y) \in \text{gph } F_t\}.$$

Theorem 5.11. If v and v^* are the optimal values of the optimization problem for ordinary discrete inclusions (P) and its dual problem (P^*), respectively, then $v \geq v^*$ for all feasible solutions of primary and dual problems. Moreover, if the regularity condition is satisfied, then the existence of a solution to one of these problems implies the existence of a solution to the other problem, where $v = v^*$ and in the case $v > -\infty$ the dual problem (P^*) has a solution.

☐ As was shown in Section 5.2, the dual problem to the primary problem in Eq. (5.50) has the form

$$\sup\{-f^*(w^*) - \delta_Q^*(-w^*)\} \tag{5.52}$$

where $\delta_Q(\cdot)$ is the indicator function of Q. Besides by the duality of the operations of addition and infimal convolution of convex functions (Theorem 3.15), if there exists a point $w_1 \in Q$ where f is continuous, the optimal value in Eq. (5.50) equals the optimal value in Eq. (5.52):

$$\inf\{f(w) : w \in Q\} = \sup\{-f^*(w^*) - \delta_Q^*(-w^*)\} \tag{5.53}$$

Note that the regularity condition guarantees that a point $w_1 \in Q$ having this property exists. In addition, if the value of the problem in Eq. (5.50) is finite, then the supremum in the problem in Eq. (5.52) is attained for all w^*. Since $\delta_Q = \sum_{t=0}^{T-1} \delta_{Q_t} + \sum_{t=1}^{T-1} \delta_{\tilde{D}_t} + \delta_{\tilde{N}} + \delta_{\tilde{M}}$, by the duality theorem, we have

$$\delta_Q^*(-w^*) \le \inf\left\{ \sum_{t=0}^{T-1} \delta_{Q_t}^*(-w^*(t)) + \sum_{t=1}^{T-1} \delta_{\tilde{D}_t}^*(-\tilde{w}^*(t)) + \delta_{\tilde{N}}^*(-w^*(-1)) \right.$$
$$\left. + \delta_{\tilde{M}}^*(-w^*(T)) : \sum_{i=-1}^{T} w^*(i) + \sum_{i=1}^{T-1} \tilde{w}(i) = w^* \right\},$$
$$w^*(i) = (x_0^*(i), \ldots, x_T^*(i)), \quad \tilde{w}^*(i) = (\tilde{x}_0{}^*(i), \ldots, \tilde{x}_T{}^*(i)),$$

where

$$\delta_{Q_t}^*(-w^*(t)) = \begin{cases} -\inf\limits_{(x_t(t), x_{t+1}(t)) \in \operatorname{gph} F} \{\langle x_t(t), x_t^*(t)\rangle + \langle x_{t+1}(t), x_{t+1}^*(t)\rangle\}, & \text{if } x_i^*(t) = 0, \\ & i \ne t,\ t+1, \\ +\infty, & \text{otherwise}, \end{cases}$$
$$t = 0, \ldots, T-1;$$

$$\delta_{\tilde{D}_t}^*(-w^*(t)) = \begin{cases} -\inf\{\langle \tilde{x}_t(t), \tilde{x}_t{}^*(t)\rangle : \tilde{x}_t(t) \in D_t\}, & \text{if } \tilde{x}_i{}^*(t) = 0,\ \ i \ne t, \\ +\infty, & \text{otherwise}, \end{cases}$$
$$t = 1, \ldots, T-1;$$

$$\delta_{\tilde{N}}^*(-\tilde{w}^*(-1)) = \begin{cases} -\inf\{\langle x_0(-1), x_0^*(-1)\rangle : x_0(-1) \in N\}, & \text{if } x_i^*(-1) = 0,\ \ i \ne 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$\delta_{\tilde{M}}^*(-\tilde{w}^*(T)) = \begin{cases} -\inf\{\langle x_T(T),\ \ x_T^*(T)\rangle : x_T(T) \in M\}, & \text{if } x_i^*(T) = 0,\ \ i \ne T, \\ +\infty, & \text{otherwise}. \end{cases}$$

Now, taking into account these relationships and the formula $f^*(w^*) = \sum_{t=0}^{T} g(u_t^*, t)$ with the preceding notations, we conclude that

$$\sup\{-f^*(w^*) - \delta_Q^*(-w^*)\} = \sup\left\{ -\sum_{t=0}^{T} g(x_t^*, t) + \sum_{t=0}^{T-1} M_{F_t}(x_t^*(t), -x_{t+1}^*(t)) \right.$$
$$- W_N(-x_0^*(-1)) - W_M(-x_T^*(T)) - \sum_{t=1}^{T-1} W_{D_t}(-\tilde{x}_t{}^*(t)) :$$
$$\left. x_t^*(t-1) + x_t^*(t) + \tilde{x}_t{}^*(t) = x_t^*, \quad t = 0, \ldots, T;\ \ \tilde{x}_T{}^*(T) = 0 \right\},$$

where supremum is attained, if $v > -\infty$. Thus, for convenience if we denote $x_t^* \equiv u_t^*$ and then $x_t^*(t-1) \equiv -x_t^*$, $\tilde{x}_t{}^*(t) \equiv v_t^*$, $t = 0, \ldots, T$ by virtue of the relations

$$x_t^*(t-1) + x_t^*(t) + \tilde{x}_t{}^*(t) = x_t^*, \quad t = 0, \ldots, T; \quad \tilde{x}_0{}^*(0) = \tilde{x}_T{}^*(T) = 0,$$

the right-hand side of the latter equality has the form of Eq. (5.51). ■

Corollary 5.1. Let $D_t = \mathbb{R}^n$, $t = 1, \ldots, T-1$ in problem (P); i.e., we consider a problem without state constraints. Then, since

$$W_{D_t}(v_t^*) = \begin{cases} 0, & \text{if } v_t^* = 0, \\ +\infty, & \text{if } v_t^* \neq 0, \end{cases}$$

the dual problem (P^*) has the form

$$\sup_{u_t^*, x_t^*,\ t=0,\ldots,T} \left\{ -\sum_{t=0}^{T} g^*(u_t^*, t) + \sum_{t=0}^{T-1} M_{F_t}(x_t^* - u_t^* + x_{t+1}^*) - W_N(x_0^*) - W_M(-u_T^* - x_T^*) \right\} \tag{5.54}$$

Theorem 5.12. Let $\{\tilde{x}_t\}_{t=0}^T$ be an optimal solution to problem (P) and suppose that the regularity condition is satisfied. Then the collection of vectors x_t^*, u_t^*, v_t^* $(v_0^* = v_T^* = 0)$, $t = 0, \ldots, T$ is an optimal solution to the dual problem (P^*) if and only if conditions (1) and (2) of Theorem 5.10 are satisfied.

☐ Suppose that conditions (1) and (2) of Theorem 5.10 are satisfied; i.e.,

$$\begin{aligned} &x_t^* - v_t^* \in F_t^*(x_{t+1}^*; (\tilde{x}_t, \tilde{x}_{t+1})) - u_t^*, \quad u_t^* \in \partial_x g(\tilde{x}_t, t), \\ &v_t^* \in K_{D_t}^*(\tilde{x}_t), \quad t = 0, \ldots, T-1, \quad u_T^* \in \partial_x g(\tilde{x}_T, T), \\ &u_T^* + x_T^* \in K_M^*(\tilde{x}_T), \quad -x_0^* \in K_N^*(\tilde{x}_0), \quad v_0^* \equiv 0. \end{aligned}$$

Moreover, the problems (P) and Eq. (5.50) are equivalent, so

$$\tilde{w} = (\tilde{x}_0, \ldots, \tilde{x}_T) \in \{w : f(w) + \delta_Q(w) = \alpha\}.$$

Hence, $0 \in \partial(f(\tilde{w}) + \delta_Q(\tilde{w}))$ or $\tilde{w} \in \partial(f + \delta_Q)^*(0) = \partial(f^* \oplus \delta_Q^*)(0)$. By Proposition 3.2, $\partial(f^* \oplus \delta_Q^*)(0) = \partial f^*(w^*) \cap \partial \delta_Q^*(-w^*)$, the latter inclusion implies that

$$\tilde{w} \in \partial f^*(w^*) \cap \partial \delta_Q^*(-w^*) \neq \emptyset. \tag{5.55}$$

Therefore, in view of Proposition 3.3, it can be deduced that w^* is a solution to the maximization problem $\sup\{-f^*(w^*) - \delta_Q^*(-w^*)\}$. Thus, from the regularity condition and Proposition 3.2, we have

$$\partial \delta_Q^*(-w^*) = \partial \left[\bigoplus_{t=0}^{T-1} \delta_{Q_t}^* \bigoplus_{t=1}^{T-1} \delta_{D_t}^* \oplus \delta_M^* \oplus \delta_N^* \right](-w^*), \quad \sum_{i=-1}^{T} w^*(i) + \sum_{i=1}^{T-1} \tilde{w}(i) = w^* \tag{5.56}$$

Now, from the relations in Eq. (5.55) and Eq. (5.56), we derive

$$\begin{aligned}&-\tilde{w}^*(t)\in\partial\delta_{Q_t}(\tilde{w}),\quad -\tilde{w}^*(t)\in\partial\delta_{\tilde{D}_t}(\tilde{w}),\ \ t=1,\dots,T-1,\\&-w^*(0)\in\partial\delta_{Q_0}(\tilde{w}),\quad -w^*(-1)\in\partial\delta_{\tilde{N}}(\tilde{w}),\quad -w^*(T)\in\partial\delta_{\tilde{M}}(\tilde{w}),\quad w^*\in\partial f(\tilde{w}).\end{aligned}\tag{5.57}$$

Since (see Eq. (1.45)) on the one hand

$$\begin{aligned}&\partial\delta_{Q_t}(\tilde{w})=-K^*_{Q_t}(\tilde{w}),\quad \partial\delta_{\tilde{D}_t}(\tilde{w})=-K^*_{\tilde{D}_t}(\tilde{w}),\\&\partial\delta_{\tilde{N}}(\tilde{w})=-K^*_{\tilde{N}}(\tilde{w}),\quad \partial\delta_{\tilde{M}}(\tilde{w})=-K^*_{\tilde{M}}(\tilde{w}),\end{aligned}$$

and on the other hand

$$K^*_{Q_t}(\tilde{w})=\{w^*(t):(x^*_t(t),x^*_{t+1}(t))\in K^*_{\text{gph }F_t}(\tilde{x}_t,\tilde{x}_{t+1}):x^*_t(t)=0,\ \ i\neq t,\ \ t+1\},$$
$$t=0,\dots,T-1.$$

$$\begin{aligned}&K^*_{D_t}(\tilde{w})=\{w^*(t):\tilde{x}_t{}^*(t)\in K^*_{D_t}(\tilde{x}_t):\tilde{x}_t{}^*(t)=0,i\neq t\},\quad t=1,\dots,T-1,\\&K^*_{\tilde{N}}(\tilde{w})=\{w^*(-1):x^*_0(-1)\in K^*_N(\tilde{x}_0),x^*_i(-1)=0,\ \ i\neq 0\},\\&K^*_{\tilde{M}}(\tilde{w})=\{w^*(T):x^*_T(T)\in K^*_M(\tilde{x}_T),x^*_i(T)=0,\ \ i\neq T\}.\end{aligned}$$

The inclusions in Eq. (5.57) imply that

$$\begin{aligned}&(x^*_t(t),x^*_{t+1}(t))\in K^*_{\text{gph }F_t}(\tilde{x}_t,\tilde{x}_{t+1}),\quad t=0,\dots,T-1;\\&\tilde{x}_t{}^*(t)\in K^*_{D_t}(\tilde{x}_t),\ \ t=1,\dots,T-1;\quad x^*_0(-1)\in K^*_N(\tilde{x}_0),\\&x^*_T(T)\in K^*_M(\tilde{x}_T),\quad x^*_t\in\partial_x g(\tilde{x}_t,t),\ \ t=0,\dots,T.\end{aligned}\tag{5.58}$$

where the vectors $w^*(t)$ $(t=0,\ \dots,\ T-1)$, $\tilde{w}^*(t)$ $(t=1,\dots,T-1)$, $w^*(-1)$, $w^*(T)$ satisfy the condition $\sum_{i=-1}^{T}w^*(i)+\sum_{i=1}^{T-1}\tilde{w}(i)=w^*$ (see Eq. (5.56)). Now, denoting again $x^*_t\equiv u^*_t$, $x^*_t(t-1)\equiv v^*_t$, $t=0,\dots,T$, where the collection of vectors x^*_t, u^*_t, v^*_t, $t=0,\dots,T$ $(v^*_0=v^*_T=0)$ is a solution to the dual problem (P^*), from Eq. (5.58), by applying the definition of LAM we may conclude that conditions (1) and (2) of Theorem 5.10 are satisfied.

Let us now prove the converse assertion. Let the collection of vectors x^*_t, u^*_t, v^*_t, $t=0,\dots,T$ be an optimal solution to the primary problem (P). Rewrite the adjoint inclusion

$$x^*_t\in F^*_t(x^*_{t+1};(\tilde{x}_t,\tilde{x}_{t+1}))+v^*_t-u^*_t,(v^*_t\in K^*_{D_t}(\tilde{x}_t),u^*_t\in\partial_x g(\tilde{x}_t,t)),$$

as follows $x^*_t-v^*_t+u^*_t\in F^*_t(x^*_{t+1};(\tilde{x}_t,\tilde{x}_{t+1}))$. By Lemma 2.6, the latter inclusion and condition

$$M_{F_t}(x^*_t-v^*_t+u^*_t,x^*_{t+1})=\langle\tilde{x}_t,x^*_t-v^*_t+u^*_t\rangle-H(\tilde{x}_t,x^*_{t+1}),\quad t=0,\dots,T-1$$

are equivalent. So by Theorem 2.1, and taking into account that $\tilde{x}_{t+1} \in F_t(\tilde{x}_t, x^*_{t+1})$ or $\langle \tilde{x}_{t+1}, x^*_{t+1} \rangle = H(\tilde{x}_t, x^*_{t+1})$, we get

$$M_{F_t}(x_t^* - v_t^* + u_t^*, x^*_{t+1}) = \langle \tilde{x}_t, x_t^* - v_t^* + u_t^* \rangle - \langle \tilde{x}_{t+1}, x^*_{t+1} \rangle, \quad t = 0, \ldots, T-1. \tag{5.59}$$

Furthermore, by Theorem 1.27, $u_t^* \in \partial g(\tilde{x}_t, t)$ is equivalent to

$$g^*(u_t^*, t) = \langle \tilde{x}_t, u_t^* \rangle - g(\tilde{x}_t, t), \quad t = 0, \ldots, T. \tag{5.60}$$

Also, as can easily be verified, the inclusions

$$v_t^* \in K^*_{D_t}(\tilde{x}_t), \ \ t = 1, \ldots, T-1; \quad x_0^* \in K_N^*(\tilde{x}_0), \quad x^* \in K_M^*(\tilde{x}_T)$$

imply

$$\begin{aligned} W_{D_t}(-v_t^*) &= -\langle \tilde{x}_t, v_t^* \rangle, \quad t = 1, \ldots, T-1, \\ W_M(-x_T^* - u_T^*) &= -\langle x_T^* + u_T^*, \tilde{x}_T \rangle, \ \ W_N(x_0^*) = \langle \tilde{x}_0, x_0^* \rangle. \end{aligned} \tag{5.61}$$

Summing up the inequalities in Eqs. (5.59) and (5.61) gives us

$$\begin{aligned} &-\sum_{t=0}^{T} g^*(u_t^*, t) + \sum_{t=0}^{T-1} M_{F_t}(x_t^* - v_t^* + u_t^*, x^*_{t+1}) + \sum_{t=1}^{T-1} W_{D_t}(-v_t^*) - W_N(x_0^*) - W_M(-u_T^* - x_T^*) \\ &= \sum_{t=0}^{T} g(\tilde{x}_t, t) - \langle \tilde{x}_T, u_T^* \rangle - \sum_{t=0}^{T-1} \langle \tilde{x}_t, u_t^* \rangle + \sum_{t=0}^{T-1} \langle \tilde{x}_t, x_t^* \rangle - \sum_{t=0}^{T-1} \langle \tilde{x}_t, v_t^* \rangle + \sum_{t=0}^{T-1} \langle \tilde{x}_t, u_t^* \rangle \\ &\quad - \sum_{t=0}^{T-1} \langle \tilde{x}_{t+1}, x^*_{t+1} \rangle + \sum_{t=1}^{T-1} \langle \tilde{x}_t, v_t^* \rangle - \langle \tilde{x}_0, x_0^* \rangle + \langle \tilde{x}_T, x_T^* + u_T^* \rangle = \sum_{t=0}^{T} g(\tilde{x}_t, t) \\ &\quad + \sum_{t=0}^{T-1} \langle \tilde{x}_t, x_t^* \rangle - \sum_{t=1}^{T} \langle \tilde{x}_t, x_t^* \rangle - \langle \tilde{x}_0, v_0^* \rangle - \langle \tilde{x}_0, x_0^* \rangle + \langle \tilde{x}_T, x_T^* \rangle = \sum_{t=0}^{T} g(\tilde{x}_t, t) - \langle \tilde{x}_0, v_0^* \rangle \end{aligned}$$

Hence, taking into account that $v_0^* = 0$, finally from the latter relation, we have

$$\begin{aligned} &-\sum_{t=0}^{T} g^*(u_t^*, t) + \sum_{t=0}^{T-1} M_{F_t}(x_t^* - v_t^* + u_t^*, x^*_{t+1}) + \sum_{t=1}^{T-1} W_{D_t}(-v_t^*) - W_N(x_0^*) \\ &\quad - W_M(-u_T^* - x_T^*) = \sum_{t=0}^{T} g(\tilde{x}_t, t). \end{aligned}$$

Then we may conclude that $v^* \geq v$. Comparing this with the opposite inequality $v \geq v^*$ (Theorem 5.11), we get $v = v^*$. Consequently, the collection of vectors x_t^*, u_t^*, v_t^* $(v_0^* = v_T^* = 0)$, $t = 0, \ldots, T$ is an optimal solution to the dual problem (P^*). ■

Theorem 5.13. Let the regularity condition for the primary problem (P), where $D_t=\mathbb{R}^n$, $t=1, \ldots, T-1$ be satisfied. Then in order for the value v^* of the dual problem in Eq. (5.54) to be finite and attainable, it is necessary that

$$\begin{aligned}&u_t^* \in \operatorname{dom} g^*(\cdot, t), \quad x_t^*+u_t^* \in F_t^*(x_{t+1}^*), \quad x_0^* \in (0^+ N)^*,\\ &-u_t^*-x_T^* \in (0^+ M)^*, \quad u_T^* \in \operatorname{dom} g^*(\cdot, T), \quad t=0, \ldots, T-1.\end{aligned}$$

In addition, if F_t, $t=0, \ldots, T-1$ and the sets N, M are quasisuperlinear, then these conditions are sufficient for the finiteness and attainability of v^*.

□ Let the value v^* of the dual problem (5.52) be finite and attainable. In view of the conjugate functions established in the proof of Theorem 5.11 in this chapter and Theorem 6.5.2 of Ref. [129:334], we can write

$$\begin{aligned}&\operatorname{dom}(f^* \oplus \delta_Q^*)=\operatorname{dom} f^*+\sum_{t=0}^{T-1} \operatorname{dom} \delta_{Q_t}^*+\operatorname{dom} \delta_{\tilde{N}}^*+\operatorname{dom} \delta_{\tilde{M}}^*\\ &=\{h^*=(h_0^*, \ldots, h_T^*): h_t^*=x_t^*-x_t^*(t-1)-x_t^*(t), \quad h_t^*=x_T^*-x_T^*(T-1)-x_T^*(T) \\ &M_{F_t}(x_t^*(t),-x_{t+1}^*(t))>-\infty, \quad W_N(-x_0^*(-1))<+\infty, \quad W_M(-x_T^*(T))<+\infty,\\ &x_t^* \in \operatorname{dom} g^*(\cdot, t), \quad x_T^* \in \operatorname{dom} g^*(\cdot, T), \quad t=0, \ldots, T-1\}\end{aligned} \tag{5.62}$$

By the argument conducted in the proof of Theorem 5.11, $v=v^*$ and $h^*=0 \in \operatorname{dom}(f+\delta_Q)^*=\operatorname{dom}(f^* \oplus \delta_Q^*)$. Then with the preceding notations, it follows from Eq. (5.62) that $x_0^* \in \operatorname{dom} W_N(x_t^*,+u_t^*, x_{t+1}^*) \in \operatorname{dom} M_{F_t}$, $-u_T^*-x_T^* \in \operatorname{dom} W_M$, $u_t^* \in \operatorname{dom} g^*(\cdot, t)$, $u_T^* \in \operatorname{dom} g^*(\cdot, T)$, $t=0, \ldots, T-1$. Now by Definitions 1.16 and 2.9 and the formula $\operatorname{dom} M_{F_t} \subseteq (0^+ \operatorname{gph} F_t)^*$, we get $x_t^*+u_t^* \in F_t^*(x_{t+1}^*)$. And in the quasisuperlinearity (Remark 2.1) case of F_t, we have $\operatorname{dom} M_{F_t}=(0^+ \operatorname{gph} F_t)^*$ and so $(x_t^*+u_t^*+x_{t+1}^*) \in \operatorname{dom} M_{F_t}$ and $(x_t^*+u_t^* \in F_t^*(x_{t+1}^*)$ are equivalent. Similarly, we observe that $x_0^* \in (0^+ N)^*$, $-u_T^*-x_T^* \in (0^+ M)^*$. Therefore, using the duality Theorem 5.11 and going in the reverse direction, we have the desired result. ■

Example 5.1. Consider the following optimal control problem for discrete inclusions

$$\begin{aligned}&\text{infimum} \quad \sum_{t=1}^{T} g(x_t, t),\\ &\text{subject to} \quad x_{t+1}=A_t x_t+B_t u_t, \quad u_t \in U_t, \quad t=0, \ldots, T-1,\\ &\qquad x_0 \in N, \quad x_T \in M,\end{aligned} \tag{5.63}$$

where A_t, B_t for each t are $n \times n$ and $n \times r$ matrices, respectively. The function $g(\cdot, t)$ and the sets N, M, U_t are proper convex function and sets, respectively. It is required to find a sequence $\{\tilde{u}_t\}_{t=0}^{T-1}$ such that the corresponding trajectory $\{\tilde{x}_t\}_{t=0}^{T-1}$ minimizes $\sum_{t=0}^{T} g(x_t, t)$. Let us introduce a multivalued function of the form

$$F_t(x_t)=A_t x_t+B_t U_t=\{A_t x_t+B_t u_t: u_t \in U_t\}, \quad t=0, \ldots, T-1, \tag{5.64}$$

Thus, the problem in Eq. (5.63) is replaced by problem (P), with Eq. (5.64) and $D_t=\mathbb{R}^n$, $t=1, \ldots, T-1$.

It is not hard to calculate that

$$M_{F_t}(x_t^*,x_{t+1}^*)=\inf\{\langle x_t,x_t^*\rangle-\langle x_{t+1},x_{t+1}^*\rangle:(x_t,x_{t+1})\in \operatorname{gph} F\}$$
$$=\inf_{x_t}\langle x_t,x_t^*-A_t^*x_{t+1}^*\rangle-\sup_{u_t\in U_t}\langle u_t,B_t^*x_{t+1}^*\rangle=\begin{cases}-W_{U_t}(B_t^*x_{t+1}^*), & \text{if } x_t^*=A_t^*x_{t+1}^*,\\ +\infty, & \text{if } x_t^*\neq A_t^*x_{t+1}^*.\end{cases}$$

Then, according to the problem in Eq. (5.54), the dual problem to Eq. (5.63) is

$$\sup_{u_t^*,x_t^*,\ t=0,\ldots,T}\left\{-\sum_{t=0}^{T}g^*(u_t^*,t)-\sum_{t=0}^{T-1}W_{U_t}(B^*x_{t+1}^*)-W_N(x_0^*)-W_M(-u_T^*-x_T^*)\right\},$$
$$x_t^*=A_t^*x_{t+1}^*-u_t^*.$$

Consider now the optimization problem for discrete inclusions with delay described in Section 4.2:

$$\begin{aligned}&\inf && \sum_{t=1}^{T}g(x_t,t)\\ &\text{subject to} && x_{t+1}\in F_t(x_t,x_{t-h}),\quad t=0,\ldots,T-1,\\ &&& x_t=\xi_t,\quad t=-h,-h+1,\ldots,0;\quad x_t\in\Phi_t,\ \ x_T\in M.\end{aligned}\tag{5.65}$$

There we have seen that the posed problem is equivalent to the minimization of the convex function $\varphi(w)=\sum_{t=1}^{T}g(x_t,t)$ over the set

$$G=N\cap\left(\bigcap_{t=0}^{T}\tilde{M}_t\right)\cap\left(\bigcap_{t=1}^{T}\tilde{\Phi}_t\right).$$

In the considered case, it can be easily calculated that

$$\varphi^*(w^*)=\begin{cases}\sum_{t=1}^{T}g^*(x_t^*,t), & \text{if } x_t^*=0,\ \ t=-h,\ldots,0,\\ +\infty, & \text{otherwise}\end{cases}$$

And the conjugates of the indicator functions of sets $\tilde{M}_t$, $\tilde{\Phi}_t$, $t=0,\ldots,T,N$ have the form

$$\delta^*_{\tilde{M}_t}(-w^*(t))=\begin{cases}-M_{F_t}(x_t(t),x_{t-h}^*(t),\ -x_{t+1}^*(t)), & \text{if } x_i^*(t)=0,\ \ i\neq t-h,t,t+1,\\ +\infty, & \text{otherwise,}\end{cases}$$
$$t=0,\ldots,T-1;$$

$$\delta^*_{\tilde{M}_T}(-w^*(T))=\begin{cases}W_M(-x_T^*(T)), & \text{if } x_i^*(T)=0,\ \ i\neq T,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$\delta^*_{\tilde{\Phi}_t}(-\tilde{w}^*(t)) = \begin{cases} W_{\Phi_t}(-\tilde{x}_t{}^*(t)), & \text{if } \tilde{x}_i{}^*(t) = 0, \; i \neq t, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$t = 1, \ldots, T-1;$$

$$\delta^*_N(-w^*(-1)) = \begin{cases} \sum_{i=-h}^{0} \langle \xi_i, x_i^*(-1) \rangle, & \text{if } x_i^*(-1) = 0, \; i \neq -h, \ldots, 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

Now, taking into account these relationships and the formula $\varphi^*(w^*) = \sum_{t=0}^{T} g(u_t^*, t)$ $x_t^* = 0, \quad t = -h, \ldots, 0$ with the notations as before, we conclude that

$$\begin{aligned} \sup\{-\varphi^*(w^*) - \delta_G^*(-w^*)\} \geq \sup\Bigg\{ & -\sum_{t=0}^{T} g(x_t^*, t) + \sum_{t=-h}^{0} \langle \xi_t, x_t^*(-1) \rangle \\ & + \sum_{t=0}^{T-1} M_{F_t}(x_t^*(t), x_{t-h}^*(t), \; -x_{t+1}^*(t)) - \sum_{t=1}^{T} W_{\Phi_t}(-\tilde{x}_t^*(t)) - W_M(-x_T^*(T)) : \\ \sum_{i=-1}^{T} w^*(i) + \sum_{i=1}^{T} \tilde{w}^*(i) = w^*; \quad & x_i^*(-1), \; i \neq -h, \ldots, 0; \quad x_i^*(t) = 0, \; i \neq t-h, t, t+1 \\ (t = 0, \ldots, T-1); \quad & \tilde{x}_i{}^*(t) = 0, \; i \neq t \; (t = 1, \ldots, T); \quad x_i^*(T) = 0, \; i \neq T; \quad x_t^* = 0, \\ t = -h, \ldots, 0\Bigg\} & \end{aligned}$$

and under the regularity condition we have an equality sign. For convenience, let us denote

$$\begin{aligned} & x_t^* = u_t^*(t = 0, \ldots, T); \quad x_t^*(t-1) = -x_t^* \; (t = 0, \ldots, T); \quad x_t^*(t+h) \\ & = \eta_{t+h}^*(t) = \eta_{t+h}^*(t = -h, \ldots, T-h); \quad \tilde{x}_t{}^*(t) = \varphi_t^*(t = 1, \ldots, T) \end{aligned}$$

Then, it is evident that there is one-to-one correspondence between each $(x_0^*, \ldots, x_T^*, \eta_0^*, \ldots, \eta_T^*, \varphi_1^*, \ldots, \varphi_T^*, u_0^*, \ldots, u_T^*)$ and $(w^*(-1), w^*(0), \ldots, w^*(T), \tilde{w}^*(1), \ldots, \tilde{w}^*(T), w^*)$. Thus, the dual problem for the primary problem in Eq. (5.65) is established as follows:

$$\begin{aligned} \sup_{\substack{x_t^*, \eta_t^*, u_t^*, \varphi_t^* \\ (\eta_T^* = u_0^* = \varphi_0^* = 0) \\ t = 0, \ldots, T}} \Bigg\{ & -\sum_{t=0}^{T} g(x_t^*, t) - \langle \xi_0, x_0^* \rangle - \sum_{t=-h}^{-1} \langle \xi_t, \eta_{t+h}^* \rangle \\ & + \sum_{t=0}^{T-1-h} M_{F_t}(x_t^* + \eta_{t+h}^* - \varphi_t^* + u_t^*, \eta_t^*, x_{t+1}^*) \\ + \sum_{t=T-h}^{T-1} M_{F_t}(x_t^* - \varphi_t^* + u_t^*, \eta_t^*, x_{t+1}^*) & - \sum_{t=1}^{T} W_{\Phi_t}(-\varphi_t^*) - W_M(\varphi_T^* - x_T^* - u_T^*) \Bigg\} \end{aligned} \tag{5.66}$$

Now, it is easy to see that the duality results, including the duality theorem for the problem in Eq. (5.65), are almost a word-for-word repetition of Theorems 5.11–5.13. The difference lies in the fact that the results here analogous to Theorems 5.11–5.13 hold for conditions (i)–(iii) in Theorem 4.1 and the problem in Eq. (5.56), whereas Theorems 5.11–5.13 refer only to conditions (1) and (2) of Theorem 5.10 and the problem in Eq. (5.51).

Example 5.2. Consider the following optimal control problem of discrete inclusion with delay:

$$\begin{aligned} &\inf && \sum_{t=1}^{T} g(x_t,t), \\ &\text{subject to} && x_{t+1} = A_0(t)x_t + A_1(t)x_{t-h} + B(t)u_t, \quad t=0,\ldots,T-1, \\ &&& x_t = \xi_t,\ t=-h,\,-h+1,\ldots,0, \\ &&& u_t \in U(t),\ x_T \in M, \end{aligned} \tag{5.67}$$

where $A_0(t)$, $A_1(t)$, and $B(t)$, as in Example 5.1, for each t are $n\times n$ and $n\times r$ matrices, respectively. The function $g(\cdot,t)$ and the sets $M,U(t)$ are proper and convex. It is required to find a sequence $\{\tilde{u}_t\}_{t=0}^{T-1}$ such that the corresponding trajectory $\{\tilde{x}_t\}_{t=0}^{T-1}$ minimizes $\sum_{t=1}^{T} g(x_t,t)$. Let us introduce a multivalued function of the form

$$F_t(x_t,x_{t-h}) = A_0(t)x_t + A_1(t)x_{t-h} + B(t)U(t), \quad t=0,\ldots,T-1$$

Taking into account that $\Phi_t = \mathbb{R}^n$, $t=1,\ldots,T$, the construction of the dual problem in Eq. (5.66) for problem in Eq. (5.67) is similar to the one for the problem in Eq. (5.63):

$$\sup_{\substack{x_t^*,\eta_t^*,u_t^*,\\ (\eta_T^*=u_0^*=0)\\ t=0,\ldots,T}} \left\{ -\sum_{t=0}^{T} g(x_t^*,t) - \langle \xi_0, x_0^*\rangle - \sum_{t=-h}^{-1} \langle \xi_t, \eta_{t+h}^*\rangle - \sum_{t=0}^{T-1} W_{U(t)}(B^*(t)x_{t+1}^*) \right.$$
$$\left. - W_M(-x_T^* - u_T^*) \right\}$$
$$\begin{aligned} x_t^* &= A_0^*(t)x_{t+1}^* - \eta_{t+h}^* - u_t^*,\ t=0,\ldots,T-1-h, \\ x_t^* &= A_0^*(t)x_{t+1}^* - u_t^*,\ t=T-h,\ldots,T-1, \\ \eta_t^* &= A_1^*(t)x_{t+1}^*, \quad t=0,\ldots,T-1. \end{aligned}$$

5.5 The Main Duality Results in Problems with Convex Differential Inclusions

In this section, a Bolza problem of optimal control theory given by convex differential inclusions are considered (see Sections 4.3 and 4.5). The dual problem is constructed and the duality theorems are proved. We pursue a twofold goal. First, we construct a dual problem for a discrete-approximation to continuous problem. Second, we use this direct method to establish a dual problem to a continuous Bolza problem. The construction of a dual problem to the latter is implemented by passing to the formal limit as the discrete step tends to zero. At first we develop this method for studying the problem considered in Section 4.3 with fixed time interval [0,1]:

$$\begin{array}{ll} \text{infimum} & J[x(\cdot)] = \int_0^1 g(x(t),t)\mathrm{d}t + \varphi_0(x(1)), \\ \text{subject to} & \dot{x}(t) \in F(x(t),t), \quad \text{a.e. } t \in [0,1], \\ & x(0) \in N_0, \quad x(1) \in M_1, \end{array} \tag{5.68}$$

where $g(\cdot,t)$ and $\varphi_0(\cdot)$ are convex continuous functions, F is a convex set-valued mapping, and N_0, $M_1 \subseteq \mathbb{R}^n$ are convex sets. It is required to find a solution $x(t)$, $t \in [0,1]$ of the differential inclusion (4.24) with boundary conditions $x(0) \in N_0$, $x(1) \in M_1$ minimizing Eq. (4.23). A feasible solution is assumed to be an absolutely continuous function $x : [0,1] \to \mathbb{R}^n$.

For the present, we consider the nonconvex case of Eq. (5.68). Let us construct a discrete (finite) approximation for the problem given in Eq. (5.68) using the replacement of the derivative in Eq. (5.68) by the Euler finite difference $\dot{x}(t) \approx [x(t+\delta) - x(t)]/\delta \equiv \Delta x(t)$. We choose a step δ and use a grid function $x_\delta(t) \equiv x(t)$ on a uniform grid on [0,1]. Then, we associate the following discrete-approximation problem with the problem in Eq. (5.68):

$$\begin{array}{ll} \text{infimum} & J_\delta[x(\cdot)] = \sum\limits_{t=0,\delta,\ldots,1-\delta} \delta g(x(t),t) + \varphi_0(x(1)), \\ \text{subject to} & x(t+\delta) \in x(t) + \delta F(x(t),t), \quad t = 0, \delta, 2\delta, \ldots, 1-\delta, \\ & x(0) \in N_0, \quad x(1) \in M_1. \end{array} \tag{5.69}$$

In what follows, introducing a new multivalued function, we rewrite this problem in a more convenient form

$$\begin{array}{ll} \text{infimum} & J_\delta[x(\cdot)] = \sum\limits_{t=0,\delta,\ldots,1-\delta} \delta g(x(t),t) + \varphi_0(x(1)), \\ \text{subject to} & x(t+\delta) \in Q(x(t),t), \quad t = 0, \delta, 2\delta, ..., 1-\delta, \\ & x(0) \in N_0, \quad x(1) \in M_1, \\ & Q(x,t) = x + \delta F(x,t) \end{array} \tag{5.70}$$

At once we observe that in more general suppositions in the problem in Eq. (5.70) (not necessarily convex) by Theorem 4.2 (see also Remark 4.4) for optimality of $\{\tilde{x}(t)\}_{t=0}^1$, an adjoint inclusion is expressed as follows:

$$x^*(t) \in Q^*(x^*(t+\delta); (\tilde{x}(t), \tilde{x}(t+1)), t) - \lambda \partial_x g(\tilde{x}(t), t), \quad t = 0, \delta, 2\delta, \ldots, 1-\delta \tag{5.71}$$

Clearly, we must find the LAM Q^* in terms of the LAM F^*. But from Theorem 4.19, we find that if $K_{\text{gph } Q}(\cdot,t)$ is a local tent for the multivalued function $Q(\cdot,t)$, then the following inclusions are equivalent

$$x^* \in Q^*(y^*; (x, y), t) \quad \text{and} \quad \frac{x^* - y^*}{\delta} \in F^*\left(y^*; \left(x, \frac{y-x}{\delta}\right), t\right).$$

Thus, by applying this equivalency, it is easy to see that Eq. (5.71) is replaced by the adjoint inclusion

$$\begin{aligned} -\Delta x^*(t) \equiv -\frac{x^*(t+\delta) - x^*(t)}{\delta} &\in F^*(x^*(t+\delta); (\tilde{x}(t), \Delta\tilde{x}(t)), t) \\ &- \lambda \partial g(\tilde{x}(t), t), \quad t = 0, \delta, 2\delta, \ldots, 1-\delta \end{aligned} \tag{5.72}$$

Now, remember that taking $g(\tilde{x}(1), 1) \equiv \varphi_0(\tilde{x}(1))$ along with Eq. (5.72), we have the condition

$$x^* - x^*(1) \in \partial\varphi_0(\tilde{x}(1)), \quad -x^*(0) \in K^*_{N_0}(\tilde{x}(0)), \quad x^* \in K^*_{M_1}(\tilde{x}(1)). \tag{5.73}$$

The obtained result is formulated in Theorem 5.14.

Theorem 5.14. Let a grid function $\{\tilde{x}(t)\}_{t=0}^{1}$ be an optimal trajectory to the nonconvex problem in Eq. (5.68). Suppose that the cones of tangent directions

$$K_{\text{gph } F(\cdot,t)}(\tilde{x}(t), \tilde{x}(t+\delta)), \quad K_{N_0}(\tilde{x}(0)), \quad K_{M_1}(\tilde{x}(1))$$

are local tents and that the functions g, φ_0 admit a convex upper approximation continuous at $\tilde{x}(t)$. Then, there exist a number $\lambda \geq 0$, vector x^*, and grid function $\{x^*(t)\}_{t=0}^{1}$ not all equal to zero such that Eqs. (5.72) and (5.73) are satisfied. In addition, if the problem in Eq. (5.68) is convex and the regularity condition is satisfied, then $\lambda = 1$ and these conditions are also sufficient for the optimality of $\{\tilde{x}(t)\}_{t=0}^{1}$.

We need the useful result found in Lemma 5.2.

Lemma 5.2. Let $F(\cdot,t): X \to P(Y)$ be a convex multivalued function and $Q(x,t) = x + \delta F(x,t)$. Then one has

$$M_{Q(\cdot,t)}(x^*, y^*) = \delta M_{F(\cdot,t)}\left(\frac{x^* - y^*}{\delta}, y^*\right).$$

□ Indeed, by definition of $M_{Q(\cdot,t)}$, we conclude that

$$\begin{aligned} M_{Q(\cdot,t)}(x^*, y^*) &= \inf\{\langle x, x^*\rangle - \langle y, y^*\rangle : (x, y) \in \text{gph } Q(\cdot, t)\} \\ &= \inf\left\{\langle x, x^*\rangle - \langle y, y^*\rangle : \left(x, \frac{y-x}{\delta}\right) \in \text{gph } F(\cdot, t)\right\} \\ &= \delta \inf\left\{\left\langle x, \frac{x^* - y^*}{\delta}\right\rangle - \left\langle \frac{y-x}{\delta}, y^*\right\rangle : \left(x, \frac{y-x}{\delta}\right) \in \text{gph } F(\cdot, t)\right\} \\ &= \delta M_{F(\cdot,t)}\left(\frac{x^* - y^*}{\delta}, y^*\right). \blacksquare \end{aligned}$$

Now, taking into account Eq. (5.54), we see that in our notations the dual problem to the convex discrete-approximation problem in Eq. (5.69) or Eq. (5.60) has the form

$$\sup_{\substack{u^*(t),x^*(t),\\ t=0,\delta\ldots,1}} \left\{ -\sum_{t=0}^{1}(\delta g)^*(u^*(t),t) + \sum_{t=0}^{1-\delta} M_{Q(\cdot,t)}(x^*(t)+u^*(t),x^*(t+\delta) \right.$$
$$\left. -W_{N_0}(x^*(0)) - W_{M_1}(-u^*(1)-x^*(1)) \right\}. \tag{5.74}$$

But by Lemma 5.2, we find that

$$\begin{aligned} M_{Q(\cdot,t)}(x^*(t)+u^*(t),x^*(t+\delta)) &= \delta M_{F(\cdot,t)}\left(\frac{x^*(t)-x^*(t+\delta)+u^*(t)}{\delta},x^*(t+\delta)\right) \\ &= \delta M_{F(\cdot,t)}\left(\frac{u^*(t)}{\delta}-\Delta x^*(t),x^*(t+\delta)\right) \end{aligned} \tag{5.75}$$

On the other hand, it is can easily be checked that the conjugate (δg^*) is defined by $(\delta g)^*(u^*,t)=\delta g^*(u^*/(\delta),t),\delta>0$. Thus, for convenience, denoting u^*/δ again by u^* in view of the formula in Eq. (5.75), we derive from Eq. (5.74) that the dual problem to the discrete-approximation problem in Eq. (5.69), $(\delta g)^*(x^*,t)=\delta g^*(x^*/\delta,t),\delta>0$, is

$$\sup_{\substack{u^*(t),x^*(t),\\ t=0,\delta\ldots,1}} \left\{ -\sum_{t=0,\delta,\ldots,1}\delta g^*(u^*(t),t) + \sum_{t=0}^{1-\delta}\delta M_{F(\cdot,t)}(u^*(t)-\Delta x^*(t),x^*(t+\delta)) \right.$$
$$\left. -W_{N_0}(x^*(0)) - W_{M_1}(-u^*(1)-x^*(1)) \right\}, \tag{5.76}$$

where we put $g^*(u^*(1),1)\equiv\varphi_0^*(u^*(1))$. Remember that in Eq. (5.76), we have two integral sums (for the functions g^* and $M_{F(\cdot,t)}$). Now by passing to the formal limit, the obtained maximization problem will be the dual problem to the previous continuous convex problem (5.68):

$$\sup_{u^*(t),x^*(t)} \left\{ \int_0^1 [M_{F(\cdot,t)}(u^*(t)-\dot{x}^*(t),x^*(t)) - g^*(u^*(t),t)]\,\mathrm{d}t \right.$$
$$\left. -\varphi_0^*(x^*(1)) - W_{N_0}(x^*(0)) - W_{M_1}(-u^*(1)-x^*(1)) \right\}. \tag{5.77}$$

Here $x^*:[0,1]\to\mathbb{R}^n$ and $u^*:[0,1]\to\mathbb{R}^n$ are absolutely continuous and summable functions, respectively, and so $\lim_{\delta\to 0}\Delta x^*(t)=\dot{x}^*(t)$, i.e., $t\in[0,1]$.

Recall that $\sum_{[0,1]}(0,N_0)$ denotes the family of solutions to the differential inclusion in Eq. (5.68), with the initial condition $x(0)\in N_0$ defined on $[0,1]$ and $S(\tau,0,N_0)$ a section of the family of solutions; i.e.,

$$S(\tau,0,N_0)=\left\{x(\tau):x(t)\in\sum_{[0,1]}(0,N_0),\quad \tau\in[0,1].\right\}$$

Moreover, we denote the affine hull of a set M by aff M.

Theorem 5.15. Let the conditions of Theorem 4.3 be satisfied, where $\lambda_0>0$. Let $\tilde{x}(t)$ be an optimal solution to the primary problem in Eq. (5.68) with convex structure. Moreover, assume that either

1. $M_1\cap \operatorname{int} S(1,0,N_0)\neq\varnothing$ or
2. $\operatorname{ri} S(1,0,N_0)\cap \operatorname{ri} M_1\neq\varnothing$

and that there is no hyperplane H containing both aff $S(1,0,N_0)$ and aff M_1. Then a pair of functions $\{\tilde{x}^*(t),\tilde{u}^*(t)\}$ is an optimal solution to the dual problem in Eq. (5.77) if and only if conditions (i)−(iii) of Theorem 4.3 are satisfied. In addition, the optimal values in the primary problem in Eq. (5.68) and the dual problem in Eq. (5.77) are equal.

□ First, we conclude that for all feasible solutions $x(t)$ and $\{x^*(t),u^*(t)\}$ of the primary problem in Eq. (5.68) and the dual problem in Eq. (5.77), respectively, the inequality holds:

$$\begin{aligned}\int_0^1 g(x(t),t)\mathrm{d}t+\varphi_0(x(1)) \quad &\geq \int_0^1[M_{F(\cdot,t)}(u^*(t)-\dot{x}^*(t),x^*(t))-g^*(u^*(t),t)]\mathrm{d}t\\ &-\varphi_0^*(x^*(1))-W_{N_0}(-x^*(0))-W_{M_1}(-u^*(1)-x^*(1)).\end{aligned}\tag{5.78}$$

Indeed by using the conjugate $g*$, φ_0^* and the definition of the Hamiltonian function, we can write

$$\int_0^1[M_{F(\cdot,t)}(u^*(t)-\dot{x}^*(t),x^*(t))-g^*(u^*(t),t)]\mathrm{d}t-\varphi_0^*(x^*(1))-W_{N_0}(x^*(0))$$

$$\begin{aligned}&-W_{M_1}(-u^*(1)-x^*(1))\leq\int_0^1[\langle x(t),u^*(t)-\dot{x}^*(t)\rangle-H(x(t),x^*(t))\\ &+g(x(t),t)-\langle x^*(t),u^*(t)\rangle]\,\mathrm{d}t+\varphi_0(x(1))-\langle x(1),u^*(1)\rangle-\langle x(0),x^*(0)\rangle\\ &+\langle x(1),u^*(1)+x^*(1)\rangle\leq-\int_0^1 d\langle x(t),x^*(t)\rangle+\int_0^1 g(x(t),t)\mathrm{d}t+\varphi_0(x(1))\\ &-\langle x(0),x^*(0)\rangle+\langle x(1),x^*(1)\rangle=\langle x(0),x^*(0)\rangle-\langle x(1),x^*(1)\rangle+\int_0^1 g(x(t),t)\mathrm{d}t\\ &+\varphi_0(x(1))-\langle x(0),x^*(0)\rangle+\langle x(1),x^*(1)\rangle=\int_0^1 g(x(t),t)\mathrm{d}t+\varphi_0(x(1)).\end{aligned}$$

As was shown in the proof of Theorem 5.9, we observe that the sets $S(1,0,N_0)$ and M_1 are not separable and $\lambda_0=1$. Furthermore, if $(\tilde{x}^*(t),\tilde{u}^*(t))$ satisfies conditions (i)−(iii) of Theorem 4.3, then in the latter relation, the inequality sign is replaced by equality, and hence for $\tilde{x}(t)$ and $(\tilde{x}^*(t),\tilde{u}^*(t))$ we have opposite inequality sign in Eq. (5.78), which ensures equality of the values of primary and dual problems. ■

Let us now construct the dual problem to the convex problem for the delay-differential inclusion in Eq. (4.67):

$$\begin{array}{ll} \text{infimum} & J(x(\cdot)) = \int_0^1 g(x(t),t)\,\mathrm{d}t + \varphi_0(x(1)), \\ \text{subject to} & \dot{x}(t) \in F(x(t),x(t-h),t), \quad \text{a.e. } t\in[0,1],\ldots \\ & x(t)=\xi(t), \quad t\in[-h,0], \;\; x(1)\in M_1. \end{array} \tag{5.79}$$

For this we must first obtain the dual problem to the corresponding discrete-approximation problem, which consists of the following:

$$\begin{array}{ll} \text{infimum} & J_\delta[x(\cdot)] = \displaystyle\sum_{t=0,\delta,\ldots,1} \delta g(x(t),t), \\ \text{subject to} & x(t+\delta)\in x(t)+\delta F(x(t),x(t-h),t), \quad t=0,\delta,\ldots,1-\delta, \\ & x(t)=\xi(t), \quad t=-h,\,-h+1,\ldots,0, \;\; x(1)\in M_1 \end{array} \tag{5.80}$$

where we set $g(x(1),1)\equiv\varphi_0(x(1))$.

Clearly, we have a multivalued function $F(\cdot,\cdot,t): X\times X\to P(Y)$. Then, as in the proof of Lemma 5.2, setting $Q(x,x_1,t)=x+\delta F(x,x_1,t)$, we determine that in the present case

$$\begin{aligned} M_{Q(\cdot,\cdot,t)}(x^*,x_1^*,y^*) &= \inf\{\langle x,x^*\rangle+\langle x_1,x_1^*\rangle-\langle y,y^*\rangle : (x,y)\in \operatorname{gph} Q(\cdot,\cdot,t)\} \\ &= \delta M_{F(\cdot,\cdot,t)}\left(\frac{x^*-y^*}{\delta},\frac{x_1^*}{\delta},y^*\right). \end{aligned} \tag{5.81}$$

Thus. in the same way as above, denoting $u^*(t)/\delta$ and $\eta^*(t)/\delta$ again by $u^*(t)$ and $\eta^*(t)$, respectively, in view of the formula in Eq. (5.81), we derive from Eq. (5.66) the dual problem to the discrete-approximation problem in Eq. (5.80):

$$\begin{aligned} \sup_{\substack{u^*(t),x^*(t),\eta^*(t)\\(\eta^*(1)=u^*(0)=0)\\t=0,\delta\ldots,1}} &\left\{-\sum_{t=0,\delta,\ldots,1}\delta g^*(u^*(t),t)-\langle\xi(0),x^*(0)\rangle\right. \\ &\quad -\sum_{t=-h,-h+1,\ldots,-1}\delta\langle\xi(t),\eta^*(t+h)\rangle \\ &\quad +\sum_{t=0}^{1-\delta-h}\delta M_{F(\cdot,\cdot,t)}(u^*(t)-\Delta x^*(t)-\eta^*(t+h),\eta^*(t),\;x^*(t+\delta)) \\ &\quad \left.+\sum_{t=1-h}^{1-\delta}\delta M_{F(\cdot,\cdot,t)}(u^*(t)-\Delta x^*(t),\;\eta^*(t),\;x^*(t+\delta))-W_{M_1}(-u^*(1)-x^*(1))\right\}. \end{aligned} \tag{5.82}$$

Here $\varphi^*(t)\equiv 0$ and so $W_{\Phi(t)}(-\varphi^*(t))\equiv 0$ because $\Phi(t)\equiv\mathbb{R}^n$, $t=\delta,\,2\delta,\,\ldots,\,1$. Consequently, following the limiting procedure in Eq. (5.82), we can formulate the dual problem to the continuous problem in Eq. (5.79) with delay-differential inclusion:

$$\sup_{\substack{u^*(t),x^*(t),\eta^*(t)\\(\eta^*(1)=u^*(0)=0)}} \left\{ -\int_0^1 g^*(u^*(t),t)\mathrm{d}t - \langle x(0),x^*(0)\rangle - \int_{-h}^{0} \langle \xi(t),\eta^*(t+h)\rangle \mathrm{d}t \right.$$
$$+\int_0^{1-h} M_{F(\cdot,\cdot,t)}(u^*(t)-\dot{x}^*(t)-\eta^*(t+h),\eta^*(t),x^*(t))\mathrm{d}t$$
$$\left. +\int_{1-h}^{1} M_{F(\cdot,\cdot,t)}(u^*(t)-\dot{x}^*(t),\eta^*(t),x^*(t))\mathrm{d}t - W_{M_1}(-u^*(1)-x^*(1))\right\}. \tag{5.83}$$

where it is assumed that $x^*:[0,1]\to\mathbb{R}^n$, $\eta^*:[0,1]\to\mathbb{R}^n$, and $u^*:[0,1]\to\mathbb{R}^n$ are absolutely continuous and summable functions, respectively.

Remark 5.3. Note that if we consider the optimization problem in Eqs. (5.68) and (5.79) with varying time interval $[0,t_1]$, then by analogy with the statements just obtained, we should replace the point $t=1$ by $t=t_1$. Hence, the formulations of the duality theorems and duality relations are the same.

Example 5.3. Let us construct the dual problem to the convex problem in Eq. (5.79) with polyhedral delay-differential inclusion and varying time interval $[0,t_1]$:

$$\begin{aligned} &\text{minimize} && J[x(\cdot),t_1]=\textstyle\int_0^{t_1} g(x(t),t)\mathrm{d}t+\varphi_0(x(t_1)),\\ &\text{subject to} && \dot{x}(t)\in F(x(t),x(t-h),t), \quad \text{a.e. } t\in[0,t_1],\\ &&& x(t)=\xi(t), \quad t\in[-h,0], \ \ x(t_1)\in M_1 \end{aligned} \tag{5.84}$$

$$F(x,x_1)=\{y: Ax+A_1x_1-By\le d\} \tag{5.85}$$

where A, A_1, B are $m\times n$ matrices and d is m-dimensional column vector. By Corollary 3.1, it is easy to establish that (x_0,x_{10}) is a solution to the minimization problem

$$\inf_{(x,x_1,y)}\left\{\langle x,x^*\rangle+\langle x_1,x_1^*\rangle-\langle y,y^*\rangle:(x,x_1,y)\in \operatorname{gph} F\right\}$$
$$=\inf_{(x,x_1)}\left\{\langle x,x^*\rangle+\langle x_1,x_1^*\rangle-H(x,x_1,y^*)\right\}$$

if and only if

$$(x^*,x_1^*)\in\partial_{(x,x_1)}H(x_0,x_{10},y^*)\equiv F^*(y^*;(x_0,x_{10},y_0))$$

where $H(x,x_1,y^*)=sup\{\langle y,y^*\rangle: y\in F(x,x_1)\}$ and $y_0\in F(x_0,x_{10}{:}y^*)$. Next, since (see Section 2.4)

$$K^*_{\text{gph } F} = \{(x^*, x_1^*, y^*) : x^* = -A^*\lambda, \quad x_1^* = -A_1^*\lambda, \quad y^* = B^*\lambda,$$
$$\lambda \geq 0, \quad \langle Ax_0 + A_1x_{10} - By_0 - d, \lambda\rangle = 0\}$$

by definition of LAM, F^* has the form

$$F^*(y^*; z_0) = \{(-A^*\lambda, -A_1^*\lambda) : \ y^* = -B^*\lambda, \quad \lambda \geq 0,$$
$$\langle Ax_0 + A_1x_{10} - By_0 - d, \lambda\rangle = 0\}, \quad z_0 = (x_0, x_{10}, y_0).$$

Thus, we find that

$$\begin{aligned} M_F(x^*, x_1^*, y^*) \quad &= \inf_{(x, x_1, y)} \left\{\langle x, x^*\rangle + \langle x_1, x_1^*\rangle - \langle y, y^*\rangle : (x, x_1, y) \in \text{gph } F\right\} \\ &= \langle x_0, -A^*\lambda\rangle + \langle x_{10}, -A_1^*\lambda\rangle - \langle y, -B^*\lambda\rangle = -\langle d, \lambda\rangle, \quad \lambda \geq 0. \end{aligned} \tag{5.86}$$

On the other hand, from Eq. (5.83) and the form of the LAM F^*, we derive that

$$\begin{aligned} &u^*(t) - \dot{x}^*(t) - \eta^*(t+h) = -A^*\lambda(t), \quad t \in [0, 1-h), \\ &u^*(t) - \dot{x}^*(t) = -A^*\lambda(t), \quad t \in (1-h, t_1], \\ &x^*(t) = -B^*\lambda(t), \ \eta^*(t) = -A_1^*\lambda(t), \quad t \in [0, t_1]. \end{aligned} \tag{5.87}$$

Now, it can easily be seen that the dual problem to the convex problem in Eqs. (5.84) and (5.85) with the polyhedral delay-differential inclusion is obtained from Eqs. (5.86) and (5.87):

$$\sup_{u^*(t), (u^*(0)=0), \lambda(t)} \left\{ -\int_0^{t_1} g^*(u^*(t), t)\mathrm{d}t - \langle x(0, x^*(0)\rangle + \int_{-h}^0 \langle \xi(t), A_1^*\lambda(t+h)\rangle \mathrm{d}t \right.$$
$$\left. - \int_0^{t_1} \langle d, \lambda(t)\rangle \mathrm{d}t - \varphi_0^*(u^*(t_1)) - W_{M_1}(-u^*(t_1) - x^*(t_1))\right\},$$
$$A^*\lambda(t) + B^*\dot{\lambda}(t) + A_1^*\lambda(t+h) + u^*(t) = 0, \quad t \in [0, t_1 - h),$$
$$A^*\lambda(t) + B^*\dot{\lambda}(t) + u^*(t) = 0, \quad t \in [1-h, t_1], \quad \lambda(t) \geq 0.$$

Note that for a problem without a delay effect, i.e., where A_1 is the zero matrix and $h = 0$, we have

$$\begin{aligned} \sup_{u^*(t), (u^*(0)=0), \lambda(t)} \quad &\left\{ -\int_0^{t_1} g^*(u^*(t), t)\mathrm{d}t - \langle x(0, x^*(0)\rangle - \varphi_0^*(u^*(t_1)) \right. \\ &\left. - \int_0^{t_1} \langle d, \lambda(t)\rangle \mathrm{d}t - W_{M_1}(-u^*(t_1) - x^*(t_1))\right\}, \end{aligned} \tag{5.88}$$

$$A^*\lambda(t) + B^*\dot{\lambda}(t) + u^*(t) = 0, \quad t \in [0, t_1], \quad \lambda(t) \geq 0.$$

It is interesting to see that if in the problem in Eqs. (5.41) and (5.42) $N_0 \equiv \mathbb{R}^n$ and in Eqs. (5.84) and (5.85) M_1 is a polyhedral set (see Eqs. (5.41) and (5.42)), then the dual problems in Eqs. (5.43) and (5.88) coincide.

6 Optimization of Discrete and Differential Inclusions with Distributed Parameters via Approximation

6.1 Introduction

This chapter is devoted to an investigation of problems described by the so-called discrete and differential inclusions with distributed parameters. The past decade has seen an ever more intensive development of the theory of extremal problems involving multivalue mappings with distributed parameters [4,13,17,21,48,62,78, 150−152,162−166,170,171,241,244,250,252].

A great many problems in economic dynamics, as well as classical problems on optimal control in vibrations, chemical, engineering, heat, diffusion processes, differential games, and so on, can be reduced to such investigations. Refer to the survey papers for more information [2,5,19,24,25,28,30,37,45,56,63,64,73,75,82,94, 95,101−103,105,106,110,111,114,122,123,131,160,174,188,189,200,203,212,217, 223,225,235−237,245,249,254,255,259,260,266]. The results we wish to obtain fall into four categories: necessary and sufficient conditions for discrete inclusions, optimization of the corresponding discrete-approximation problems, sufficient conditions for partial differential inclusions, and the construction of duality relations.

The principal method that we use is that of LAM, which facilitates obtaining necessary and sufficient conditions for all types of discrete and differential inclusions; we use one of the constructions of convex and nonsmooth analysis. Moreover, it appears that the use of the CUAs for nonconvex functions and local tents [38,224] is very suitable for obtaining the optimality conditions for posed problems. Optimization of different types of discrete inclusions can be reduced to finite-dimensional problems of mathematical programming, namely, to minimization of functions on the intersection of a finite number of sets. The adjoint inclusions that arise are stated in Euler−Lagrange form [50,63,133,136,207,208], and this form automatically implies the Weierstrass−Pontryagin maximum condition.

Note that this happens because the LAM is not the same as in Refs. [188,192, 214,226]. Another definition of the LAM is introduced by Mordukhovich and is called the coderivative of multivalued functions at a given point [193,194,198,214].

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Moreover it appears that the use of the CUA for nonconvex functions and local tents [38,224] is very suitable for obtaining the optimality conditions for posed problems. Observe that the main successful application of local approximations and transition to convex approximations of sets is the establishment of necessary conditions for nonconvex optimization problems. For related and additional material in the field of different convex and nonconvex approximations of functions and sets, also consult Borwein et al. [42], Clarke [52,54,58], Demyanov [65,66], Frankowska [83,87,88], Ioffe and Penot [109], Outrata and Mordukhovich [218], Mordukhovich [188,192,194,198,214], Pshenichnyi [226], Rockafellar [229−232], and Rockafellar and Zagrodny [232].

We use difference approximations of partial derivatives and grid functions on a uniform grid to approximate partial differential inclusions and to derive necessary and sufficient conditions for optimality for discrete-approximation problems. The latter is possible by passing to the necessary conditions for an extremum of the corresponding discrete inclusion. It turns out that this requires some special equivalence theorems of a LAM, which arise in discrete and discrete-approximation problems. These equivalence theorems allow us to make a bridge between problems (P_D) and (P_C). Obviously, such difference problems, in addition to being of independent interest, can play an important role also in computational procedures.

Thus, we are able to use the results for discrete-approximation problems with distributed parameters to get sufficient conditions of optimality for partial differential inclusions. We associate the discrete-approximation problem and derive necessary and sufficient conditions for optimality. The key tools of our investigations are based on the extremal principle and its modifications together with generalized differential calculus [162−171,188,195,196,229,231]. Using the method of discrete approximations and concepts of generalized differentiation, we establish optimality conditions in both Euler−Lagrange and Hamiltonian forms. The relationship between continuous and discrete systems is one of the central objects of this chapter. Transition to the optimality conditions for discrete-approximation problems from their discrete counterparts is realized by using special equivalence theorems of the LAM, which play a substantial role in the next investigations and without which few necessary or sufficient conditions would be obtained. The point is that discrete and discrete-approximation problems naturally are described by different (say F and G, respectively) multivalued functions, and in order to formulate the optimality conditions for each discrete-approximation problem and then for its corresponding continuous problem, we must express the LAM G^* by F^*. In fact, many necessary and sufficient conditions for optimality appearing in the survey papers of Mahmudov [162−173] for partial differential inclusions inevitably require the development of new forms of equivalence results.

The derivation of sufficient conditions is implemented by passing to the formal limit as the discrete steps tend to zero. Of course, by using the suggested methods for ordinary differential inclusions of Mordukhovich [190−192,196,199] or Pshenichnyi [226], it can be proved that the obtained sufficient conditions are also necessary for optimality. At the end of each section, we consider linear-type optimal control problems. These examples show that in known problems the adjoint inclusions coincide with the adjoint equations, which are traditionally obtained with the help of the Hamiltonian function.

In addition to many works devoted to optimization of partial differential inclusions, the qualitative problems for boundary-value problems for systems of first-order partial differential inclusions are considered. Benchohra and Ntouyas [34] studied nonlocal Cauchy problems for neutral functional differential and integro-differential inclusions in Banach spaces. Gatsori's main goal [96] was to establish sufficient conditions for the existence of solutions for semilinear differential inclusions with nonlocal conditions. He relied on Covitz and Nadler's fixed-point theorem for contraction of multivalued maps and on Schaefar's fixed-point theorem combined with lower semicontinuous multivalued operators with decomposable values. Furthermore, the well-posedness of degenerate Cauchy problems treated as Cauchy problems for a differential inclusion with a multivalued linear operator is considered. Using a new approach to the definition of degenerate integrated semigroups and their generators in a Banach space, a well-posedness criterion for the problem is obtained. Moreover, Melnikova [179] considers the Cauchy problem for a differential inclusion in the space of abstract distributions and given necessary and sufficient conditions for well-posedness in the distribution space. Aubin and Frankowska [17] devotes themself to set-valued solutions to the Cauchy problem for hyperbolic differential inclusions.

Sections 6.2 and 6.3 are devoted to optimal control problems described by first-order partial differential inclusions, which is an interesting and not well-investigated class of control problems. Such research on the optimization of partial differential inclusions can be applied in the theory of differential games, dynamic optimization, gas dynamics, heat and mass transfer, wave theory, and much more. It appears that optimization of the Cauchy problem is significantly different from the optimization problems of the Dirichlet or Neumann types considered in Section 6.5. The main distinction is that in the motivation of the two-parameter optimization for discrete inclusions, a Hilbert space consideration is demanded. A major role is played by the LAM calculus that is crucial for applications to very different problems [164−171]. The LAM apparatus was introduced by Mahmudov [164,166,168,171−173] and Pshenichnyi [226] and was applied successfully in Refs. [140−164]. Note that in the related definition of LAM by Mordukhovich [204,205,214], the normal cone construction is used and is called the coderivative of a set-valued map at a given point. In addition to the LAM for convex set-valued maps, F as defined in Refs. [164,166,168,171] is the same as the basic subdifferential of the Hamiltonian concave function $H_F(u,v^*) = \sup\{\langle v,v^*\rangle : v \in F(x)\}$ on the first argument. There are different approaches and various results in this field using one or another tool in nonsmooth analysis. The primary goals of the book by Mordukhovich [214] are to present basic concepts and principles of variational analysis and to develop a comprehensive generalized differential theory in the framework of Asplund spaces. In continuous problems, it is required that in some way the coordinate-wise limit of the conjugate variables be equal to zero.

Thus, sufficient conditions for optimality of boundary-value problems (where feasible solutions belong to the Sobolev space $W^{1,1}(Q)$—i.e., the distributional derivatives of the first order belong to $L_1(Q)$)—and Cauchy problems both for convex and nonconvex DFIs with first-order partial derivatives are formulated. Naturally, the latter problems are converted to the computation problems of multiple improper integrals. By passing to the formal limit in the intermediate discrete-approximation problem with the use of LAM, we derive sufficient conditions for continuous problem both in the extended Euler–Lagrange and Hamiltonian forms. This method of optimization can be useful for other classes of discrete and partial differential inclusions, namely, properly defined Cauchy problems.

The optimality conditions obtained by Mahmudov [164–171] are more precise, since they involve useful forms of the Weierstrass–Pontryagin condition and the Euler–Lagrange type of adjoint inclusions. We emphasize that in the problem under consideration, the adjoint inclusion involves only one conjugate variable; i.e., there are no auxiliary adjoint variables in the conjugate differential inclusions. Apparently, this occurs because a set-valued map does not depend on partial derivatives.

Section 6.4 is devoted to an optimal control problem given by hyperbolic discrete inclusion (DSI) and DFI of the Darboux type. Note that problems for DSI and DFI can be applied in the field of mathematical economics, in differential games, and in the Fornosini–Marchesini [81] model which plays an essential role in the theory of automatic control systems with two independent variables [81,113]. Finally, in addition to optimization problems for DFI, existence problems for hyperbolic differential inclusions have been extensively studied in the literature [21,49,62,219]. Aubin and Frankowska, and Domachawski investigate first-order hyperbolic systems of partial differential inclusions [21,68]. Papageorgiou considers the existence of solutions for Darboux hyperbolic differential inclusions (6.81) in Banach spaces [219]. Some second-order necessary conditions for an analogous nonconvex hyperbolic differential inclusion problem are considered in Ref. [48].

In Sections 6.5 and 6.6, an extremal Dirichlet problem and a first boundary value problem are formulated for elliptic and hyperbolic DSI and DFI with elliptic Laplace's operator, respectively. In addition, the results obtained generalize to the multidimensional case with a second-order elliptic operator. Therefore, at the end, we indicate general ways of extending the results to the case of generalized solutions [6,7,27,29,44,46,92,104,126–128,131].

Section 6.7 is devoted to optimization of parabolic-type DSI and DFI. At the end of this section, we indicate how to extend the results to multicriteria optimization.

Next, we construct the dual problems of primary convex problems for partial differential inclusions. The duality theorems for partial differential inclusions allow us to conclude that a sufficient condition for optimality is an extremal relation for the primary and dual problems. Here, the dual problem to the convex problem obtained by using the infimal convolution of convex functions is the starting point of the construction of duality theory. Thus, the duality problems for corresponding discrete and discrete-approximation problems allow us to formulate this problem for continuous problems. But to avoid long calculations, we omit it and formulate only dual problems constructed for different partial DFIs.

Some duality relations and optimality conditions for an extremum of different control problems with partial differential inclusions can be found in Mahmudov [159,162−166,168−171].

6.2 The Optimality Principle of Boundary-Value Problems for Discrete-Approximation and First-Order Partial Differential Inclusions and Duality

This section is devoted to an investigation of problems with distributed parameters of the first kind, where the treatment is in a finite-dimensional Euclidean space. Aubin and Cellina, and Barbu [20,27] obtained necessary conditions for an extremum for some control problems with distributed parameters in abstract Hilbert spaces. As a rule, the methods of these authors require the introduction of operators with a maximal monotone graph.

Section 6.2 is divided into four parts. In the first part (Subsection 1), a certain extremal problem is formulated for discrete inclusions. For such problems, we use constructions of convex and nonsmooth analysis in terms of CUAs, local tents, and LAMs [38,148,152,160−173] for both convex and nonconvex problems to get necessary (and sufficient, in the convex case under a regularity condition) conditions for optimality that are based on some subtle computations with the help of the LAM apparatus.

In Subsection 2, we use difference approximations of derivatives and grid functions on a uniform grid to approximate problems with first-order partial differential inclusions and to derive a necessary and sufficient condition for optimality for the discrete-approximation problem. The latter is possible by passing to necessary conditions for an extremum of a discrete inclusion in Subsection 1.

In Subsection 3, we will be able to use the results from Subsection 2 to get sufficient conditions for optimality for convex differential inclusions. The derivation of sufficient conditions will be implemented by passing to the formal limit as the discrete steps tend to zero.

At the end of Subsection 3, we consider an optimal control problem described by a first-order differential equation with distributed parameters. This example shows that in known problems, the adjoint inclusion coincides with the conjugate equation, which is traditionally obtained with the help of the Hamiltonian function.

The basic concepts and definitions can be found in Sections 2.2 and 3.6. For a multivalued mapping $F:\mathbb{R}^{2n}\to P(\mathbb{R}^n)$, we introduce the following notation:

$$H_F(p,x,v^*) = \sup_v\{\langle v,v^*\rangle : v\in F(p,x)\},\quad v^*\in\mathbb{R}^n,$$
$$F(p,x;v^*) = \{v\in F(p,x) : \langle v,v^*\rangle = H_F(p,x,v^*)\}.$$

For convex F we set,

$$\text{if } H_F(p,x,v^*) = -\infty,\quad \text{if } F(p,x)=\varnothing.$$

A mapping $F^*(v^*;(p^0,x^0,v^0))=\{(p^*,x^*):(p^*,x^*,-u^*)\in K^*_{\text{gph }F}(p^0,x^0,u^0)\}$ is called the LAM to F at a point (p^0,x^0,v^0), where $K^*_{\text{gph }F}(p^0,x^0,v^0)$ is the cone dual to the cone $K_{\text{gph }F}(p^0,x^0,v^0)$. For nonconvex multivalued functions, obviously a cone of tangent vectors is defined differently. But the wider this cone is, the more effective optimality conditions are.

At first we consider the following optimization problem for discrete inclusions with distributed parameters [168]:

$$\inf \sum_{\substack{t=1,\ldots,T \\ \tau=0,\ldots,L-1}} g_{t,\tau}(x_{t,\tau}) \tag{6.1}$$

subject to

$$x_{t+1,\tau}\in F(x_{t,\tau+1},x_{t,\tau}),\quad (t,\tau)\in H_1\times L_1, \tag{6.2}$$

$$x_{t,L}=\alpha_{tL},\quad t\in H_1,\quad x_{0,\tau}=\beta_{0\tau},\quad \tau\in L_0\ (\alpha_{0L}=\beta_{0L}), \tag{6.3}$$

where $H=\{0,\ldots,T\}$, $H_1=\{0,\ldots,T-1\}$, $L_0=\{0,\ldots,L\}$, $L_1=\{0,\ldots,L-1\}$, and $g_{t,\tau}:\mathbb{R}^n\to\mathbb{R}\cup\{\pm\infty\}$ are functions taking values on the extended line, F is multivalued mapping, and α_{tL}, $\beta_{0\tau}$ are fixed vectors.

A set of points $\{x_{t,\tau}\}_{(t,\tau)\in H\times L_0}=\{x_{t,\tau}:(t,\tau)\in H\times L_0,(t,\tau)\neq(T,L)\}$ is called an admissible solution for the problem in Eqs. (6.1)–(6.3) if it satisfies the inclusion in Eq. (6.2) and boundary conditions in Eq. (6.3). It is easy to see that for fixed natural numbers T and L, the conditions in Eq. (6.3) enable us to choose some admissible solution, and the number of points to be determined coincides with the number of discrete inclusions in Eq. (6.2). The following condition is assumed below for the functions $g_{t,\tau}$, $t=1,\ldots,T$, $\tau\in L_1$, and the mapping F.

Condition N. Suppose that in the problem in Eqs. (6.1)–(6.3), the mapping F is such that the cone $K_{\text{gph }F}(\tilde{x}_{t,\tau+1},\tilde{x}_{t,\tau},\tilde{x}_{t+1,\tau})$ of tangent directions is a local tent, where $\tilde{x}_{t,\tau}$ are the points of the optimal solution $\{\tilde{x}_{t,\tau}\}_{(t,\tau)\in H\times L_0}$. Suppose, moreover, that the functions $g_{t,\tau}(x)$ admit a CUA $h_{t,\tau}(\bar{x},\tilde{x}_{t,\tau})$ at the points $\tilde{x}_{t,\tau}$ that is continuous with respect to $\bar{x}$. The latter means that the subdifferentials $\partial g_{t,\tau}(\tilde{x}_{t,\tau})=\partial h_{t,\tau}(0,\tilde{x}_{t,\tau})$ are defined.

The problem in Eqs. (6.1)–(6.3) is said to be convex if the mapping F is convex and the $g_{t,\tau}$ are proper convex functions. For a convex problem, we introduce Definition 6.1.

Definition 6.1. We say that the convex problem in Eqs. (6.1)–(6.3) satisfies the regularity condition if for some feasible (admissible) solution $\{x^0_{t,\tau}\}_{(t,\tau)\in H\times L_0}$, we have either (a) or (b):

a. $(x^0_{t,\tau+1},x^0_{t,\tau},x^0_{t+1,\tau})\in \text{ri gph } F$, $(t,\tau)\in H_1\times L_1$,
$x^0_{t,\tau}\in \text{ri dom } g_{t,\tau},(t,\tau)\in H\times L_0$.

b. $(x^0_{t,\tau+1},x^0_{t,\tau},x^0_{t+1,\tau})\in \text{int gph } F$, $(t,\tau)\in H_1\times L_1$

$(t,\tau) \neq (t_0,\tau_0)$ ((t_0,τ_0) is the fixed pair) and $g_{t,\tau}$ are continuous at the point $x^0_{t,\tau}$. In Subsection 3, we study the convex problem for differential inclusions with distributed parameters:

$$\inf I(x(\cdot,\cdot)) = \iint\limits_Q g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau + \int_0^1 g_0(x(1,\tau),\tau)\mathrm{d}\tau, \tag{6.4}$$

$$\text{subject to } \frac{\partial x(t,\tau)}{\partial t} \in a\left(\frac{\partial x(t,\tau)}{\partial \tau}, x(t,\tau)\right), \quad 0 < t \leq 1, \; 0 \leq \tau < 1, \tag{6.5}$$

$$x(t,1) = \alpha(t), \quad x(0,\tau) = \beta(\tau), \quad \alpha(0) = \beta(1), \quad Q = [0,1] \times [0,1]. \tag{6.6}$$

Here, F is a convex multivalued mapping, g is a continuous function that is convex with respect to x, $g : \mathbb{R}^n \times Q \to \mathbb{R}$, $g_0 : \mathbb{R}^n \times [0,1] \to \mathbb{R}$, and $\alpha(t)$ and $\beta(\tau)$ are absolutely continuous functions, $\alpha : [0,1] \to \mathbb{R}^n$, $\beta : [0,1] \to \mathbb{R}^n$. The problem is to find a solution $\tilde{x}(t,\tau)$ of the boundary value problem in Eqs. (6.5) and (6.6) that minimizes Eq. (6.4). Here, an admissible solution is understood to be an absolutely continuous function with summable first order partial derivatives. In other words, an admissible solution belongs to the Sobolev space $W^{1,1}(Q)$; i.e., the distributional derivatives of the first order belong to $L_1(Q)$. In general, a Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space [131,212].

1 Necessary and Sufficient Conditions for an Extremum for Discrete Inclusions

At first we consider the convex problem in Eqs. (6.1)–(6.3).

Theorem 6.1. Suppose that F is a convex multivalued mapping and $g_{t,\tau}$ are convex proper functions continuous at the points of some admissible solution $\{x^0_{t,\tau}\}_{(t,\tau)\in H\times L_0}$. Then for $\{\tilde{x}_{t,\tau}\}_{(t,\tau)\in H\times L_0}$ to be an optimal solution of the problem in Eqs. (6.1)–(6.3), it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{x^*_{t,\tau}\}$ and $\{\varphi^*_{t,\tau}\}$, not all zero, such that

1. $(\varphi^*_{t,\tau+1}, x^*_{t,\tau} - \varphi^*_{t,\tau}) \in F^*(x^*_{t,\tau}; (\tilde{x}_{t,\tau+1}, \tilde{x}_{t,\tau}, \tilde{x}_{t+1,\tau}))$
   $+ \{0\} \times \{-\lambda \partial g_{t,\tau}(\tilde{x}_{t,\tau})\}, \quad \partial g_{0,\tau}(\tilde{x}_{0,\tau}) \equiv 0, \quad (t,\tau) \in H_1 \times L_1,$
2. $-x^*_{T,\tau} \in \lambda \partial g_{T,\tau}(\tilde{x}_{T,\tau}), \quad \varphi^*_{t,0} = 0.$

In addition, under the regularity condition, (1) and (2) are also sufficient for the optimality of $\{\tilde{x}_{t,\tau}\}_{(t,\tau)\in H\times L_0}$.

□ One of the essential points in the proof is the use of convex programming results. With this goal, we form the $m = n(L+1)$-dimensional vector $x_t = (x_{t,0}, x_{t,1}, \ldots, x_{t,L}) \in \mathbb{R}^m$ for any $t \in H_1$. Further, let $x_T = (x_{T,0}, \ldots, x_{T,L-1}) \in \mathbb{R}^{nL}$. Then $w = (x_0, \ldots, x_T) \in \mathbb{R}^{mT+nL}$. We consider the following convex sets defined on the space $\mathbb{R}^{mT+nL}$:

$$M_{t,\tau} = \{w = (x_0, \ldots, x_T) : (x_{t,\tau+1},\ x_{t,\tau},\ x_{t-1,\tau}) \in \operatorname{gph} F\} (t,\tau) \in H_1 \times L_1,$$

$$M_0 = \{w = (x_0, \ldots, x_T) : x_{t,L} = \alpha_{tL}, t \in H_1\},$$

$$N_0 = \{w = (x_0, \ldots, x_T) : x_{0,\tau} = \beta_{0\tau}, \tau \in L_0\}.$$

Thus, setting $g(w) = \sum_{\substack{t=1,\ldots,T \\ \tau=1,\ldots,L-1}} g_{t,\tau}(x_{t,\tau})$, we can easily show that the boundary problem in Eqs. (6.1)–(6.3) is equivalent to the following convex minimization problem in the space $\mathbb{R}^{mT+nL}$:

$$\inf g(w) \ \text{ subject to } \ w \in P = \left(\bigcap_{(t,\tau) \in H_1 \times L_1} M_{t,\tau} \right) \cap M_0 \cap N_0. \tag{6.7}$$

We apply Theorem 3.4 to this problem. For this, it is necessary to compute the cones $K^*_{M_{t,\tau}}(w),\ K^*_{M_0}(w),\ K^*_{N_0}(w),\ w \in P$ as in Lemma 6.1.

Lemma 6.1. Let $K_{M_{t,\tau}}(w)$ be a convex cone of tangent directions at $w \in M_{t,\tau}$. Then

$$K^*_{M_{t,\tau}}(w) = \big\{ w^* = (x^*_0, \ldots, x^*_T) : (x^*_{t,\tau+1}, x^*_{t,\tau}, x^*_{t+1,\tau}) \in K^*_{\operatorname{gph} F}(x_{t,\tau+1}, x_{t,\tau}, x_{t+1,\tau}),$$
$$x^*_{i,j} = 0,\ \ (i,j) \neq (t,\tau+1),\ \ (t,\tau),\ \ (t+1,\tau) \big\}, \quad (t,\tau) \in H_1 \times L_1.$$

□ Let $\overline{w} \in K_{M_{t,\tau}}(w)$. This means that $w + \lambda\overline{w} \in M_{t,\tau}$ for sufficiently small $\lambda > 0$, equivalently,

$$(x_{t,\tau+1} + \lambda\overline{x}_{t,\tau+1}, x_{t,\tau} + \lambda\overline{x}_{t,\tau}, x_{t+1,\tau} + \lambda\overline{x}_{t+1,\tau},) \in \operatorname{gph} F.$$

Therefore,

$$K_{M_{t,\tau}}(w) = \big\{ \overline{w} = (\overline{x}_0, \ldots, \overline{x}_T) : (\overline{x}_{t,\tau+1}, \overline{x}_{t,\tau}, \overline{x}_{t+1,\tau}) \in K_{\operatorname{gph} F}(x_{t,\tau+1}, x_{t,\tau}, x_{t+1,\tau}) \big\}. \tag{6.8}$$

On the other hand, $w^* \in K^*_{M_{t,\tau}}(w)$ is equivalent to the condition

$$\langle \overline{w}, w^* \rangle = \sum_{(i,j) \in H \times L_0} \langle \overline{x}_{i,j}, x^*_{i,j} \rangle \geq 0, \quad \overline{w} \in K_{M_{t,\tau}}(w),\ \ (i,j) \neq (T,L),$$

where the components $\overline{x}_{i,j}$ of the vector $\overline{w}$ (see Eq. (6.8)) are arbitrary. Therefore, the last relation is valid only for $x^*_{i,j} = 0$, $(i,j) \neq (t,\tau+1)$, (t,τ), $(t+1,\tau)$. This completes the proof of the lemma. ■

It is also not hard to show that

$$
\begin{aligned}
K^*_{M_0}(w) &= \{w^* = (x^*_0, \ldots, x^*_T) : x^*_{t,\tau} = 0, \tau \neq L, t \in H\}, \\
K^*_{N_0}(w) &= \{w^* = (x^*_0, \ldots, x^*_T) : x^*_{t,\tau} = 0, t \neq 0, \tau \in L_0, (t,\tau) \neq (T,L)\}.
\end{aligned} \tag{6.9}
$$

Further, by the hypothesis of Theorem 6.1, $\{\tilde{x}_{t,\tau}\}_{(t,\tau)\in H\times L_0}$ is an optimal solution. Consequently, $\tilde{w} = (\tilde{x}_0, \ldots, \tilde{x}_T)$ is an optimal solution of the problem in Eq. (6.7). Moreover, $g(w)$ is continuous at the point $w^0 = (x^0_0, \ldots, x^0_T)$. Then applying Theorem 3.4 to the minimization problem in Eq. (6.7), we can assert the existence of vectors

$$
\begin{gathered}
w^*(t,\tau) \in K^*_{M_{t,\tau}}(\tilde{w}), \;\; (t,\tau) \in H_1 \times L_1, \;\; \tilde{w}^* = (\tilde{x}^*_0, \ldots, \tilde{x}^*_T) \in K^*_{M_0}(\tilde{w}), \\
\overline{w}^* = (\overline{x}^*_0, \ldots, \overline{x}^*_T) \in K^*_{N_0}(\overline{w}), \;\; w^{0*} \in \partial_w g(\tilde{w})
\end{gathered}
$$

and of a number λ (equal to 0 or 1), not all zero, such that

$$
\lambda w^{0*} = \sum_{(t,\tau)\in H_1 x L_1} w^*(t,\tau) + \tilde{w}^* + \overline{w}^*. \tag{6.10}
$$

This equality plays a central role in the investigations to follow. Using the fact that

$$
\begin{gathered}
w^*(t,\tau) = (x^*_0(t,\tau), \ldots, x^*_T(t,\tau)), \;\; x^*_t(t,\tau) = (x^*_{t,0}(t,\tau), \ldots, x^*_{t,L}(t,\tau)), \quad t \in H_1 \\
x^*_T(t,\tau) = (x^*_{T,0}(t,\tau), \ldots, x^*_{T.L-1}(t,\tau))
\end{gathered}
$$

and Lemma 6.1, we get

$$
\left[\sum_{(t,\tau)\in H_1 \times L_1} w^*(t,\tau) \right]_{t,\tau} = x^*_{t,\tau}(t,\tau) + x^*_{t,\tau}(t-1,\tau) + x^*_{t,\tau}(t,\tau-1) \tag{6.11}
$$

$$
x^*_{t,\tau}(t,\tau-1) = 0, \quad \tau = 0, \;\; t \in H_1,
$$

where $[w^*]_{t,\tau}$ denotes the components of the vector w^* for the given pair (t,τ).

On the other hand, using Eq. (6.9), we can write

$$
\left[\sum_{(t,\tau)\in H_1 \times L_1} w^*(t,\tau) + \tilde{w}^* + \overline{w}^* \right]_{t,\tau} = \begin{cases} x^*_{t,0}(t,0) + x^*_{t,0}(t-1,0), & t \in H_1 \\ x^*_{T,\tau}(T-1,\tau), & \tau \in L_1, \end{cases} \tag{6.12}
$$

where it is taken into account that $[\tilde{w}^*]_{t,0} = \tilde{x}^*_{t,0} = 0, \; t \in H, \quad [\overline{w}^*]_{t,0} = \overline{x}^*_{t,0} = 0,$ $t = 1, \ldots, T$, and $[\tilde{w}^*]_{t,\tau} = \tilde{x}^*_{t,\tau} = 0, \quad [\overline{w}^*]_{T,\tau} = 0, \;\; \tau \in L_1$. Also from the arbitrariness of $\tilde{x}^*_{t,L}$, it is clear that the equalities

$$
x^*_{t,L}(t,L) + x^*_{t,L}(t-1,L) + x^*_{t,L}(t,L-1) + \tilde{x}^*_{t,L} = \lambda x^{0*}_{t,L}, \quad t = 1, \ldots, T-1
$$

always hold, where $[w^{0*}]_{t,L} = x^{0*}_{t,L}$. The same applies to the component-wise representation in Eq. (6.10) for $(0,\tau)$, $\tau \in L_1$ in view of the arbitrariness of $[\overline{w}^*]_{0,\tau} = \overline{x}^*_{0,\tau}$. Thus, by Eqs. (6.11) and (6.12), it follows from Eq. (6.10) that

$$\begin{aligned}
&\lambda x^{0*}_{t,\tau} = x^*_{t,\tau}(t,\tau) + x^*_{t,\tau}(t-1,\tau) + x^*_{t,\tau}(t,\tau-1),\\
&x^*_{t,\tau}(t,\tau-1) = 0, \quad \tau = 0, \ t \in H_1, \qquad (6.13)\\
&\lambda x^{0*}_{T,\tau} = x^*_{T,\tau}(T-1,\tau), \quad \tau \in L_1.
\end{aligned}$$

Using Lemma 6.1 and the definition of a LAM, it can be concluded that

$$(x^*_{t,\tau+1}(t,\tau), \ x^*_{t,\tau}(t,\tau)) \in F^*(-x^*_{t+1,\tau}(t,\tau); \ (\tilde{x}_{t,\tau+1}, \tilde{x}_{t,\tau}, \tilde{x}_{t+1,\tau})). \qquad (6.14)$$

Then, introducing the new notation $x^*_{t,\tau+1}(t,\tau) \equiv \varphi^*_{t,\tau+1}$, $x^*_{t,\tau}(t-1,\tau) \equiv -x^*_{t,\tau}$, we see from Eqs. (6.13) and (6.14) that the first part of the theorem is valid. As for the sufficiency of the conditions is obtained, it is clear that by Theorems 1.11 and 1.30 and the regularity condition, the representation in Eq. (6.10) holds with $\lambda = 1$ for the point $w^{0*} \in \partial_w g(\tilde{w}) \cap K^*_P(\tilde{w})$. ■

Note that if in the problem in Eqs. (6.1)–(6.3) the functions and mappings are polyhedral, then by virtue of Lemma 1.22, the regularity condition in Theorem 6.1 is superfluous.

Theorem 6.2. Assume condition (N) for the problem in Eqs. (6.1)–(6.3). Then for $\{\tilde{x}_{t,\tau}\}_{(t,\tau)\in H\times L}$ to be a solution of this nonconvex problem, it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{x^*_{t,\tau}\}$, $\{\varphi^*_{t,\tau}\}$ not all zero, satisfying conditions (1) and (2) of Theorem 6.1.

□ In this case, condition (N) implies the hypotheses of Theorem 3.25 for the problem in Eq. (6.7). Therefore, we get the necessary condition as in Theorem 6.1 by starting from the relation in Eq. (6.10), written out for the nonconvex problem. ■

2 Approximation of the Continuous Problem and a Necessary Condition for the Discrete-Approximation Problem

In this section, we use difference derivatives to approximate the problem in Eqs. (6.4)–(6.6), and with the help of Theorem 6.1 we formulate a necessary and sufficient condition for it. We choose steps δ and h on the t-axis and the τ-axis, respectively, using the grid function $x_{t,\tau} = x_{\delta h}(t,\tau)$ on a uniform grid on Q. We introduce the following difference operators, defined on the two-point models [246] $A_1 = A_{1\delta}$, $A_2 = A_{2h}$:

$$A_1 x(t+\delta,\tau) = \frac{x(t+\delta,\tau) - x(t,\tau)}{\delta}; \quad A_2 x(t+\tau,h) = \frac{x(t,\tau+h) - x(t,\tau)}{h},$$
$$t = 0, \delta, \ldots, \ 1-\delta, \ \tau = 0, \ldots, 1-h.$$

With the problem in Eqs. (6.4)–(6.6), we now associate the following difference boundary value problem, approximating it:

$$\inf I_{\delta h}(x(\cdot,\cdot)) = \sum_{\substack{t=0,\dots,1-\delta \\ \tau=h,\dots,1-h}} \delta h g(x(t,\tau),t,\tau) + \sum_{\tau=0,\dots,1-h} h g_0(x(1,\tau),\tau), \tag{6.15}$$

$$\begin{aligned} &\text{subject to } A_1 x(t+\delta,\tau) \in a(A_2 x(t,\tau+h), x(t,\tau)), \\ &\qquad x(t,1)=\alpha(t), \quad x(0,\tau)=\beta(\tau), \ \ t=0,\dots,1-\delta, \ \ \tau=0,\dots,1-h. \end{aligned} \tag{6.16}$$

We reduce the problem in Eqs. (6.15) and (6.16) to a problem of the form in Eqs. (6.1)–(6.3). To do this, we introduce a new mapping

$$\tilde{F}(p,x) = x + \delta F\left(\frac{p-x}{h}, x\right) \tag{6.17}$$

and rewrite the problem in Eqs. (6.15) and (6.16) as follows:

$$\begin{aligned} &\inf && I_{\delta h}(x(\cdot,\cdot)), \\ &\text{subject to} && x(t+\delta,\tau) \in \tilde{F}(x(t,\tau+h), x(t,\tau)), \\ &&& x(t,1)=\alpha(t), x(0,\tau)=\beta(\tau), \\ &&& t=0,\dots,1-\delta, \quad \tau=0,\dots,1-h. \end{aligned} \tag{6.18}$$

By Theorem 6.1, for optimality of the trajectory $\{\tilde{x}(t,\tau)\}$, $t=0,\dots,1$, $\tau=0,\dots,1$, $(t,\tau)\neq(1,1)$ in Eq. (6.18), it is necessary that there exist vectors $\{x^*(t,\tau)\}$, $\{\varphi^*(t,\tau)\}$, and a number $\lambda=\lambda_{\delta h}\in\{0,1\}$, not all zero, such that

$$\begin{aligned} (\varphi^*(t,\tau+h), x^*(t,\tau)-\varphi^*(t,\tau)) \in \tilde{F}^*&(x^*(t+\delta,\tau), \tilde{x}(t,\tau+h), \tilde{x}(t,\tau), \tilde{x}(t+\delta,\tau)) \\ &+ \{0\}\times\{\delta h\lambda\partial g(\tilde{x}(t,\tau),t,\tau)\}, \end{aligned} \tag{6.19}$$

$$x^*(1,\tau) \in \lambda h \partial g_0(\tilde{x}(1,\tau),\tau), \quad \varphi^*(t,0)=0, \ \ t=0,\dots,1-\delta, \ \ \tau=0,\dots,1-h. \tag{6.20}$$

In Eq. (6.19), $\tilde{F}^*$ must be expressed in terms of F^*.

Theorem 6.3. Suppose that the mapping $\tilde{F}$ is such that the cones $K_{\text{gph}\,\tilde{F}}(p,x,v)$, $(p,x,v)\in \text{gph}\,\tilde{F}$ of tangent directions determine a local tent. Then the following inclusions are equivalent:

1. $(p^*,x^*)\in \tilde{F}^*(v^*;(p,x,v))$,
2. $\left(\frac{p^*}{\delta}, \frac{p^*+x^*-v^*}{\delta h}\right) \in F^*\left(\frac{v^*}{h}; \left(\frac{p-x}{h}, x, \frac{v-x}{\delta}\right)\right), \quad v^*\in\mathbb{R}^n.$

☐ By Definition 3.4 of a local tent (Section 3.4), there exist functions $r_i(\overline{z})$, $i = 1, 2$ and $r(\overline{z})$, $\overline{z} = (\overline{p}, \overline{x}, \overline{v})$, such that $r_i(\overline{z})\|\overline{z}\|^{-1} \to 0$ and $r(\overline{z})\|\overline{z}\|^{-1} \to 0$ as $\overline{z} \to 0$, and

$$\begin{aligned} v + \overline{v} + r(\overline{z}) &\in x + \overline{x} + r_1(\overline{z}) \\ &+ \delta F\left(\frac{p + \overline{p} + r_2(\overline{z}) - x - \overline{x} - r_1(\overline{z})}{h}, x + \overline{x} + r_1(\overline{z})\right) \end{aligned}$$

for sufficiently small $\overline{z} \in Q$, $Q \subseteq \text{ri}\, K_{\text{gph}\ \tilde{F}}(z)$.

Transforming this inclusion, we get

$$\begin{aligned} &\frac{v - x}{\delta} + \frac{\overline{v} - \overline{x}}{\delta} + \frac{r(\overline{z}) - r_1(\overline{z})}{\delta} \\ &\in F\left(\frac{p - x}{h} + \frac{\overline{p} - \overline{x}}{h} + \frac{r_2(\overline{z}) - r_1(\overline{z})}{h}, x + \overline{x} + r_1(\overline{z})\right). \end{aligned}$$

From this, it is clear that $K_{\text{gph}\ F}((p - x)/h, x, (v - x)/\delta)$ is a local tent of gph F, and

$$\left(\frac{\overline{p} - \overline{x}}{h}, \overline{x}, \frac{\overline{v} - \overline{x}}{\delta}\right) \in K_{\text{gph}\ \tilde{F}}\left(\frac{p - x}{h}, x, \frac{v - x}{\delta}\right). \tag{6.21}$$

Now, by going in the reverse direction, it is also not hard to see from Eq. (6.21) that

$$(\overline{p}, \overline{x}, \overline{v}) \in K_{\text{gph}\ \tilde{F}}(p, x, v). \tag{6.22}$$

This means that Eqs. (6.21) and (6.22) are equivalent.

Suppose now that $(p^*, x^*) \in \tilde{F}^*(v^*; (p, x, v))$ or, equivalently,

$$\langle \overline{p}, p^* \rangle + \langle \overline{x}, x^* \rangle - \langle \overline{v}, v^* \rangle \geq 0, \quad (\overline{p}, \overline{x}, \overline{v}) \in K_{\text{gph}\ \tilde{F}}(p, x, v). \tag{6.23}$$

We rewrite this in the form

$$\left\langle \frac{\overline{p} - \overline{x}}{h}, \psi_1^* \right\rangle + \langle \overline{x}, \psi_2^* \rangle - \left\langle \frac{\overline{v} - \overline{x}}{\delta}, \psi^* \right\rangle \geq 0, \quad (\overline{p}, \overline{x}, \overline{v}) \in K_{\text{gph}\ \tilde{F}}(p, x, v), \tag{6.24}$$

where ψ_1^*, ψ_2^*, and ψ^* are to be determined. Carrying out the necessary transformations in Eq. (6.24) and comparing it with Eq. (6.23), we see that

$$\psi_1^* = \frac{hp^*}{\delta}, \quad \psi_2^* = \frac{x^* + p^* - v^*}{\delta}, \quad \psi^* = v^*.$$

Then it follows from the equivalence of Eqs. (6.21) and (6.22) that

$$\left(\frac{hp^*}{\delta},\frac{x^*+p^*-\upsilon^*}{\delta}\right)\in F^*\left(\upsilon^*;\left(\frac{p-x}{h},x,\frac{\upsilon-x}{\delta}\right)\right). \tag{6.25}$$

This conclude the proof of the theorem. ■

Remark 6.1. In Eq. (6.25), it is taken into account that the LAM F^* is a positive homogeneous mapping with respect to the first argument:

$$F^*(\mu\upsilon^*;(p,x,\upsilon))=\mu F^*(\upsilon^*;(p,x,\upsilon)),\quad \mu>0.$$

Remark 6.2. If the mapping F is convex, then we can establish the equivalence of the inclusions in Theorem 6.3 in another way, namely, by computing $\partial H_{\tilde{F}}(p,x,\upsilon^*)$, and expressing it in terms of $\partial H_F((p-x)/h,x,\upsilon^*)$.

Let us return to the conditions in Eqs. (6.19) and (6.20). By Theorem 6.3, the condition in Eq. (6.19) takes the form

$$\begin{aligned}&\left(\frac{\varphi^*(t,\tau+h)}{\delta},\frac{x^*(t,\tau)-x^*(t+\delta,\tau)+\varphi^*(t,\tau+h)-\varphi^*(t,\tau)}{\delta h}\right)\\&\quad\in F^*\left(\frac{x^*(t+\delta,\tau)}{h};(A_2\tilde{x}(t,\tau+h),\tilde{x}(t,\tau),A_1\tilde{x}(t+\delta,\tau))\right)\\&\quad+\{0\}\times\{-\lambda\partial g(\tilde{x}(t,\tau),t,\tau)\},\quad t=0,\ldots,1-\delta,\quad \tau=0,\ldots,1-h.\end{aligned} \tag{6.26}$$

Here, denoting the expressions $(\varphi^*(t,\tau))/\delta$ and $(x^*(t,\tau))/h$ again by $\varphi^*(t,\tau)$ and $x^*(t,\tau)$, respectively, it is not hard to verify the following representation of the second quotient:

$$\begin{aligned}&\frac{x^*(t,\tau)-x^*(t+\delta,\tau)+\varphi^*(t,\tau+h)-\varphi^*(t,\tau)}{\delta h}=A_2\varphi^*(t,\tau+h)-A_1x^*(t+\delta,\tau),\\&\quad t=0,\ldots,1-\delta,\quad \tau=0,\ldots,1-h.\end{aligned}$$

Then it follows from Eqs. (6.20) and (6.26) that

$$\begin{aligned}&(\varphi^*(t,\tau+h),A_2\varphi^*(t,\tau+h))\in F^*(x^*(t+\delta,\tau);A_2\tilde{x}(t,\tau+h),\tilde{x}(t,\tau),A_1\tilde{x}(t+\delta,\tau))\\&\quad+\{0\}\times\{\lambda\partial g(\tilde{x}(t,\tau),t,\tau)+A_1x^*(t+\delta,\tau)\},\end{aligned} \tag{6.27}$$

$$\varphi^*(t,0)=0,\qquad -x^*(1,\tau)\in\lambda\partial g_0(x(1,\tau),\tau),\quad t=0,\ldots,1-\delta,\quad \tau=0,\ldots,1-h. \tag{6.28}$$

We formulate the result just obtained as in Theorem 6.4.

Theorem 6.4. Suppose that F is convex and g and g_0 are proper functions convex with respect to x and continuous at the points of some admissible trajectory $\{x^0\,(t,\tau)\}$, $t=0,\,\delta,\,\ldots,\,1,\ \tau=0,h,\,\ldots,\,1\ (t,\tau)\neq(1,1)$. Then for the optimality of the trajectory $\{\tilde{x}(t,\tau)\}$ in the discrete-approximation problem in Eqs. (6.15) and (6.16), it is necessary that there exist a number $\lambda=\lambda_{\delta h}=\{0,1\}$ and vectors $\{v^*(t,\tau)\}$ and $\{x^*(t,\tau)\}$, not all zero, satisfying Eqs. (6.27) and (6.28). And under the regularity condition, Eqs. (6.27) and (6.28) are also sufficient for the optimality of $\{\tilde{x}(t,\tau)\}$.

Remark 6.3. As in Theorem 2.2, Eqs. (6.27) and (6.28) are necessary conditions for optimality in the case of nonconvexity for the problem in Eqs. (6.15) and (6.16) under condition (N).

3 Sufficient Conditions of Optimality for Differential Inclusions

Using the results of Subsection 2, we formulate a sufficient condition for optimality for the continuous problem in Eqs. (6.4)–(6.6). Setting $\lambda=1$ in Eqs. (6.27) and (6.28), we find, by passing to the formal limit as δ and h tend to 0, that

1. $$\left(\varphi^*(t,\tau),\frac{\partial\varphi^*(t,\tau)}{\partial\tau}\right)\in F^*\left(x^*(t,\tau);\left(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau},\tilde{x}(t,\tau),\frac{\partial\tilde{x}(t,\tau)}{\partial t}\right)\right)$$
$$+\{0\}\times\left\{\frac{\partial x^*(t,\tau)}{\partial t}-\partial g(\tilde{x}(t,\tau),t,\tau)\right\},$$

2. $\varphi^*(t,0)=0,\qquad -x^*(1,\tau)\in\partial g_0(\tilde{x}(1,\tau),\tau).$

Along with this we get one more condition ensuring that the LAM F^* is nonempty:

3. $\dfrac{\partial\tilde{x}(t,\tau)}{\partial t}\in F\left(\dfrac{\partial\tilde{x}(t,\tau)}{\partial\tau},\tilde{x}(t,\tau);x^*(t,\tau)\right).$

The arguments in Subsection 2 suggest the sufficiency of conditions (1)–(3) for optimality. It turns out that the assertion in Theorem 6.5 is true.

Theorem 6.5. Suppose that $g(x,t,\tau)$ and $g_0(x,\tau)$ are jointly continuous functions convex with respect to x and F is a convex closed mapping; i.e., gph F is a convex closed subset of $\mathbb{R}^{3n}$. Then for the optimality of the solution $\tilde{x}(\cdot,\cdot)$ among all admissible solutions, it is sufficient that there exist absolutely continuous functions $\{\varphi^*(t,\tau),\,x^*(t,\tau)\}$ such that the conditions (1) and (3) hold almost everywhere on Q.
☐ By Theorem 2.1,

$$F^*(v^*;(p,x,v))=\partial_{(p,x)}H_F(p,x,v^*),\qquad v\in F(p,x;v^*).$$

Then by using the Moreau−Rockafellar theorem (Theorem 1.29), from condition (1) we obtain the inclusion

$$\left(\varphi^*(t,\tau),\frac{\partial\varphi^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}\right)\in\partial_{(p,x)}H_F\left(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau},\tilde{x}(t,\tau),x^*(t,\tau)\right)$$
$$+g_1\left(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau},\tilde{x}(t,\tau),t,\tau\right),\ \ g_1(p,x,t,\tau)\equiv-g(x,t,\tau),\ (t,\tau)\in Q.$$

Using the definitions of a subdifferential and H_F, we rewrite the last relation in the form

$$g(x(t,\tau),t,\tau)-g(\tilde{x}(t,\tau),t,\tau)-\left\langle\frac{\partial x(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle+\left\langle\frac{\partial\tilde{x}(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle$$
$$\geq-\left\langle\varphi^*(t,\tau),\frac{\partial x(t,\tau)}{\partial\tau}-\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle+\left\langle\frac{\partial x^*(t,\tau)}{\partial t}-\frac{\partial\varphi^*(t,\tau)}{\partial\tau},x(t,\tau)-\tilde{x}(t,\tau)\right\rangle,$$

or

$$g(x(t,\tau),t,\tau)-g(\tilde{x}(t,\tau),t,\tau)\geq\left\langle\frac{\partial}{\partial t}(x(t,\tau)-\tilde{x}(t,\tau)),x^*(t,\tau)\right\rangle$$
$$-\frac{\partial}{\partial\tau}\langle\varphi^*(t,\tau),x(t,\tau)-\tilde{x}(t,\tau)\rangle+\left\langle\frac{\partial x^*(t,\tau)}{\partial t},x(t,\tau)-\tilde{x}(t,\tau)\right\rangle.$$

On the other hand, by the second condition in (2),

$$g_0(x(1,\tau),\tau)-g_0(\tilde{x}(1,\tau),\tau)\geq-\langle x^*(1,\tau),x(1,\tau)-\tilde{x}(1,\tau)\rangle.$$

Integrating the preceding relation over the domain Q, and the latter over the interval [0,1] and then adding them, we get

$$\iint_Q[g(x(t,\tau),t,\tau)-g(\tilde{x}(t,\tau),t,\tau)]\mathrm{d}t\,\mathrm{d}\tau+\int_0^1[g_0(x(1,\tau),\tau)-g_0(\tilde{x}(1,\tau),\tau)]\mathrm{d}\tau$$
$$\geq\iint_Q\frac{\partial}{\partial t}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau-\iint_Q\frac{\partial}{\partial\tau}\langle\varphi^*(t,\tau),x(t,\tau)-\tilde{x}(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau$$
$$-\int_0^1\langle x^*(1,\tau),x(1,\tau)-\tilde{x}(1,\tau)\rangle\mathrm{d}\tau. \tag{6.29}$$

It is clear that

$$\iint\limits_Q \frac{\partial}{\partial t}\langle x(t,\tau)-\tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau$$
$$= \int_0^1 \langle x^*(1,\tau), x(1,\tau)-\tilde{x}(1,\tau)\rangle \mathrm{d}\tau - \int_0^1 \langle x^*(0,\tau), x(0,\tau)-\tilde{x}(0,\tau)\rangle \mathrm{d}\tau \tag{6.30}$$

where, since $\tilde{x}(0,\tau)=x(0,\tau)$ (see Eq. (6.6)),

$$\int_0^1 \langle x^*(0,\tau), x(0,\tau)-\tilde{x}(0,\tau)\rangle \mathrm{d}\tau = 0.$$

Analogously,

$$\int_0^1\int_0^1 \frac{\partial}{\partial \tau}\langle \varphi^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau$$
$$= \int_0^1 \langle \varphi^*(t,1), x(t,1)-\tilde{x}(t,1)\rangle \mathrm{d}t - \int_0^1 \langle \varphi^*(t,0), x(t,0)-\tilde{x}(t,0)\rangle \mathrm{d}t, \tag{6.31}$$

and since $x(t,1)=\tilde{x}(t,1)$and $\varphi^*(t,0)=0$ by condition (2),

$$\int_0^1 \langle \varphi^*(t,1), x(t,1)-\tilde{x}(t,1)\rangle \mathrm{d}t = 0, \qquad \int_0^1 \langle \varphi^*(t,0), x(t,0)-\tilde{x}(t,0)\rangle \mathrm{d}t = 0.$$

Then from Eqs. (6.30) and (6.31), we obtain that the right-hand side of Eq. (6.29) is equal to zero. Thus, we have finally

$$\iint\limits_Q g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau + \int_0^1 g_0(x(1,\tau),\tau)\mathrm{d}\tau$$
$$\geq \iint\limits_Q g(\tilde{x}(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau + \int_0^1 g_0(\tilde{x}(1,\tau),\tau)\mathrm{d}\tau$$

for all admissible solutions $x(t,\tau)$, $(t,\tau)\in Q$.

The theorem is proved. ■

Remark 6.4. Recall from Sections 4.5 and 4.6 that the multivalued mapping defined by the following inequality is called the LAM to nonconvex mapping F at a point:

$$F^*(v^*;(\tilde{p},\tilde{x},\tilde{v})) = \{(p^*,x^*): H_F(p,x,v^*) - H_F(\tilde{p},\tilde{x},\tilde{v}^*) \\ \leq \langle p^*, p-\tilde{p}\rangle + \langle x^*, x-\tilde{x}\rangle \quad \forall (p,x)\in\mathbb{R}^n\times\mathbb{R}^n\}.$$

Obviously, F^* is a convex closed set at a given point.

Note that for smooth functions $H_F(\cdot,\cdot,v^*)$, the inequality in this definition can be given by the Weierstrass function [111:124], which plays an important role in the classical variational calculation's problems. By using this definition of LAM and the following conditions (a)−(c) in Theorem 6.5, it easily be seen that Eq. (6.29) holds and hence a similar result to this theorem is valid for such a nonconvex case.

a. $\left(\varphi^*(t,\tau), \frac{\partial\varphi^*(t,\tau)}{\partial\tau} - \frac{\partial x^*(t,\tau)}{\partial t} + x^*(t,\tau)\right) \in F^*\left(x^*(t,\tau); \frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), \frac{\partial\tilde{x}(t,\tau)}{\partial t}\right),$

b. $g(x,t,\tau) - g(\tilde{x}(t,\tau),t,\tau) \geq \langle x^*(t,\tau), x-\tilde{x}(t,\tau)\rangle \quad \forall x\in\mathbb{R}^n,$

c. $\varphi^*(t,0)=0,\ \ g_0(x,\tau) - g_0(\tilde{x}(1,\tau),\tau) \geq -\langle x^*(1,\tau), x-\tilde{x}(1,\tau)\rangle \quad \forall x\in\mathbb{R}^n.$

Example 6.1. In the conclusion, we consider an example:

$$\inf I(x(t,\tau)),$$

$$\text{subject to } \frac{\partial x(t,\tau)}{\partial t} = A_1\frac{\partial x(t,\tau)}{\partial\tau} + A_2 x(t,\tau) + Bu(t,\tau), \quad u(t,\tau)\in U$$

$$x(t,1)=\alpha(t), \quad x(0,\tau)=\beta(\tau), \tag{6.32}$$

where A_1 and A_2 are $n\times n$ matrices, B is a rectangular $n\times r$ matrix, $U\subset\mathbb{R}^r$ is a convex closed set, and g and g_0 are continuously differentiable functions of x. It is required to find a controlling parameter $\tilde{u}(t,\tau)\in U$ such that the solution $\tilde{x}(t,\tau)$ corresponding to it minimizes $I(x(\cdot,\cdot))$. In this case, $F(p,x)=A_1p+A_2x+BU$.

By elementary computations, we find that

$$F^*(v^*;(p,x,v)) = \begin{cases} (A_1^*v^*, A_2v^*), & \text{if } -B^*v^*\in[\operatorname{cone}(U-u)]^*, \\ \varnothing, & \text{if } -B^*v^*\notin[\operatorname{cone}(U-u)]^*. \end{cases}$$

$$v = A_1p + A_2x + Bu.$$

Then, using Theorem 6.5, we get the relations

$$\varphi^*(t,\tau) = A_1^*x^*(t,\tau), \quad \varphi^*(t,0)=0, \tag{6.33}$$

$$\frac{\partial \varphi^*(t,\tau)}{\partial \tau} - \frac{\partial x^*(t,\tau)}{\partial t} = A_2^* x^*(t,\tau) - g'(\tilde{x}(t,\tau),t,\tau), \tag{6.34}$$

$$\langle u - \tilde{u}(t,\tau), -B^* x^*(t,\tau)\rangle \geq 0, \quad u \in U, \tag{6.35}$$

$$x^*(1,\tau) = -g_0'(\tilde{x}(1,\tau),\tau). \tag{6.36}$$

Substituting Eq. (6.33) into Eq. (6.34), we have

$$-\frac{\partial x^*(t,\tau)}{\partial t} = -A_1^* \frac{\partial x^*(t,\tau)}{\partial \tau} + A_2^* x^*(t,\tau) - g'(\tilde{x}(t,\tau),t,\tau). \tag{6.37}$$

Obviously, Eq. (6.35) and the second condition of Eq. (6.33) can be written in the form

$$\langle B\tilde{u}(t,\tau), x^*(t,\tau)\rangle = \sup_{u \in U}\langle Bu, x^*(t,\tau)\rangle \tag{6.38}$$

$$x^*(t,0) = 0.$$

Thus, we have obtained the following theorem.

Theorem 6.6. The solution $\tilde{x}(t,\tau)$ corresponding to the control $\tilde{u}(t,\tau)$ minimizes $I(x(\cdot,\cdot))$ in the problem in Eq. (6.32) if there exists an absolutely continuous function $x^*(t,\tau)$ satisfying almost everywhere the differential equation in Eq. (6.37) and conditions in Eqs. (6.36) and (6.38).

4 Duality in the Problems of Optimal Control Described by First-Order Partial Differential Inclusions

On the basis of the apparatus of locally conjugate mappings, a sufficient condition for optimality is derived for the nonconvex problem and duality theorems are proved. The duality theorems proved allow us to conclude that a sufficient condition for an extremum is an extremal relation for the direct and dual problems.

In this section, at first we establish the dual problem for the first-order differential inclusions in Eqs. (6.4)–(6.6) with homogeneous boundary value problems ($x(t,1) = 0$, $x(0,\tau) = 0$).

The problem of determining the supremum

$$\sup_{\substack{\varphi^*(t,\tau),x^*(t,\tau),z^*(t,\tau)\\ \varphi^*(t,0)=0}} I_*[\varphi^*(t,\tau), x^*(t,\tau), z^*(t,\tau)] \tag{6.39}$$

is called the dual problem to the convex problem in Eqs. (6.4)–(6.6), where

$$I_*[\varphi^*(t,\tau),x^*(t,\tau),z^*(t,\tau)]=\iint\limits_Q\left[M\left(\varphi^*(t,\tau),\frac{\partial\varphi^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}+z^*(t,\tau),x^*(t,\tau)\right)\right.$$
$$\left.-g^*(z^*(t,\tau),t,\tau)\right]\mathrm{d}t\,\mathrm{d}\tau-\int_0^1 g_0^*(x^*(1,\tau),\tau)\mathrm{d}\tau.$$

It is assumed that the functions $\varphi^*(t,\tau)$, $x^*(t,\tau)$ are absolutely continuous functions with summable first partial derivatives on Q; i.e., they belong to the Sobolev space $W^{1,1}(Q)$ and $z^*(t,\tau)$ is an absolutely continuous function on Q.

Theorem 6.7. The inequality

$$I(x(t,\tau))\geq I_*[\varphi^*(t,\tau),x^*(t,\tau),z^*(t,\tau)]$$

holds for all feasible solutions $x(t,\tau)$ and $\{u^*(t,\tau),x^*(t,\tau),z^*(t,\tau)\}$ of the convex problem in Eqs. (6.4)–(6.6) and the dual problem in Eq. (6.39), respectively.

□ It is clear from the definitions of the functions M, g^*, and g_0^* that

$$M\left(\varphi^*(t,\tau),\frac{\partial\varphi^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}+z^*(t,\tau),x^*(t,\tau)\right)\leq\left\langle\varphi^*(t,\tau),\ \frac{\partial x(t,\tau)}{\partial\tau}\right\rangle$$
$$+\left\langle\frac{\partial\varphi^*(t,\tau)}{\partial\tau}-\frac{\partial x^*(t,\tau)}{\partial t}+z^*(t,\tau),x(t,\tau)\right\rangle-\left\langle\frac{\partial x(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle$$
$$=\frac{\partial}{\partial\tau}\langle\varphi^*(t,\tau),x(t,\tau)\rangle-\frac{\partial}{\partial t}\langle x^*(t,\tau),x(t,\tau)\rangle+\langle z^*(t,\tau),x(t,\tau)\rangle \tag{6.40}$$

$$-g^*(z^*(t,\tau),t,\tau)\leq g(x(t,\tau),t,\tau)-\langle x(t,\tau),z^*(t,\tau)\rangle, \tag{6.41}$$
$$-g_0^*(x^*(1,\tau),\tau)\leq g_0(x(1,\tau),\tau)-\langle x(1,\tau),x^*(1,\tau)\rangle.$$

Using Eqs. (6.40) and (6.41), we have

$$I_*[\varphi^*(t,\tau),x^*(t,\tau),z^*(t,\tau)]$$
$$\leq\iint\limits_Q\frac{\partial}{\partial\tau}\langle\varphi^*(t,\tau),x(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau-\iint\limits_Q\frac{\partial}{\partial t}\langle x^*(t,\tau),x(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau \tag{6.42}$$
$$+\iint\limits_Q g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau+\int_0^1 g_0(x(1,\tau),\tau)\mathrm{d}\tau+\int_0^1\langle x^*(1,\tau),x(1,\tau)\rangle\mathrm{d}\tau.$$

But since

$$\iint\limits_Q \frac{\partial}{\partial \tau}\langle \varphi^*(t,\tau), x(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau = \int_0^1 [\langle \varphi^*(t,1), x(t,1)\rangle - \langle \varphi^*(t,0), x(t,0)\rangle]\mathrm{d}t,$$

$$\iint\limits_Q \frac{\partial}{\partial t}\langle x^*(t,\tau), x(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau = \int_0^1 [\langle x^*(1,\tau), x(1,\tau)\rangle - \langle x^*(0,\tau), x(0,\tau)\rangle]\mathrm{d}\tau,$$

the solution $x(t,\tau)$ $(x(0,\tau)=0, x(t,1)=0)$ is admissible, and $u^*(t,0)=0$, we get from the inequality in Eq. (6.42):

$$\begin{aligned} I_*(u^*(t,\tau), x^*(t,\tau), z^*(t,\tau)) \le &\int_0^1 \langle x^*(1,\tau), x(1,\tau)\rangle \mathrm{d}\tau + I(x(t,\tau)) \\ &- \int_0^1 \langle x^*(1,\tau), x(1,\tau)\rangle \mathrm{d}\tau = I(x(t,\tau)). \end{aligned}$$

Thus,

$$I_*(u^*(t,\tau), x^*(t,\tau), z^*(t,\tau)) \le I(x(t,\tau)),$$

which is what was required. ■

Theorem 6.8. If the functions $\tilde{x}(t,\tau)$ and $\{\tilde{u}^*(t,\tau), \tilde{x}^*(t,\tau), \tilde{z}^*(t,\tau)\}$, $\tilde{z}^*(t,\tau) \in \partial g(\tilde{x}(t,\tau), t, \tau)$, where $-\tilde{x}^*(1,\tau) \in \partial g_0(\tilde{x}(1,\tau),\tau)$, satisfy conditions (1)–(3) of Theorem 6.5, then they are solutions of the direct and dual problems, respectively, and their values coincide.

□ The fact that $\tilde{x}(t,\tau)$ is a solution of the direct problem was proved in Theorem 6.5. We study the remaining assertions. By the definition of the LAM, condition (1) of Theorem 6.5 is equivalent to the inequality

$$\begin{aligned} &\left\langle \varphi^*(t,\tau), p - \frac{\partial \tilde{x}(t,\tau)}{\partial \tau}\right\rangle + \left\langle \frac{\partial \varphi^*(t,\tau)}{\partial \tau} - \frac{\partial \tilde{x}^*(t,\tau)}{\partial t} + \tilde{z}^*(t,\tau), x - \tilde{x}(t,\tau)\right\rangle \\ &\qquad + \left\langle \tilde{x}^*(t,\tau), \upsilon - \frac{\partial \tilde{x}(t,\tau)}{\partial \tau}\right\rangle \ge 0, \quad (p,x,\upsilon) \in \operatorname{gph} F, \end{aligned}$$

which means that

$$\left(\tilde{u}^*(t,\tau), \frac{\partial \tilde{u}^*(t,\tau)}{\partial \tau} - \frac{\partial \tilde{x}^*(t,\tau)}{\partial t} + \tilde{z}^*(t,\tau), \tilde{x}^*(t,\tau)\right) \in \operatorname{dom} M, \tag{6.43}$$

where

$$\operatorname{dom} M = \{(p^*, x^*, -v^*) : M(p^*, x^*, v^*) > -\infty\}.$$

Further, since $\partial g(x,t,\tau) \subset \operatorname{dom} g^*(\cdot,t,\tau)$ and $\partial g_0(x,\tau) \subset \operatorname{dom} g_0^*(\cdot,\tau)$, it is clear that

$$\tilde{z}^*(t,\tau) \in \operatorname{dom} g^*(\cdot,t,\tau), \quad \tilde{x}^*(1,\tau) \in \operatorname{dom} g_0^*(\cdot,\tau). \tag{6.44}$$

Then it can be concluded from Eqs. (6.43) and (6.44) that the indicated functions $\{\tilde{u}^*(t,\tau), \tilde{x}^*(t,\tau), \tilde{z}^*(t,\tau)\}$ form an admissible solution. It remains to show that it is optimal. Using Lemma 2.6, we get

$$\begin{aligned} &M\left(\tilde{u}^*(t,\tau), \frac{\partial \tilde{u}^*(t,\tau)}{\partial \tau} - \frac{\partial \tilde{x}^*(t,\tau)}{\partial t} + \tilde{z}^*(t,\tau), \tilde{x}^*(t,\tau)\right) \\ &= -H_F\left(\frac{\partial \tilde{x}(t,\tau)}{\partial \tau}, \tilde{x}(t,\tau), \tilde{x}^*(t,\tau)\right) \end{aligned} \tag{6.45}$$

$$\left\langle \tilde{u}^*(t,\tau), \frac{\partial \tilde{x}(t,\tau)}{\partial t}\right\rangle + \left\langle \frac{\partial \tilde{u}^*(t,\tau)}{\partial \tau} - \frac{\partial \tilde{x}^*(t,\tau)}{\partial t} + \tilde{z}^*(t,\tau), \tilde{x}^*(t,\tau)\right\rangle.$$

By condition (3) of Theorem 6.5,

$$H_F\left(\frac{\partial \tilde{x}(t,\tau)}{\partial \tau}, \tilde{x}(t,\tau), \tilde{x}^*(t,\tau)\right) = \left\langle \frac{\partial \tilde{x}(t,\tau)}{\partial t}, \tilde{x}^*(t,\tau)\right\rangle. \tag{6.46}$$

Moreover, we can write

$$\langle \tilde{x}(t,\tau), \tilde{z}^*(t,\tau)\rangle - g(\tilde{x}(t,\tau),t,\tau) = g^*(\tilde{z}^*(t,\tau),t,\tau), \tag{6.47}$$

$$\langle \tilde{x}(1,\tau), \tilde{x}^*(1,\tau)\rangle - g_0(\tilde{x}(1,\tau),\tau) = g_0^*(\tilde{x}^*(1,\tau)\tau),$$

where it is taken into account that

$$\tilde{z}^*(t,\tau) \in \partial g(\tilde{x}(t,\tau),t,\tau), \ \tilde{x}^*(1,\tau) \in \partial g_0(\tilde{x}(1,\tau),\tau).$$

Then in view of Eqs. (6.45)–(6.47), it is easy to establish, as in the proof of Theorem 6.7, that

$$I(\tilde{x}(t,\tau)) = I_*(\tilde{u}^*(t,\tau), \tilde{x}^*(t,\tau), \tilde{z}^*(t,\tau)).$$

The proof is complete. ■

Let us construct the dual problem for the convex problem in Eq. (6.32), where $x(t,1)=0$, $x(0,\tau)=0$. It is easy to see that

$$M(x_1^*, x_2^*, v^*) = \begin{cases} -W_U(B^*v^*), & \text{if } x_1 = A_1^* v^*, x_2 = A_2^* v^*, \\ -\infty, & \text{otherwise,} \end{cases}$$

$$g_0^*(x^*, \tau) = \begin{cases} 0, & x^* = 0, \\ \infty, & x^* \neq 0. \end{cases}$$

Then taking into account the form of M, we conclude that the required dual problem is transformed as follows:

$$\sup_{\substack{x^*(t,\tau),\ z(t,\tau) \\ x^*(1,\tau)=0, x^*(t,0)=0}} I_*[x^*(t,\tau), z^*(t,\tau)],$$

$$-\frac{\partial x^*(t,\tau)}{\partial t} = -A_1^* \frac{\partial x^*(t,\tau)}{\partial \tau} + A_2^* x^*(t,\tau) + z^*(t,\tau),$$

where

$$I_*[x^*(t,\tau), z^*(t,\tau)] = -\iint_Q [W_U(-B^*x^*(t,\tau)) + g^*(z^*(t,\tau), t, \tau]\mathrm{d}t\, \mathrm{d}\tau.$$

6.3 Optimal Control of the Cauchy Problem for First-Order Discrete and Partial Differential Inclusions

Optimization of the Cauchy problem for discrete inclusions is reduced to a problem with geometric constraints in the Hilbert space ℓ_2, and necessary and sufficient conditions for optimality are derived. The obtained results are generalized to the multidimensional case.

In Subsection 1, on the basis of convex minimization programming, necessary and sufficient conditions of Cauchy optimality problems for two-parameter discrete inclusions are proved.

In Subsection 3, we associate a discrete-approximation problem to the Cauchy problem for partial differential inclusions and derive necessary and sufficient conditions for optimality. We establish optimality conditions in both the Euler–Lagrange and Hamiltonian forms. It should be pointed out that in the present case, we do not use the local tent apparatus and CUAs (see Refs. [163,166,169,170,224]). The optimality conditions obtained involve useful forms of the Weierstrass–Pontryagin condition and the Euler–Lagrange type of adjoint inclusions [50,63,136,162,166,169, 170,197,214]. Now, let us formulate the two-parameter Cauchy optimization problem for discrete inclusions. In Subsection 2 on the basis of this problem, we study optimization of the discrete-approximation analogue of the continuous Cauchy problem. The problem consists of the following:

$$\inf \sum_{\tau \in N} \sum_{t \in H} g_{t,\tau}(x_{t,\tau}) \tag{6.48}$$

$$\text{subject to } x_{t+1,\ \tau} \in F_{t,\tau}(x_{t,\tau+1}, x_{t,\tau}), \quad (t,\tau) \in H_1 \times N, \tag{6.49}$$

$$x_{0,\tau} = \varphi_{0\tau}, \quad \tau \in N, \tag{6.50}$$

where $H = \{0, \ldots, T\}$, $H_1 = \{0, \ldots, T-1\}$, $N = \{0,1,2,3, \ldots\}$ is a set of natural numbers including zero, $g_{t,\tau}: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ are functions taking values on the extended line, $F_{t,\tau} : \mathbb{R}^{2n} \to P(\mathbb{R}^n)$ is a set-valued mapping, and $\varphi_{0\tau}$ are fixed vectors. A set of points $\{x_{t,\tau}\}_{(t,\tau)\in H\times N} = \{x_{t,\tau} : (t,\tau) \in H \times N\}$ is called a feasible solution for the problem in Eqs. (6.48)–(6.50) if it satisfies the inclusion in Eqs. (6.49) and the conditions in Eq. (6.50). Obviously, the problem in Eqs. (6.48)–(6.50) can be interpreted as an optimization of a Cauchy problem. Observe that feasible solutions to the discrete inclusions in Eq. (6.49) can always be defined because of the Cauchy condition in Eq. (6.50).

In Subsection 3, we set up an optimization of the Cauchy problem for differential inclusions with distributed parameters:

$$\inf \quad I(x(\cdot,\cdot)) = \int_{-\infty}^{+\infty}\int_{t_0}^{t_1} g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau + \int_{-\infty}^{+\infty} g_0(x(t_1,\tau),\tau)\mathrm{d}\tau \tag{6.51}$$

$$\begin{aligned}&\text{subject to } \frac{\partial x(t,\tau)}{\partial t} + \frac{\partial x(t,\tau)}{\partial \tau} \in F(x(t,\tau),t,\tau)\\ &\text{a.e. } (t,\tau),\ t_0 < t \le t_1,\ -\infty < \tau < +\infty,\end{aligned} \tag{6.52}$$

$$x(t_0,\tau) = \varphi(\tau). \tag{6.53}$$

Here, $F(\cdot,t,\tau) : \mathbb{R}^{2n} \to P(\mathbb{R}^n)$ is a set-valued map, $g : \mathbb{R}^n \times Q \to \mathbb{R}$, and $g_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are continuous functions; $\varphi : \mathbb{R} \to \mathbb{R}^n$ is an absolutely continuous function; and $Q = [t_0,t_1] \times \mathbb{R}$, where t_0,t_1 are fixed points. The problem is to find a solution $\tilde{x}(\cdot,\cdot)$ among all the set of feasible solutions of the problem in Eqs. (6.52) and (6.53) that minimizes Eq. (6.51). Here, a feasible solution is understood to be an absolutely continuous function satisfying almost everywhere (a.e.) the differential inclusion in Eq. (6.52) and the initial condition in Eq. (6.53). The problem in Eqs. (6.51)–(6.53) is called convex if the functions $g(\cdot,t,\tau)$ $g_0(\cdot,\tau)$ and set-valued map $F(\cdot,t,\tau)$ are convex. We label this problem (CP). Such research on the optimization of partial differential inclusions can be applied in the theory of differential games, dynamic optimization, gas dynamics, heat and mass transfer, wave theory, and much more (see also Ref. [242]).

By analogy to the second part of Subsection 3, we study a generalized optimization of a Cauchy problem:

$$\inf \quad J(x(\cdot,\cdot)) = \int_{\mathrm{R}^n}\int_{t_0}^{t_1} f(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau + \int_{\mathrm{R}^n} f_0(x(t_1,\tau),\tau)\mathrm{d}\tau \tag{6.54}$$

$$\text{subject to } \frac{\partial x(t,\tau)}{\partial t} + \sum_{k=1}^{n} \frac{\partial x(t,\tau)}{\partial \tau_k} \in G(x(t,\tau),t,\tau) \quad \text{a.e. } (t,\tau) \in [t_0,t_1] \times \mathbb{R}^n \tag{6.55}$$

$$x(t_0,\tau) = \alpha(\tau), \tag{6.56}$$

where $G : \mathbb{R}^n \times [t_0,t_1] \times \mathbb{R}^n \to P(\mathbb{R}^n)$, $\tau = (\tau_1,\ldots,\tau_n)$ is n-dimensional vector; $d\tau = d\tau_1, \ldots, d\tau_n$, $f : \mathbb{R}^n \times [t_0,t_1] \times \mathbb{R}^n \to \mathbb{R}$ and $f_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions, and $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is an absolutely continuous function. It is required to find an absolutely continuous function $\tilde{x}(t,\tau)$ of $n+1$ variables of the problem in Eqs. (6.55) and (6.56) that minimizes a functional $J(x(\cdot,\cdot))$. We label this problem (GCP).

1 Optimization Problem for Discrete Inclusions with Distributed Parameters

Mahmudov [163] showed the necessary and sufficient conditions for another boundary value problem with discrete inclusions (6.49) in the finite-dimensional Euclidean space both for the convex and the nonconvex cases. Theorem 6.9 is proved by analogy to Theorem 6.1, where you can find all the details. However, for the problem in Eqs. (6.48) and (6.50), it is assumed only that $\{x_{t,0}, x_{t,1}, x_{t,2}, x_{t,3}, \ldots\} \in \ell_2$ or $\{x_{t,\tau}\}_{(t,\tau)\in H\times N} \in \ell_2$ for all $t \in H$. Remember that ℓ_2 is a space of numerical sequences, such that if $x = \{x_i\}$ then $\sum_{i=1}^{\infty} x^2{}_i < \infty$. In fact, ℓ_2 is an infinity-dimensional coordinate-wise Hilbert space with the corresponding inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$. Endowing it with the associated relevant norm, we have a Banach space. Obviously, optimization of the Cauchy problem can be reduced to a problem with geometric constraints in this infinite-dimensional Hilbert space. For convenience, let $\|x_{t,\tau}\|$ be the Euclidean norm of the n-dimensional vector $x_{t,\tau}$, $(t,\tau) \in H \times N$. Then we remark that in our case

$$\sum_{\tau=0}^{\infty} \|x_{t,\tau}\|^2 < \infty, \quad t = 0,1,\ldots,T,$$

and for

$$x_t = (x_{t,0}, x_{t,1}, x_{t,2}, \ldots) \in \ell_2 \quad \text{and} \quad x_t^* = (x_{t,0}^*, x_{t,1}^*, x_{t,2}^*, \ldots) \in \ell_2^*,$$

the inner product $\langle x_t, x_t^*\rangle = \sum_{\tau=0}^{\infty} \langle x_{t,\tau}, x_{t,\tau}^*\rangle$ is finite for all fixed $t \in H$, since the series $\sum_{\tau=0}^{\infty} \langle x_{t,\tau}, x_{t,\tau}^*\rangle$ is convergent. In addition, it is known [139] that ℓ_p is a self-adjoint space; i.e., $\ell_p = \ell_q^*$, $\frac{1}{p} + \frac{1}{q} = 1$, so $\ell_2^* = \ell_2$. Thus, the dual cone constructed in Theorem 6.9 can be defined.

Let us give Theorem 6.9 for the convex case. In what follows, a function $g: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$.

Theorem 6.9. Suppose that $F_{t,\tau}$ are convex set-valued mappings and $g_{t,\tau}$ are convex proper functions continuous at the points of some feasible solution of the problem in Eqs. (6.48)–(6.50), $\{x^0_{t,\tau}\}_{(t,\tau)\in H\times N}$. Then for its optimality, it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $\{x_{t,\tau}{}^*\}$ and $\{\varphi^*_{t,\tau}\}$, not all zero, such that

i. $(\varphi^*_{t,\tau+1}, x^*_{t,\tau} - \varphi^*_{t,\tau}) \in F^*_{t,\tau}(x^*_{t,\tau}; (\tilde{x}_{t,\tau+1}, \tilde{x}_{t,\tau}, \tilde{x}_{t+1,\tau})) + \{0\} \times \{-\lambda \partial g_{t,\tau}(\tilde{x}_{t,\tau})\}, \quad (t,\tau) \in H_1 \times N,$

ii. $-x^*_{T,\tau} \in \lambda \partial g_{T,\tau}(\tilde{x}_{T,\tau})$.

In addition, if $\lambda = 1$, then these conditions are also sufficient for optimality of $\{\tilde{x}_{t,\tau}\}_{(t,\tau)\in H\times N}$.

□ Let us reduce the problem in Eqs. (6.48)–(6.50) to one of convex programming with geometric constraints in ℓ_2 space. With this goal, we set $x_t = (x_{t,0},\ x_{t,1}\ ,x_{t,2}, \ldots) \in \ell_2$ for any $t \in H$ and consider the following convex sets defined in the space ℓ_2:

$$M_{t,\tau} = \{w = (x_0, \ldots, x_T) : (x_{t,\tau+1}, x_{t,\tau}, x_{t+1,\tau}) \in \text{ gph } F(\cdot, t, \tau)\}, \quad (t,\tau) \in H_1 \times N$$
$$M_0 = \{w = (x_0, \ldots, x_T) : x_{0,\tau} = \varphi_{0\tau}, \tau \in N\}.$$

Thus, setting $g(w) = \sum_{\tau\in N}\sum_{t\in H} g_{t,\tau}(x_{t,\tau})$, we can easily show that the problem in Eqs. (6.48)–(6.50) is equivalent to the following convex minimization problem in the space ℓ_2:

$$\inf \quad g(w), \ \text{ subject to } \ w \in P = \left(\bigcap_{(t,\tau)\in H\times N} M_{t,\tau} \right) \cap M_0. \tag{6.57}$$

In order to write the necessary and sufficient conditions of the problem in Eq. (6.57), first we must compute the cones $K^*_{M_{t,\tau}}(w)$, $K^*_{M_0}(w)$, $w \in P$. Obviously, $\overline{w} \in K_{M_{t,\tau}}(w)$ means that $w + \lambda\overline{w} \in M_{t,\tau}$ for sufficiently small $\lambda > 0$, or equivalently,

$$(x_{t,\tau+1} + \lambda\overline{x}_{t,\tau+1}, x_{t,\tau} + \lambda\overline{x}_{t,\tau}, x_{t+1,\tau} + \lambda\overline{x}_{t+1,\tau},) \in \text{gph } F_{t,\tau}.$$

Hence,

$$K_{M_{t,\tau}}(w) = \{\overline{w} = (\overline{x}_0, \ldots, \overline{x}_T) : (\overline{x}_{t,\tau+1}, \overline{x}_{t,\tau}, \overline{x}_{t+1,\tau}) \in K_{F_{t,\tau}}(x_{t,\tau+1}, x_{t,\tau}, x_{t+1,\tau})\}. \tag{6.58}$$

Then $w^* \in K^*_{M_{t,\tau}}(w)$ is equivalent to the condition

$$\langle \overline{w}, w^* \rangle = \sum_{i\in H} \sum_{j=0}^{\infty} \langle \overline{x}_{i,j}, x^*_{i,j} \rangle \geq 0, \quad \overline{w} \in K_{M_{t,\tau}}(w).$$

We notice that convergence of the series $\sum_{j=0}^{\infty}\langle \overline{x}_{i,j}, x^*_{i,j}\rangle$ provides that for all $\overline{w}\in K_{M_{t,\tau}}(w)$ and $w^*\in K^*_{M_{t,\tau}}(w)$, the inner product $\langle \overline{w}, w^*\rangle$ is defined, where the components $\overline{x}_{i,j}$ of the vector $\overline{w}$ (see Eq. (6.58)) are arbitrary.

The rest of the proof is the same as in Theorem 6.1. The difference lies in the fact that in this case, $K^*_{M_0}(w) = \{w^* = (x^*_0, \ldots, x^*_T) : x^*_{t,\tau} = 0, t \neq 0, \tau \in N\}$. ■

Remark 6.5. If instead of N in the problem in Eqs. (6.48)–(6.50) we take the set of integer numbers $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ by direct verification without loss of generality, it can be shown that Theorem 6.9 is valid in this case.

2 Necessary and Sufficient Conditions for the Discrete-Approximation Problem Associated with the Continuous Problem

In this section, we associate a discrete-approximation problem to the problem (CP) and formulate necessary and sufficient conditions of optimality for it. Using the grid function $x_{t,\tau} = x_{\delta h}(t,\tau)$ on a uniform grid on Q, we introduce the difference operators A, B defined on the two-point models [20]:

$$Ax(t+\delta,\tau) = \frac{x(t+\delta,\tau) - x(t,\tau)}{\delta}; \quad Bx(t,\tau+h) = \frac{x(t,\tau+h) - x(t,\tau)}{h},$$

$$t = t_0, t_0+\delta, \ldots, t_1-\delta, \quad \tau \in Z,$$

where δ, h are steps on the t-axis and τ-axis, respectively, and Z is a set of integers.

Then discretization analogy of the partial differential inclusion in Eq. (6.52) consists of the following:

$$Ax(t+\delta,\tau) + Bx(t,\tau+h) \in F(x(t,\tau),t,\tau), \quad t = t_0, t_0+\delta, \ldots, t_1-\delta, \quad \tau \in Z.$$

After some simplification of this inclusion due to the above-defined operators, we associate the following discrete-approximation problem to the problem in Eqs. (6.51)–(6.53):

$$\inf \quad I_{\delta h}(x(\cdot,\cdot)) = \sum_{\tau=-\infty}^{\tau=+\infty} \sum_{t=t_o,\ldots,t_1-\delta} \delta h g(x(t,\tau),t,\tau) + \sum_{\tau=-\infty}^{\tau=+\infty} h g_0(x(1,\tau),\tau)$$

$$\text{subject to } x(t+\delta,\tau) \in Q(x(t,\tau+h), x(t,\tau), t, \tau) \tag{6.59}$$

$$x(t_0,\tau) = \varphi(\tau),$$

$$t = t_0, t_0+\delta, \ldots, t_1-\delta; \quad \tau \in Z,$$

where

$$Q(x(t,\tau+h),x(t,\tau),t,\tau)=(1+\beta)x(t,\tau)-\beta x(x(t,\tau+h)+\delta F(x(t,\tau),t,\tau),\frac{\delta}{h}=\beta. \tag{6.60}$$

Obviously, $v\in Q(p,x,t,\tau)$ means that

$$v\in(1+\beta)x-\beta p+\delta F(x,t,\tau),$$

or, equivalently,

$$\frac{v-x}{\delta}+\frac{p-x}{h}\in F(x,t,\tau).$$

Thus, by Theorem 6.9 (see also Remark 6.5), for optimality of the solution $\{\tilde{x}(t,\tau)\}$, $t=t_0,\ldots,t_1$, $\tau\in Z$ in the problem in Eq. (6.59) it is necessary that there exist vectors $\{x^*(t,\tau)\}$, $\{\varphi^*(t,\tau)\}$ and a number $\lambda=\lambda_{\delta h}\in\{0,1\}$, not all zero, such that

$$\begin{aligned}(\varphi^*(t,\tau+h),x^*(t,\tau)-\varphi^*(t,\tau))\in Q^*(x^*(t+\delta,\tau),\tilde{x}(t,\tau+h),\tilde{x}(t,\tau),\tilde{x}(t+\delta,\tau),t,\tau)\\+\{0\}\times\{-\delta h\lambda\partial g(\tilde{x}(t,\tau),t,\tau)\},\end{aligned}$$

$$-x^*(t_1,\tau)\in\lambda h\partial g_0(\tilde{x}(t_1,\tau),\tau),\quad t=t_0,t_0+\delta,\ldots,t_1-\delta,\ \ \tau\in Z. \tag{6.61}$$

In Eq. (6.61), Q^* must be expressed in terms of F^*.

Theorem 6.10. Let a set-valued mapping $Q(\cdot,t,\tau)$ be a convex multivalued function defined by Eq. (6.60). Then $K_{Q(\cdot,t,\tau)}(p,x,v)$, $(p,x,v)\in\operatorname{gph} Q(\cdot,t,\tau)$ is a cone of tangent directions if and only if $K_{F(\cdot,t,\tau)}(x,(v-x)/\delta+(p-x)/h),(x,(v-x)/\delta+(p-x)/h)\in\operatorname{gph} F(\cdot,t,\tau)$ is a cone of tangent directions. Furthermore, the following inclusions are equivalent:

1. $(p^*,x^*)\in Q^*(v^*,(p,x,v),t,\tau)$,

2. $(p^*+x^*-v^*/\delta)\in F^*(v^*;(x,(v-x)/\delta+(p-x)/h),t,\tau)$, $v^*\in R^n$,

where $p^*=-\beta v^*$.

☐ First, let us justify the equivalency of the remembered cones of tangent directions. Assume that

$$\left(\overline{x},\frac{\overline{v}-\overline{x}}{\delta}+\frac{\overline{p}-\overline{x}}{h}\right)\in K_{F(\cdot,t,\tau)}\left(x,\frac{v-x}{\delta}+\frac{p-x}{h}\right);\quad\left(x,\frac{v-x}{\delta}+\frac{p-x}{h}\right)\in\operatorname{gph} F(\cdot,t,\tau). \tag{6.62}$$

By the definition of a cone of tangent directions (condition (2)), this means that for a sufficiently small $\lambda > 0$

$$\left(x, \frac{v-x}{\delta} + \frac{p-x}{h}\right) + \lambda\left(\overline{x}, \frac{\overline{v}-\overline{x}}{\delta} + \frac{\overline{p}-\overline{x}}{h}\right) \in \operatorname{gph} F(\cdot, t, \tau),$$

or

$$\frac{v-x}{\delta} + \lambda\frac{\overline{v}-\overline{x}}{\delta} + \frac{p-x}{h} + \lambda\frac{\overline{p}-\overline{x}}{h} \in F(x+\lambda\overline{x}, t, \tau).$$

Multiplying the left- and right-hand sides of this inclusion by δ and taking into account that $(\delta/h) = \beta$, we have

$$v + \lambda\overline{v} \in (1+\beta)(x+\lambda\overline{x}) - \beta(p+\lambda\overline{p}) + \lambda F(x+\lambda\overline{x}, t, \tau).$$

But it follows from Eq. (6.60) that the right-hand side of this relation is $Q(p+\lambda\overline{p}, x+\lambda\overline{x}, t, \tau)$, i.e.,

$$v + \lambda\overline{v} \in Q(p+\lambda\overline{p}, x+\lambda\overline{x}, t, \tau)$$

or

$$(p+\lambda\overline{p}, x+\lambda\overline{x}, v+\lambda\overline{v}) \in \operatorname{gph} Q(\cdot, \cdot, t, \tau).$$

In turn, this means that

$$(p, x, v) + \lambda(\overline{p}, \overline{x}, \overline{v}) \in \operatorname{gph} Q(\cdot, \cdot, t, \tau).$$

Consequently, by the definition of a cone of tangent directions, the last inclusion yields

$$(\overline{p}, \overline{x}, \overline{v}) \in K_{Q(\cdot, t, \tau)}(p, x, v), \quad (p, x, v) \in \operatorname{gph} Q(\cdot, t, \tau). \tag{6.63}$$

Conversely, it is also not hard to see that Eq. (6.63) implies Eq. (6.62), so Eqs. (6.62) and (6.63) are equivalent.

Moreover, now suppose that

$$(p^*, x^*) \in Q^*(v^*, (p, x, v), t, \tau),$$

or, equivalently,

$$\langle\overline{p}, p^*\rangle + \langle\overline{x}, x^*\rangle - \langle\overline{v}, v^*\rangle \geq 0, \quad (\overline{p}, \overline{x}, \overline{v}) \in K_{Q(., t, \tau)}(p, x, v). \tag{6.64}$$

Let us rewrite this inequality in the form

$$\langle\overline{x}, \psi_1^*\rangle - \left\langle\frac{\overline{v} - (1+\beta)\overline{x} + \beta\overline{p}}{\delta}, \psi^*\right\rangle \geq 0, \quad (\overline{p}, \overline{x}, \overline{v}) \in K_{Q(\cdot, t, \tau)}(p, x, v)$$

and determine ψ_1^*, ψ^*. Since δ is a positive number, this inequality is equivalent to

$$\langle \bar{x}, \delta\psi_1^* \rangle - \langle \bar{v} - (1+\beta)\bar{x} + \beta\bar{p}, \psi^* \rangle \geq 0, \quad (\bar{p}, \bar{x}, \bar{v}) \in K_{Q(\cdot,t,\tau)}(p, x, v). \tag{6.65}$$

After some simple transformations of Eq. (6.64) and comparing it with Eq. (6.63), we have

$$p^* = -\beta v^*, \quad \psi^* = v^*, \quad \psi_1^* = \frac{x^* + p^* - v^*}{\delta}.$$

Then it follows from the equivalence of Eqs. (6.62) and (6.63) that

$$\frac{x^* + p^* - v^*}{\delta} \in F^*\left(v^*; \left(x, \frac{v-x}{\delta} + \frac{p-x}{h}\right), t, \tau\right),$$

which ends the proof of the theorem. ■

In Theorem 6.11, sufficient conditions for the discrete-approximation problem in Eqs. (6.59) and (6.60) are formulated.

Theorem 6.11. Let a set-valued map $F(\cdot,t,\tau)$ be convex, and g and g_0 be proper functions convex with respect to x and continuous at the points of some feasible solution $\{x^0\ (t,\tau)\}$, $t = t_0,\ t_0 + \delta,\ \ldots,\ t_1 - \delta,\ t_1{:}\tau \in Z$ of the problem in Eqs. (6.59) and (6.60). Then for the optimality of the solution $\{\tilde{x}(t,\tau)\}$ in the discrete-approximation Cauchy problem in Eqs. (6.59) and (6.60), it is necessary that there exist a number $\lambda = \lambda_{\delta h} \in \{0,1\}$ and vectors $\{x^*(t,\tau)\}$, not all zero, satisfying

1. $-Bx^*(t+\delta,\tau) - Ax^*(t+\delta,\tau) \in F^*(x^*(t+\delta,\tau); (\tilde{x}(t,\tau), A\tilde{x}(t+\delta,\tau) + B\tilde{x}(t,\tau+h)), t, \tau) - \lambda\partial g(\tilde{x}(t,\tau), t, \tau)$,
2. $-x^*(t_1,\tau) \in \lambda\, \partial g_0(\tilde{x}(t_1,\tau),\tau), \quad t = t_0, t_0+\delta, \ldots, t_1 - \delta; \ \ \tau \in Z.$

In addition, if $\lambda = 1$, then these conditions are sufficient for the optimality of $\{\tilde{x}(t,\tau)\}$.

□ Let us return to the condition in Eq. (6.61) for problem (CP). Because of condition (1) of Theorem 6.10, the difference inclusion in Eq. (6.61) has the form $(p^*, x^*) \in Q^*\ (v^*,(p,x,v),t,\tau)$, and since this inclusion is equivalent to condition (2) of Theorem 6.10, the relation in Eq. (6.61) implies that

$$\frac{x^*(t,\tau) - x^*(t+\delta,\tau) + \varphi^*(t,\tau+h) - \varphi^*(t,\tau)}{\delta}$$
$$\in F^*(x^*(t+\delta,\tau), (\tilde{x}(t,\tau), A\tilde{x}(t+\delta,\tau) + B\tilde{x}(t,\tau+h)), t, \tau) - \lambda h\partial g(\tilde{x}(t,\tau), t, \tau),$$
$$t = t_0, t_0+\delta, \ldots, t_1 - \delta; \quad \tau \in Z. \tag{6.66}$$

It is not hard to verify the following representation of the quotient on the left-hand side of the inclusion in Eq. (6.66):

$$\frac{x^*(t,\tau) - x^*(t+\delta,\tau) + \varphi^*(t,\tau+h) - \varphi^*(t,\tau)}{\delta} = \frac{1}{\beta} B\varphi^*(t,\tau+h) - Ax^*(t+\delta,\tau),$$
$$t = t_0, t_0+\delta, \ldots, t_1-\delta; \quad \tau \in Z.$$

Now, remember that a LAM is a positive homogeneous mapping with respect to the first argument. Then dividing Eq. (6.66) and the second relation in Eq. (6.61) by h and denoting $(\varphi^*(t,\tau))/h$ and $(x^*(t,\tau))/h$ again by $\varphi^*(t,^\tau)$ and $x^*(t,\tau)$, respectively, we have

$$\frac{1}{\beta} B\varphi^*(t,\tau+h) - Ax^*(t+\delta,\tau)$$
$$\in F^*(x^*(t+\delta,\tau); (\tilde{x}(t,\tau), A\tilde{x}(t+\delta,\tau) + B\tilde{x}(t,\tau+h)), t,\tau) - \lambda\partial g(\tilde{x}(t,\tau),t,\tau), \tag{6.67}$$

$$-x^*(t_1,\tau) \in \lambda\partial g_0(\tilde{x}(t_1,\tau),\tau), \quad t = t_0, t_0+\delta, \ldots, t_1-\delta; \quad \tau \in Z. \tag{6.68}$$

Because of the condition $p^* = -\beta v^*$, we set $(1/\beta)\varphi^*(t,\tau+h) = -x^*(t+\delta,\tau)$, so

$$\frac{1}{\beta} B\varphi^*(t,\tau+h) = -Bx^*(t+\delta,\tau).$$

Then the relation in Eq. (6.66) has the form

$$-Bx^*(t+\delta,\tau) - Ax^*(t+\delta,\tau)$$
$$\in F^*(x^*(t+\delta,\tau); (\tilde{x}(t,\tau), A\tilde{x}(t+\delta,\tau) + B\tilde{x}(t,\tau+h)), t,\tau) - \lambda\partial g(\tilde{x}(t,\tau),t,\tau). \tag{6.69}$$

Now, taking into account Eqs. (6.68) and (6.69), we see that the proof of the theorem is completed. ■

Remark 6.6. Observe that the conditions of Theorems 6.9 and 6.11 are necessary for optimality in the corresponding nonconvexity cases. It must be pointed out, however, that only in the nonconvexity case we can use the definition of LAM. In Remark 6.4, we assume that the cone of tangent directions are local tents and the functions $g(\cdot,t,\tau)$ and $g_0(\cdot,\tau)$ admit CUAs.

3 Sufficient Conditions in the Cauchy Problem for First-Order Partial Differential Inclusions

Using the results in Subsection 2, we formulate sufficient conditions for optimality for the two-parameter Cauchy problem in Eqs. (6.51)–(6.53). The formulation of sufficient conditions for continuous problems is based on the passage to the formal limit in conditions (1) and (2) of Theorem 6.11 as discrete steps δ, h approach zero. In this section, by the nonconvexity of a set-valued mapping, we have in mind that for such mappings the Hamiltonian function satisfies the inequality of the definition given in Remark 6.4.

Theorem 6.12. For the optimality of the solution $\tilde{x}(\cdot,\cdot)$ in the nonconvex problem (CP), it is sufficient that there exists an absolutely continuous function $x^*(\cdot,\cdot)$ with summable first-order partial derivatives such that conditions (a)−(e), below, hold.

a. $-\frac{\partial x^*(t,\tau)}{\partial t} - \frac{\partial x^*(t,\tau)}{\partial \tau} + x^*(t,\tau) \in F^*\left(x^*(t,\tau); (\tilde{x}(t,\tau), (\partial\tilde{x}(t,\tau))/(\partial t) + (\partial\tilde{x}(t,\tau))/(\partial t)), t, \tau\right)$,
b. $g(x,t,\tau) - g(\tilde{x}(t,\tau),t,\tau) \geq \langle x^*(t,\tau), x-\tilde{x}(t,\tau)\rangle \quad \forall x \in \mathrm{R}^n$ a.e. $(t,\tau)\in Q$,
c. $g_0(x,\tau) - g_0(\tilde{x}(t_1,\tau),\tau) \geq \langle x^*(t_1,\tau), x-\tilde{x}(t_1,\tau)\rangle \quad \forall x \in \mathrm{R}^n$ a.e. $\tau \in \mathbb{R}$,
d. $(\partial\tilde{x}(t,\tau))/(\partial t) + (\partial\tilde{x}(t,\tau))/(\partial t) \in F(\tilde{x}(t,\tau), x^*(t,\tau), t, \tau)$ a.e $(t,\tau)\in Q$,
e. $\lim\limits_{\tau\to+\infty} x^*(t,\tau) = \lim\limits_{\tau\to-\infty} x^*(t,\tau) = 0$ uniformly with respect to $t \in [t_0, t_1]$.

☐ It is easy to observe that the definition of LAM described in Remark 6.4 for all feasible solutions $x(\cdot,\cdot)$ implies that

$$\begin{aligned} &H_F(x(t,\tau),x^*(t,\tau),t,\tau) - H_F(\tilde{x}(t,\tau),x^*(t,\tau),t,\tau) \\ &\quad \leq \left\langle -\frac{\partial x^*(t,\tau)}{\partial t} - \frac{\partial x^*(t,\tau)}{\partial \tau}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle + \langle x^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\rangle. \end{aligned}$$

Using the definition of Hamiltonian function, and condition (d), we rewrite the last relation in the form

$$\begin{aligned} &\left\langle \frac{\partial(x(t,\tau)-\tilde{x}(t,\tau))}{\partial t}, x^*(t,\tau)\right\rangle + \left\langle \frac{\partial(x(t,\tau)-\tilde{x}(t,\tau))}{\partial \tau}, x^*(t,\tau)\right\rangle \\ &\quad + \left\langle \frac{\partial x^*(t,\tau)}{\partial t}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle + \left\langle \frac{\partial x^*(t,\tau)}{\partial \tau}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle \\ &\quad \leq \left\langle x^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\right\rangle. \end{aligned}$$

Therefore, we have

$$\frac{\partial}{\partial t}\langle x^*(t,\tau),x(t,\tau)-\tilde{x}(t,\tau)\rangle + \frac{\partial}{\partial \tau}\langle x^*(t,\tau),x(t,\tau)-\tilde{x}(t,\tau)\rangle \leq \langle x^*(t,\tau),x(t,\tau)-\tilde{x}(t,\tau)\rangle.$$

Thus, taking into account condition (b) for all feasible solutions,

$$\begin{aligned} g(x(t,\tau),t,\tau) - g(\tilde{x}(t,\tau),t,\tau) &\geq \left\langle \frac{\partial}{\partial t}(x(t,\tau)-\tilde{x}(t,\tau)), x^*(t,\tau)\right\rangle \\ &\quad + \left\langle \frac{\partial}{\partial \tau}(x(t,\tau)-\tilde{x}(t,\tau)), x^*(t,\tau)\right\rangle. \end{aligned}$$

Now integrating the last relation over the domain Q, and the inequality (c) of Theorem 6.12 over the interval $(-\infty, +\infty)$ and then adding them, we get

$$\int_{-\infty}^{+\infty}\int_{t_0}^{t_1}[g(x(t,\tau),t,\tau)-g(\tilde{x}(t,\tau),t,\tau)]\mathrm{d}t\,\mathrm{d}\tau+\int_{-\infty}^{+\infty}[g_0(x(t_1,\tau),\tau)-g_0(\tilde{x}(t_1,\tau),\tau)]\mathrm{d}\tau$$
$$\geq\int_{-\infty}^{+\infty}\int_{t_0}^{t_1}\frac{\partial}{\partial t}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau+\int_{-\infty}^{+\infty}\int_{t_0}^{t_1}\frac{\partial}{\partial \tau}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau \tag{6.70}$$
$$-\int_{-\infty}^{+\infty}\langle x^*(t_1,\tau),x(t_1,\tau)-\tilde{x}(t_1,\tau)\rangle\mathrm{d}\tau.$$

Now, let us denote

$$\Phi_{+\infty}=\int_{0}^{+\infty}\int_{t_0}^{t_1}\frac{\partial}{\partial t}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau+\int_{0}^{+\infty}\int_{t_0}^{t_1}\frac{\partial}{\partial \tau}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau$$
$$-\int_{0}^{+\infty}\langle x^*(t_1,\tau),x(t_1,\tau)-\tilde{x}(t_1,\tau)\rangle\mathrm{d}\tau$$

and

$$\Phi_{-\infty}=\int_{-\infty}^{0}\int_{t_0}^{t_1}\frac{\partial}{\partial t}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau+\int_{-\infty}^{0}\int_{t_0}^{t_1}\frac{\partial}{\partial \tau}\langle x(t,\tau)-\tilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau$$
$$-\int_{-\infty}^{0}\langle x^*(t_1,\tau),x(t_1,\tau)-\tilde{x}(t_1,\tau)\rangle\mathrm{d}\tau.$$

By elementary computations of improper integrals of the first type, we have

$$\Phi_{+\infty}=\int_{0}^{+\infty}[\langle x(t_1,\tau)-\tilde{x}(t_1,\tau),x^*(t_1,\tau)\rangle-\langle x(t_0,\tau)-\tilde{x}(t_0,\tau),x^*(t_0,\tau)\rangle]\mathrm{d}\tau$$
$$+\int_{t_0}^{t_1}\lim_{b_1\to+\infty}[\langle x^*(t,b_1),x(t,b_1)-\tilde{x}(t,b_1)\rangle-\langle x^*(t,0),x(t,0)-\tilde{x}(t,0)\rangle]\mathrm{d}t$$
$$-\int_{0}^{+\infty}\langle x^*(t_1,\tau),x(t_1,\tau)-\tilde{x}(t_1,\tau)\rangle\mathrm{d}\tau=\int_{0}^{+\infty}\langle\tilde{x}(t_0,\tau)-x(t_0,\tau),x^*(t_0,\tau)\rangle\mathrm{d}\tau$$
$$+\int_{t_0}^{t_1}\lim_{b_1\to+\infty}\langle x^*(t,b_1),x(t,b_1)-\tilde{x}(t,b_1)\rangle\mathrm{d}t-\int_{t_0}^{t_1}\langle x^*(t,0),x(t,0)-\tilde{x}(t,0)\rangle\mathrm{d}t.$$

By analogy,

$$\Phi_{-\overline{\infty}} \int_{-\infty}^{0} [\langle x(t_1,\tau)-\tilde{x}(t_1,\tau), x^*(t_1,\tau)\rangle - \langle x(t_0,\tau)-\tilde{x}(t_0,\tau), x^*(t_0,\tau)\rangle]\mathrm{d}\tau$$
$$+\int_{t_0}^{t_1} \lim_{b_2\to-\infty} [\langle x^*(t,0), x(t,0)-\tilde{x}(t,0)\rangle - \langle x^*(t,b_2), x(t,b_2)-\tilde{x}(t,b_2)\rangle]\mathrm{d}t$$
$$-\int_{-\infty}^{0} \langle x^*(t_1,\tau), x(t_1,\tau)-\tilde{x}(t_1,\tau)\rangle \mathrm{d}\tau = \int_{-\infty}^{0} \langle \tilde{x}(t_0,\tau)-x(t_0,\tau), x^*(t_0,\tau)\rangle \mathrm{d}\tau$$
$$-\int_{t_0}^{t_1} \lim_{b_2\to-\infty} \langle x^*(t,b_2), x(t,b_2)-\tilde{x}(t,b_2)\rangle \mathrm{d}t + \int_{t_0}^{t_1} \langle x^*(t,0), x(t,0)-\tilde{x}(t,0)\rangle \mathrm{d}t.$$

Since $\tilde{x}(t_0,\tau)=x(t_0,\tau)=\varphi(\tau)$ (see Eq. (6.53)), it is clear that

$$\Phi_{+\infty}+\Phi_{-\infty} = \int_{t_0}^{t_1} \lim_{b_1\to+\infty} \langle x^*(t,b_1), x(t,b_1)-\tilde{x}(t,b_1)\rangle \mathrm{d}t - \int_{t_0}^{t_1} \lim_{b_2\to-\infty} \langle x^*(t,b_2), x(t,b_2)-\tilde{x}(t,b_2)\rangle \mathrm{d}t. \tag{6.71}$$

Thus, taking into account condition (e) in Eq. (6.71), we have $\Phi_{+\infty}+\Phi_{-\infty}=0$. Note that the right-hand side of the inequality in Eq. (6.70) is a sum $\Phi_{+\infty}+\Phi_{-\infty}$, which is equal to zero. Consequently, $I(x(t,\tau))\ge I(\tilde{x}(t,\tau))$ for all feasible solutions $x(\cdot,\cdot)\in Q$. The proof of the theorem is completed. ■

Remark 6.7. Obviously, for the convex problem (CP), conditions (a)–(c) of Theorem 6.12 consist of the following:

i. $-(\partial x^*(t,\tau))/(\partial t)-(\partial x^*(t,\tau))/(\partial\tau)\in F^*(x^*(t,\tau);(\tilde{x}(t,\tau),(\partial\tilde{x}(t,\tau))/(\partial t)+(\partial\tilde{x}(t,\tau))/(\partial\tau)),t,\tau) - \partial g(\tilde{x}(t,\tau),t,\tau)$ a.e. $(t,\tau)\in Q$,
ii. $-x^*(t_1,\tau)\in\partial g_0(\tilde{x}(t_1,\tau),\tau)$,
iii. $(\partial\tilde{x}(t,\tau))/(\partial t)+(\partial\tilde{x}(t,\tau))/(\partial\tau)\in F(\tilde{x}(t,\tau),x^*(t,\tau),t,\tau)$ a.e. $(t,\tau)\in Q$.

Note that sufficient conditions for a convex problem can be formulated by setting $\lambda=1$ and then by passing to the formal limit in Eq. (6.69) as $\delta, h\to 0$.

If $F(\cdot,t,\tau)$ is a closed multivalued mapping then results (i)–(iii) of Remark 6.7 can be rewritten more symmetrically in terms of the Hamiltonian function.

Corollary 6.1. In addition to the assumptions of Remark 6.7, let $F(\cdot,t,\tau)$ be a closed multivalued mapping, which of course is not the case for graph F. Then conditions (i) and (iii) can be rewritten as follows:

1. $-\frac{\partial x^*(t,\tau)}{\partial t}-\frac{\partial x^*(t,\tau)}{\partial\tau}\in\partial_x H_F(\tilde{x}(t,\tau),x^*(t,\tau),t,\tau))-\partial g(\tilde{x}(t,\tau),t,\tau)$,
2. $\frac{\partial\tilde{x}(t,\tau)}{\partial t}+\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\in\partial_{v^*}H_F(\tilde{x}(t,\tau),x^*(t,\tau),t,\tau)$ a.e. $(t,\tau)\in Q$.

☐ The corollary follows immediately from Theorem 2.1 and Lemma 2.6. ■

Now we consider the multidimensional Cauchy problem in Eqs. (6.54)–(6.56). By analogy, conditions (a′)–(e′) are formulated, which required that the limit of the conjugate variable be equal to zero uniformly with respect to $t \in [t_0, t_1]$ and τ_i as $\tau_k \to \pm\infty$ $(i \neq k)$.

Theorem 6.13. For the optimality of the solution $\tilde{x}(t,\tau)$ in the nonconvex problem (GCP) among all admissible solutions, it is sufficient that there exists an absolutely continuous function $x^*(t,\tau)$ with summable first partial derivatives such that conditions, (a′)–(e′), below, hold:

a′. $-(\partial x^*(t,\tau))/(\partial t) - \sum_{k=1}^{n}(\partial x^*(t,\tau))/(\partial \tau_k) + x^*(t,\tau) \in G^*\left(x^*(t,\tau); (\tilde{x}(t,\tau), (\partial \tilde{x}(t,\tau))/(\partial t) + \sum_{k=1}^{n}(\partial \tilde{x}(t,\tau))/(\partial \tau_k)), t, \tau\right)$,
b′. $f(x,t,\tau) - f(\tilde{x}(t,\tau),t,\tau) \geq \langle x^*(t,\tau), x - \tilde{x}(t,\tau)\rangle \quad \forall x \in \mathbb{R}^n$,
c′. $f_0(x,\tau) - f_0(\tilde{x}(t_1,\tau),\tau) \geq \langle x^*(t_1,\tau), x - \tilde{x}(t_1,\tau)\rangle \quad \forall x \in \mathbb{R}^n$,
d′. $(\partial \tilde{x}(t,\tau))/(\partial t) + \sum_{k=1}^{n}(\partial \tilde{x}(t,\tau))/(\partial \tau_k) \in G(\tilde{x}(t,\tau), x^*(t,\tau), t, \tau)$ a.e. $(t,\tau) \in [t_0,t_1] \times \mathbb{R}^n$,
e′. $\lim\limits_{\tau_k \to +\infty} x^*(t,\tau) = \lim\limits_{\tau_k \to -\infty} x^*(t,\tau) = 0$ uniformly with respect to $t \in [t_0, t_1]$ and

$$\tau_i (i \neq k), \quad k = 1, 2, \ldots, n.$$

☐ Proceeding as in the proof of Theorem 6.12 by using conditions (a′) and (b′), we get a corresponding inequality in terms of the Hamiltonian function:

$$\begin{aligned} &H_G(x(t,\tau), x^*(t,\tau), t, \tau) - H_G(\tilde{x}(t,\tau), x^*(t,\tau), t, \tau) \\ &\quad \leq \left\langle -\frac{\partial x^*(t,\tau)}{\partial t} - \sum_{k=1}^{n} \frac{\partial x^*(t,\tau)}{\partial \tau_k}, x(t,\tau) - \tilde{x}(t,\tau) \right\rangle + \langle x^*(t,\tau), x(t,\tau) - \tilde{x}(t,\tau)\rangle. \end{aligned}$$

Thus, because of the definition of the Hamiltonian function, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle + \sum_{k=1}^{n} \frac{\partial}{\partial \tau_k}\langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \\ &\quad \leq \langle x^*(t,\tau), x(t,\tau) - x(t,\tau)\rangle. \end{aligned} \tag{6.72}$$

Now, integrating the inequalities (b′), Eq. (6.72), and (c′) over $[t_0, t_1] \times \mathbb{R}^n$ and $\mathbb{R}^n$, respectively, we deduce that

$$\begin{aligned} &\int_{R^n}\int_{t_0}^{t_1} [f(x(t,\tau),t,\tau) - f(\tilde{x}(t,\tau),t,\tau]\mathrm{d}t\,\mathrm{d}\tau + \int_{R^n} [f_0(x(t_1,\tau),\tau) - f_0(\tilde{x}(t_1,\tau),\tau]\mathrm{d}\tau \\ &\quad \geq \int_{R^n}\int_{t_0}^{t_1} \frac{\partial}{\partial t}\langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau + \sum_{k=1}^{n}\int_{R^n}\int_{t_0}^{t_1} \frac{\partial}{\partial \tau_k}\langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}\tau\,\mathrm{d}t \\ &\quad - \int_{R^n} \langle x(t_1,\tau) - \tilde{x}(t_1,\tau), x^*(t_1,\tau)\rangle \mathrm{d}\tau. \end{aligned} \tag{6.73}$$

Let us simplify the right-hand side of this inequality. Using the fact that $\tilde{x}(t_0, \tau) = x(t_0, \tau) = \alpha(\tau)$ for all feasible solutions, we get

$$\begin{aligned}
&\int\limits_{R^n} \int\limits_{t_0}^{t_1} \frac{\partial}{\partial t} \langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}\tau - \int\limits_{R^n} \langle x(t_1,\tau) - \tilde{x}(t_1,\tau), x^*(t_1,\tau)\rangle \mathrm{d}\tau \\
&\quad = \int\limits_{R^n} \langle x(t_1,\tau) - \tilde{x}(t_1,\tau), x^*(t_1,\tau)\rangle \mathrm{d}\tau - \int\limits_{R^n} \langle x(t_0,\tau) - \tilde{x}(t_0,\tau), x^*(t_0,\tau)\rangle \mathrm{d}\tau \\
&\qquad - \int\limits_{R^n} \langle x(t_1,\tau) - \tilde{x}(t_1,\tau), x^*(t_1,\tau)\rangle \mathrm{d}\tau = 0.
\end{aligned}$$

So the inequality in Eq. (6.73) can be written as follows:

$$J(x(t,\tau)) - J(\tilde{x}(t,\tau)) \geq \sum_{k=1}^{n} \int\limits_{R^n} \int\limits_{t_0}^{t_1} \frac{\partial}{\partial \tau_k} \langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}\tau \, \mathrm{d}t. \tag{6.74}$$

Moreover, obviously

$$\begin{aligned}
&\sum_{k=1}^{n} \int\limits_{R^n} \int\limits_{t_0}^{t_1} \frac{\partial}{\partial \tau_k} \langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}\tau \, \mathrm{d}t \\
&= \int\limits_{t_0}^{t_1} \left[\int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} \sum_{k=1}^{n} \frac{\partial}{\partial \tau_k} \langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}\tau \right] \mathrm{d}t.
\end{aligned}$$

Let us transform the multiple improper integral in the square brackets of this equality. Then taking into account that in the present case the value obtained is independent of the order of integration, we get the following chain of equalities:

$$\begin{aligned}
&\int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} \sum_{k=1}^{n} \frac{\partial}{\partial \tau_k} \langle x(t,\tau_1,\cdots,\tau_n) - \tilde{x}(t,\tau_1,\cdots,\tau_n), x^*(t,\tau_1,\cdots,\tau_n)\rangle \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \\
&= \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} \frac{\partial}{\partial \tau_1} \langle x(t,\tau_1,\cdots,\tau_n) - \tilde{x}(t,\tau_1,\cdots,\tau_n), x^*(t,\tau_1,\cdots,\tau_n)\rangle \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \\
&\quad + \cdots + \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} \frac{\partial}{\partial \tau_n} \langle x(t,\tau_1,\cdots,\tau_n) - \tilde{x}(t,\tau_1,\cdots,\tau_n), x^*(t,\tau_1,\cdots,\tau_n)\rangle \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \\
&= \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} \left[\lim_{\substack{a_1 \to -\infty \\ b_1 \to +\infty}} \int\limits_{a_1}^{b_1} \frac{\partial}{\partial \tau_1} \langle x(t,\tau_1,\cdots,\tau_n) - \tilde{x}(t,\tau_1,\cdots,\tau_n), x^*(t,\tau_1,\cdots,\tau_n)\rangle \mathrm{d}\tau_1 \right] \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n
\end{aligned}$$

$$+\cdots+\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\left[\lim_{\substack{a_n\to-\infty\\ b_n\to+\infty}}\int\limits_{a_n}^{b_n}\frac{\partial}{\partial\tau_n}\langle x(t,\tau_1,\cdots,\tau_n)-\tilde{x}(t,\tau_1\cdots,\tau_n),x^*(t,\tau_1,\cdots,\tau_n)\rangle\mathrm{d}\tau_n\right]\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{n-1}$$

$$=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\lim_{\substack{a_1\to-\infty\\ b_1\to+\infty}}[\langle x(t,b_1,\tau_2,\cdots,\tau_n)-\tilde{x}(t,b_1,\cdots,\tau_n),x^*(t,b_1,\tau_2,\cdots,\tau_n)\rangle$$

$$-\langle x(t,a_1,\tau_2,\cdots,\tau_n)-\tilde{x}(t,a_1,\tau_2,\cdots,\tau_n),x^*(t,a_1,\tau_2,\cdots,\tau_n)\rangle]\mathrm{d}\tau_2\cdots\mathrm{d}\tau_n$$

$$+\cdots+\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\lim_{\substack{a_n\to-\infty\\ b_n\to+\infty}}[\langle x(t,\tau_1,\tau_2,\cdots,\tau_{n-1},b_n)-\tilde{x}(t,\tau_1,\cdots,\tau_{n-1},b_n),x^*(t,\tau_1,\tau_2\cdots,\tau_{n-1},b_n)\rangle$$

$$-\langle x(t,\tau_1,\tau_2,\cdots,\tau_{n-1},a_n)-\tilde{x}(t,\tau_1,\cdots,\tau_{n-1},a_n),x^*(t,\tau_1,\tau_2,\cdots,\tau_{n-1},\alpha_n)\rangle]\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{n-1}.$$

Then, since the feasible solutions are bounded, according to condition (e′) of the theorem, the right-hand side of this relation is zero, so Eq. (6.74) implies that $J(x(t,\tau))\geq J(\tilde{x}(t,\tau))$. ■

Example 6.2. We now consider the convex problem:

$$\inf \quad J(x(t,\tau)),$$

$$\text{subject to}\quad \frac{\partial x(t,\tau)}{\partial t}+\frac{\partial x(t,\tau)}{\partial\tau}=Ax(t,\tau)+Bu(t,\tau),\quad u(t,\tau)\in U \tag{6.75}$$

$$x(t_0,\tau)=\varphi(\tau),$$

where A is an $n\times n$ matrix, B is a rectangular $n\times r$ matrix, $U\subset\mathbb{R}^r$ is a convex closed set, for simplicity $g_0\equiv 0$, and g is a continuously differentiable function in x. It is required to find a controlling parameter $\tilde{u}(t,\tau)\in U$ such that the solution $\tilde{x}(t,\tau)$ corresponding to it minimizes $I(x(\cdot,\cdot))$. In this case, $F(x,t,\tau)=Ax+BU$. By elementary computations, we find that

$$H_F(x,v^*)=\langle x,A^*v^*\rangle-W_U(B^*v^*),$$

where W_U is a support function of the set U.

Then by Theorem 2.1, we have

$$F^*(v^*;(\tilde{x},\tilde{v}))=\begin{cases}A^*v^*, & \langle\tilde{u},B^*v^*\rangle=M_U(B^*v^*),\\ \varnothing, & \langle\tilde{u},B^*v^*\rangle\neq M_U(B^*v^*),\end{cases}$$

where $\tilde{v}=A\tilde{x}+B\tilde{u},\ \ \tilde{u}\in U$ and here, as before, an asterisk denotes the adjoint (transpose) matrix.

Then, using Remark 6.7, we get the relations

$$
\begin{aligned}
&-\frac{\partial x^*(t,\tau)}{\partial t}-\frac{\partial x^*(t,\tau)}{\partial \tau}=A^*x^*(t,\tau)-g'(\tilde{x}(t,\tau),t,\tau),\\
&x^*(t_1,\tau)=0,\\
&\langle B\tilde{u}(t,\tau),x^*(t,\tau)\rangle=\sup_{u\in U}\langle Bu,x^*(t,\tau)\rangle.
\end{aligned}
\tag{6.76}
$$

Thus, we have obtained the result as in Theorem 6.14.

Theorem 6.14. The solution $\tilde{x}(\cdot,\cdot)$ corresponding to the control $\tilde{u}(\cdot,\cdot)$ is optimal in the problem in Eq. (6.75) if there exists a function $x^*(\cdot,\cdot)$ satisfying the conditions in Eq. (6.76) and (e) of Theorem 6.12.

These results can be divided into two parts. In the first part, a new optimization problem for discrete and discrete-approximation inclusions associated with the continuous problem (CD) for first-order partial differential inclusions are investigated. It appears that for the stated Cauchy problem, a set of feasible solutions must be taken in the Hilbert space ℓ_2. An important role of Theorem 6.10 on the LAMs to discrete and discrete-approximation problems is that this theorem provides the formulation of sufficient conditions for the discrete analogy of a continuous problem.

Further, by passing to the formal limit in conditions (1) and (2) of Theorem 6.11, sufficient conditions for two-parameter and generalized optimization problems of Cauchy types (GCP) are proved.

We note that the given method of optimization here can be useful for other types of discrete and partial differential inclusions, namely, properly defined Cauchy problems for hyperbolic and parabolic differential inclusions.

6.4 Optimal Control of Darboux-Type Discrete-Approximation and Differential Inclusions with Set-Valued Boundary Conditions and Duality

This section is devoted to an optimal control problem given by Darboux-type hyperbolic discrete (P_D) and differential inclusions and ordinary discrete inclusions (P_C). An approach concerning necessary and sufficient conditions for optimality is proposed. Next we will present some applications of the results obtained to problems with single-valued mappings and set-valued boundary conditions. The formulation of these conditions is based on LAMs.

First- and second-order necessary conditions for differential inclusions are well known [48,49]. In particular, Mahmudov [162,165] established the duality theorems for Goursat–Darboux-type problems with state constraints.

In Subsection 2, we use discrete approximations of partial and ordinary derivatives and grid functions on a uniform grid to derive necessary and sufficient conditions for a discrete-approximation problem.

In Subsection 3, we are able to use the results of Subsection 2 to get sufficient conditions for a Darboux DFI. The adjoint inclusions that arise are stated in the Euler–Lagrange and Hamiltonian forms. This form automatically implies the Weierstrass–Pontryagin maximum condition. A distinctive property of the investigated problem is that it can involve discrete and differential equations given by single-valued mappings and controlling parameters included in boundary conditions. It should be expected that the optimality conditions for hyperbolic and ordinary DFI involved both adjoint hyperbolic and ordinary DFI represented by the LAM, respectively. But it appears that the optimality conditions for problem (P_C) are not simply a combination of the optimality conditions for a partial DFI and an ordinary DFI (wherein the ordinary adjoint DFI takes the place of the LAM).

Finally, note that the more general problem, where the right-hand side of the hyperbolic DFI depends not only on the required function but also on the first partial derivatives of this function, is considered separately. One can draw the important conclusion that, in the more general case, despite the existence of some auxiliary adjoint variables in the adjoint DSI or DFI, the main adjoint variable $u^*(\cdot,\cdot)$ is essential.

Let us now consider the following discrete optimization problem:

$$\inf \sum_{(t,x)\in H_0\times L_0} g_{t,x}(u_{t,x}) \tag{6.77}$$

$$(P_D)\ \text{ subject to }\ u_{t,x}\in F_{t,x}(u_{t,x-1}, u_{t-1,x}, u_{t-1,x-1}), \quad (t,x)\in H_1\times L_1, \tag{6.78}$$

$$\begin{aligned} &u_{t,0}\in F_{t,0}(u_{t-1,0}), \quad t\in H_1,\\ &u_{0,x}\in F_{0,x}(u_{0,x-1}), \quad x\in L_1. \end{aligned} \tag{6.79}$$

$$H_i=\{t: t=i,\ldots,T\}, \quad L_i=\{x: x=i,\ldots,L),\ \ i=0,1,$$

where $u_{t,x}\in\mathbb{R}^n$, $g_{t,x}$ are real-valued functions, $g_{t,x}:\mathbb{R}^n\to\mathbb{R}\cup\{\pm\infty\}$, $F_{t,x}$ are multivalued mappings, $F_{t,x}:\mathbb{R}^{3n}\to P(\mathbb{R}^n)$, and T and L are fixed natural numbers. The conditions in Eq. (6.79) are simply set-valued boundary conditions. A sequence $\{u_{t,x}\}_{H_0\times L_0}=\{u_{t,x}:(t,x)\in H_0\times L_0\}$ is called a feasible solution for the stated problem (P_D). Obviously, this sequence consists of $(T+1)(L+1)$ points of the space $\overline{\mathrm{R}}^n$.

In Subsection 3, we will study the following problem (P_C), for Darboux-type differential inclusions:

$$\inf \quad J[u(\cdot,\cdot)]:=\iint_Q g(u(t,x),t,x)\mathrm{d}t\,\mathrm{d}x+\int_0^1 g_1(u(1,x),x)\mathrm{d}t\,\mathrm{d}x+\int_0^1 g_2(u(t,1),t)\mathrm{d}t \tag{6.80}$$

$$\text{subject to } u_{tx}(t,x)\in F(u(t,x),t,x),(t,x)\in Q=[0,1]\times[0,1], \tag{6.81}$$

$$u_t(t,0)\in F_1(u(t,0),t),\ \ t\in[0,1],\quad u_x(0,x)\in F_2(x(0,x),x),\ \ x\in[0,1],\ \ u(0,0)=u_0. \tag{6.82}$$

Here, $F(\cdot,t,x):\mathbb{R}^n\to P(\mathbb{R}^n)$, $F_1(\cdot,t):\mathbb{R}^n\to P(\mathbb{R}^n)$, and $F_2(\cdot,x):\mathbb{R}^n\to P(\mathbb{R}^n)$, are multivalued functions for all fixed $(t,x)\in Q$, $t\in[0,1]$, and $x\in[0,1]$, respectively, and $g(\cdot,t,x)$, $g_1(\cdot,x)$, and $g_2(\cdot,t)$ are continuous functions from $\mathbb{R}^n$ into $\mathbb{R}^n$. The problem is to find a solution $\tilde{u}(t,x)$ of the boundary value problem (P_C) that minimizes $J[u(\cdot,\cdot)]$.

Here, an admissible solution is understood to be an absolutely continuous function defined on Q with an integrable mixed derivative $u_{tx}(\cdot,\cdot)$ satisfying Eq. (6.81) almost everywhere on Q and the boundary conditions in Eq. (6.82) on [0,1]. Note that a function $u(\cdot,\cdot)$ is said to be absolutely continuous on Q if there exist f: $Q\to\mathbb{R}^n$, h:$[0,1]\to\mathbb{R}^n$, z:$[0,1]\to\mathbb{R}^n$ such that

$$u(t,x)=\int_0^t\int_0^x f(\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta+\int_0^t h(\xi)\mathrm{d}\xi+\int_0^x z(\eta)\mathrm{d}\eta+u(0,0),\quad (t,x)\in Q.$$

It is known that the space B of absolutely continuous functions u:$Q\to\mathbb{R}^n$ is a Banach space endowed with the norm

$$\|u(\cdot,\cdot)\|_B=\iint_Q \|u_{tx}(t,x)\|\mathrm{d}t\,\mathrm{d}x+\int_0^1\|u_t(t,0)\|\mathrm{d}t+\int_0^1\|u_x(0,x)\|\mathrm{d}x+\|u(0,0)\|.$$

1 Necessary and Sufficient Conditions for the Discrete Inclusion Problem (P_D)

First, we consider a convex problem (P_D). In what follows, we say that for a convex problem (P_D), the regularity condition is satisfied if there are $u^0_{t,x}\in\mathbb{R}^n$, $(t,x)\in H_0\times L_0$ such that either

a. $(u^0_{t,x-1},u^0_{t-1,x},u^0_{t-1,x-1},u^0_{t,x})\in$ ri gph $F_{t,x}$,$(u^0_{t-1,0},u^0_{t,0})\in$ri gph $F_{t,0}$,$(u^0_{0,x-1},u^0_{0,x})\in$ ri gph $F_{0,x}$$(t,x)\in H_1\times L_1$

or

b. $(u^0_{t,x-1},u^0_{t-1,x},u^0_{t-1,x-1},u^0_{t,x})\in$ int gph $F_{t,x}$,$(u^0_{t-1,0},u^0_{t,0})\in$ int gph $F_{t,0}$,

$(u^0_{0,x-1},u^0_{0,x})\in$int gph $F_{0,x}$; $(t,x)\in H_1\times L_1$ (with the possible exception of one fixed pair (t_0,x_0)), and the $g_{t,x}$ are continuous at $u^0_{t,x}$.

Theorem 6.15. Suppose $F_{t,x}$, $(t,x) \neq (0,0)$ are convex multivalued mappings, and $g_{t,x}$ are convex proper functions, continuous at the points of some feasible solution $\{u^0_{t,x}\}_{H_0 \times L_0}$. Then for the optimality of the solution $\{\tilde{u}_{t,x}\}_{H_0 \times L_0}$ in the problem (P_D), it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $\{u^*_{t,x}\}, \{\varphi^*_{t,x}\}, \{\eta^*_{t,x}\}$ $(u^*_{0,0} = \eta^*_{T+1,L} = \varphi^*_{T,L+1} = 0)$, $(t,x) \in H_0 \times L_0$, not all zero, such that

1. $(\varphi^*_{t,x}, \eta^*_{t,x}, u^*_{t-1,x-1}) \in F^*_{t,x}(u^*_{t,x}, (\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1}, \tilde{u}_{t,x}))$ $+ (0,0) \times \{\varphi^*_{t-1,x} + \eta^*_{t,x-1} - \lambda \partial g_{t-1,x-1}(\tilde{u}_{t-1,x-1})\}$,
2. $\eta^*_{t,0} \in F^*_{t,0}(u^*_{t,0}, (\tilde{u}_{t-1,0}, \tilde{u}_{t,0}))$, $\quad \varphi^*_{0,x} \in F^*_{0,x}$, $(u^*_{0,x}, (\tilde{u}_{0,x-1}, \tilde{u}_{0,x}))$, $(t,x) \in H_1 \times L_1$,
3. $\varphi^*_{T,x+1} - u^*_{T,x} \in \lambda \partial g_{T,x}(\tilde{u}_{T,x})$, $x \in L_0$, $\quad \eta^*_{t+1,L} - u^*_{t,L} \in \lambda \partial g_{t,L}(\tilde{u}_{t,L})$, $t \in H_0$.

Under regularity conditions, (1)−(3) are also sufficient for the optimality of the solution $\{\tilde{u}_{t,x}\}_{H_0 \times L_0}$.

□ Obviously, the problem (P_D) can be reduced to a problem with geometric constraints in a finite-dimensional Euclidean space. With this aim, we form an $m = n(L+1)$ dimensional vector $u_t = (u_{t,0}, u_{t,1}, \ldots, u_{t,L}) \in \mathbb{R}^m$. Let us introduce $w = (u_0, u_1, \ldots, u_T) \in \mathbb{R}^{m(T+1)}$ and define in the space $\mathbb{R}^{m(T+1)}$ the following convex sets:

$$M_{t,x} = \{w = (u_0, \ldots, u_t) : (u_{t,x-1}, u_{t-1,x}, u_{t-1,x-1}, u_{t,x} \in \mathrm{gph} F_{t,x}\} \quad \forall (t,x) \in H_1 \times L_1,$$

$$N^1_t = \{w = (u_0, \ldots, u_t) : (u_{t,x-1}, u_{t,0}) \in \mathrm{gph}\, F_{t,0}\}, \quad t \in H_1,$$
$$N^2_x = \{w = (u_0, \ldots, u_t) : (u_{0,x-1}, u_{0,x}) \in \mathrm{gph}\, F_{0,x}\}, \quad x \in L_1.$$

Then denoting $g(w) = \sum_{(t,x) \in H_0 \times L_0} g_{t,x}(u_{t,x})$, we see that the convex problem (P_D) is equivalent to the following convex minimization problem in $\mathbb{R}^{m(T+1)}$:

$$\text{infimum } g(w) \text{ subject to } w \in P = \left(\bigcap_{(t,x) \in H_1 \times L_1} M_{t,x} \right) \cap \left(\bigcap_{t=1}^{T} N^1_t \right) \cap \left(\bigcap_{x=1}^{L} N^2_x \right). \tag{6.83}$$

Now we write out the optimality conditions (Section 3.2) for the convex minimization problem (6.83). Therefore, first we should calculate the dual cones to the cones of tangent directions for the above-defined sets. It is an elementary way to show that

$$K^*_{M_{t,x}}(w) = \{w^* = u^*_0, \ldots, u^*_T) : (u^*_{t,x-1}, u^*_{t-1,x}, u^*_{t-1,x-1}, u^*_{t,x})$$

$$\in K^*_{\mathrm{gph}\, F_{t,x}}(u_{t,x-1}, u_{t-1,x}, u_{t-1,x-1}, u_{t,i}), \quad u^*_{i,j} = 0, \quad i \neq t, \ t-1; j \neq x, x-1\},$$
$$(t,x) \in H_1 \times L_1, K^*_{N^1_t}(w) = \{w^* = (u^*_0, \ldots, u^*_T) : (u^*_{t-1,0}, u^*_{t,0}) \in K^*_{\mathrm{gph}\, F_{t,0}}(u_{t-1,0}, u_{t,0}),$$
$$u^*_{i,j} = 0, i \neq t, t-1, j \neq 0\}, \quad t \in H_1, \tag{6.84}$$

$$K^*_{N^2_x}(w) = \{w^* = (u^*_0, \ldots, u^*_T) : (u^*_{0,x-1}, u^*_{0,x}) \in K^*_{\mathrm{gph}\, F_{0,x}}(u_{0,x-1}, u_{0,x}),$$
$$u^*_{i,j} = 0, \ i \neq 0, \ j \neq x, \ x-1\}, \ x \in L_1; \quad u^*_t = (u^*_{t,0}, u^*_{t,1}, \ldots u^*_{t,L}), \ t \in H_0.$$

By the hypothesis, $g(w)$ is continuous at $w^0 = (u_0^0, \ldots, u_T^0)$, $u_t^0 = (u_{t,0}^*, \ldots u_{t,L}^0)$ and the point $\tilde{w} = (\tilde{u}_0, \ldots, \tilde{u}_T)$ is a solution of the problem in Eq. (6.83). Then by Theorem 3.3, there are vectors $w^*(t,x) \in K_{M_{t,x}}^*(\tilde{w})$, $(t,x) \in H_1 \times L_1$; $w^*(t) \in K_{N_t^1}^*(\tilde{w})$, $t \in H_1$; $\overline{w}^*(x) \in K_{N_x^2}^*(\tilde{w})$, $x \in L_1$; $w^{0*} \in \partial_w g(\tilde{w})$and $\lambda \in \{0,1\}$, such that

$$\lambda w^{0*} = \sum_{(t,x)\in H_1 \times L_1} w^*(t,x) + \sum_{t=1}^{T} w^*(t) + \sum_{x=1}^{L} \overline{w}^*(x). \tag{6.85}$$

Here the indicated vectors and λ are not all equal to zero.

According to Eq. (6.84), we have

$$\begin{gathered}
w^*(t,x) = (0,\ldots,0,u_{t-1,x-1}^*(t,x),u_{t-1,x}^*(t,x),0,\ldots,0,u_{t,x-1}^*(t,x),u_{t,x}^*(t,x),0,\ldots,0),\\
(u_{t,x-1}^*(t,x),u_{t-1,x}^*(t,x),u_{t-1,x-1}^*(t,x),u_{t,x}^*(t,x)) \in K_{\mathrm{gph}\,F_{t,x}}^*(\tilde{u}_{t,x-1},\tilde{u}_{t-1,x},\tilde{u}_{t-1,x-1},\tilde{u}_{t,x})\\
(t,x) \in H_1 \times L_1,\\
w^*(t) = (u_{t-1,0}^*(t),0,\ldots,0,u_{t,0}^*(t),0,\ldots,0),\\
\overline{w}^*(x) = (u_{0,x-1}^*(x),u_{0,x}^*(x),0,\ldots,0),(u_{t-1,0}^*(t),u_{t,0}^*(t)) \in K_{\mathrm{gph}\,F_{t,0}}^*(\tilde{u}_{t-1,0},\tilde{u}_{t,0}),\ t \in H_1,
\end{gathered} \tag{6.86}$$

$$(u_{0,x-1}^*(x),u_{0,x}^*(x)) \in K_{\mathrm{gph}\ F_{0,x}}^*(\tilde{u}_{0,x-1},\tilde{u}_{0,x}),\quad x \in L_1.$$

By the definition of LAM, it follows from Eq. (6.86) that

$$\begin{gathered}
(u_{t,x-1}^*(t,x),u_{t-1,x}^*(t,x),u_{t-1,x-1}^*(t,x)) \in F_{t,x}^*(-u_{t,x}^*(t,x),(\tilde{u}_{t,x-1},\tilde{u}_{t-1,x},\tilde{u}_{t-1,x-1}\tilde{u}_{t,x})),\\
(t,x) \in H_1 \times L_1,\\
u_{t-1,0}^*(t) \in F_{t,0}^*(-u_{t,0}^*(t),(\tilde{u}_{t-1,0},\tilde{u}_{t,0})),\ t \in H_1;\\
u_{0,x-1}^*(x) \in F_{0,x-1}^*(-u_{0,x}^*(x),(\tilde{u}_{0,x-1},\tilde{u}_{0,x})),\ x \in L_1.
\end{gathered}$$

Now, transforming Eq. (6.85), we obtain that the component-wise representation of Eq. (6.85) consists of the following:

$$u_{t,x}^*(t,x) + u_{t,x}^*(t,x+1), u_{t,x}^*(t+1,x) + u_{t,x}^*(t+1,x+1) = \lambda u_{t,x}^{0*}, \tag{6.87}$$

$$u_{T,x}^*(T,x+1) + u_{T,x}^*(T,x) = \lambda u_{T,x}^{0*},\ x \in L_0;\quad u_{t,L}^*(t+1,L) + u_{t,L}^*(t,L) = \lambda u_{t,L}^*,\ t \in H_0,$$

where $w^{0*} = (u_0^{0*}, \ldots, u_T^{0*})$, $\quad u_t^{0*} = (u_{t,0}^{0*}, \ldots, u_{t,L}^{0*})$, $t \in H_0$, $u_{t,x}^{0*} \in \partial_u g(\tilde{u}_{t,x})$.

Now, denoting

$$\begin{gathered}
u_{t,x-1}^*(t,x) = \varphi_{t,x}^*,\ u_{t-1,x}^*(t,x) = \eta_{t,x}^*,\quad -u_{t,x}^*(t,x) = u_{t,x}^*,\ (t,x) \in H_0 \times L_0;\\
u_{t-1,0}^*(t,0) = u_{t-1,0}^*(t),\ u_{t,0}^*(t,0) = u_{t,0}^*(t),\ t \in H_1;\quad u_{0,x-1}^*(0,x) = u_{0,x-1}^*(x),\\
u_{0,x}^*(0,x) = u_{0,x}^*(x),\ x \in L_1;\quad u_{t,L}^*(t+1,L+1) = 0,\\
u_{t,L}^*(t,L+1) = 0,\ t \in H_0, u_{T,x}^*(T+1,x+1) = 0,\ u_{T,x}^*(T+1,x) = 0,\ x \in L_0
\end{gathered}$$

and in view of Eq. (6.87), we have

$$(\varphi^*_{t,x}, \eta^*_{t,x}, \lambda u^{0*}_{t-1,x-1} - \eta^*_{t,x-1} - \varphi^*_{t-1,x} + u^*_{t-1,x-1}),$$

$$\in F^*_{t,x}(u^*_{t,x}, (\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1}, \tilde{u}_{t,x})), \quad (t,x) \in H_1 \times L_1, \tag{6.88}$$

$$\eta^*_{t,0} \in F^*_{t,0}(u^*_{t,0}, (\tilde{u}_{t-1,0}, \tilde{u}_{t,0})), \quad t \in H_1 \, \varphi^*_{0,x} \in F^*_{0,x}(u^*_{0,x}, (\tilde{u}_{0,x-1}, \tilde{u}_{0,x})), \quad x \in L_1, \tag{6.89}$$

$$\varphi^*_{T,x+1} - u^*_{T,x} = \lambda u^{0*}_{T,x}, \quad x \in L_0; \quad \eta^*_{t+1,L} - u^*_{t,L} = \lambda u^*_{t,L}, \quad t \in H_0. \tag{6.90}$$

As a result of the regularity condition, the dual cone $K^*_P(\tilde{w})$ is a sum of the cones in Eq. (6.84). Then it follows by Theorem 3.2 that for $w^{0*} \in \partial_w g(\tilde{w}) \cap K^*_P(\tilde{w})$ in the representation in Eq. (6.85), $\lambda = 1$. Therefore, taking into account Eqs. (6.88)–(6.90), the proof is concluded. ■

Remark 6.8. In addition to the assumptions of Theorem 6.15, let $F_{t,x}(\cdot)$ be a closed-valued mapping (which of course is not the case for gph $F_{t,x}$). Then $\partial_{v^*} H_{F_{t,x}}(u, v^*) = F_{t,x}(u, v^*)$, $\quad u = (u_1, u_2, u_3)$ and, therefore, both conditions (1) and (2) can be written in a more symmetrical form:

1. $\tilde{u}_{t,x} \in \partial_{v^*} H_{F_{t,x}}(\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1}, u^*_{t,x})$,
   $(\varphi^*_{t,x}, \eta^*_{t,x}, u^*_{t-1,x-1}) \in \partial_u H_{F_{t,x}}(\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1}, u^*_{t,x})$
   $+ (0,0) \times \{\varphi^*_{t-1,x} + \eta^*_{t,x-1} - \lambda \partial g_{t-1,x-1}(\tilde{u}_{t-1,x-1})\}$,
2. $\tilde{u}_{t,0} \in \partial_{v^*} H_{F_{t,0}}(\tilde{u}_{t-1,0}, u^*_{t,0})$, $\quad \eta^*_{t,0} \in \partial_u H_{F_{t,0}}(\tilde{u}_{t-1,0}, u^*_{t,0})$,
   $\tilde{u}_{0,x} \in \partial_{v^*} H_{F_{0,x}}(\tilde{u}_{0,x-1}, u^*_{0,x})$, $\quad \varphi^*_{0,x} \in \partial_u H_{F_{0,x}}(\tilde{u}_{0,x-1}, u^*_{0,x})$.

Theorem 6.16. Assume that in the problem in Eqs. (6.77)–(6.79), the mappings $F_{t,x}$, $(t,x) \neq (0,0)$ are such that the cones of tangent directions

$$K_{\text{gph } F_{t,x}}(\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1}, \tilde{u}_{t,x}), \quad K_{\text{gph } F_{t,0}}(\tilde{u}_{t-1,0}, \tilde{u}_{t,0}), \quad K_{\text{gph } F_{0,x}}(\tilde{u}_{0,x-1}, \tilde{u}_{0,x})$$

are local tents and the functions $g_{t,x}$ admit a CUA $h_{t,x}(\bar{u}, \tilde{u}_{t,x})$ at points $\tilde{u}_{t,x}$ that are continuous with respect to $\bar{u}$. Then for the optimality of the solution $\{\tilde{u}_{t,x}\}_{H_0 \times L_0}$ in such a nonconvex problem, it is necessary that there exist vectors $\{u^*_{t,x}\}, \{\varphi^*_{t,x}\}$, $\{\eta^*_{t,x}\}$ $(u^*_{0,0} = \eta^*_{T+1,L} = \varphi^*_{T,L+1} = 0)$, $(t,x) \in H_0 \times L_0$, not all equal to zero, and a number $\lambda \in \{0,1\}$ satisfying conditions (1)–(3) of Theorem 6.15.

□ In this case, we have the nonconvex problem in Eq. (6.83) and the subdifferentials $\partial g_{t,x}(\tilde{u}_{t,x}) = \partial h_{t,x}(0, \tilde{u}_{t,x})$ are defined. Therefore, we get a necessary condition as in Theorem 6.15 by starting from the relation in Eq. (6.83), written out for the nonconvex problem (P_D). Here the hypothesis of the theorem for the problem in Eqs. (6.1)–(6.3) ensures the condition of Theorem 3.25 for the nonconvex minimization problem (P_D). ■

2 Approximation of a Continuous Problem (P_C) and Optimization of a Discrete-Approximation Problem

Let us introduce the following difference operator on a uniform grid on Q [246]:

$$Au(t+\delta, x+h) = \frac{1}{\delta h}[u(t+\delta, x+h) - u(t+\delta, x) - u(t, x+h) + u(t,x)],$$
$$t = 0, \delta, \ldots, 1-\delta; \quad x = 0, h, \ldots, 1-h,$$

where λ and h are steps on the t- and x-axes, respectively.

Then to the problem (P_C), we associate the following discrete-approximation problem:

$$\inf \quad J_{\delta h}[u(t,x)] = \sum_{\substack{t=0,\ldots,1-\delta \\ x=0,\ldots,1-h}} \delta h g(u(t,x),t,x) + \sum_{x=0,\ldots,1-h} h g_1(u(1,x),x)$$
$$+ \sum_{t=0,\ldots,1-\delta} \delta g_2(u(t,1),t)$$

$$\text{subject to } Au(t+\delta, x+h) \in F(u(t,x),t,x),$$
$$\Delta_\delta u(t,0) \in F_1(u(t,0),t); \quad \Delta_h u(0,x) \in F_2(u(0,x),x), \tag{6.91}$$
$$u(0,0) = u_0, t = 0, \delta, \ldots, 1-\delta; \quad x = 0, h, \ldots, 1-h$$

where $\Delta_\delta u(t,0) = (1/\delta)[u(t+\delta,0) - u(t,0)]$; $\Delta_h u(0,x) = (1/h)[u(0,x+h) - u(0,x)]$.

Let us introduce the new multivalued functions

$$G(p,q,u,t,x) = p + q - u + \delta h F(u,t,x), \quad G_1(u,t) = u + \delta F_1(u,t),$$
$$G_2(u,x) = u + hF_2(u,x)$$

and rewrite the problem in Eq. (6.91) as follows:

$$\inf \quad J_{\delta h}[u(\cdot,\cdot)],$$

$$\begin{aligned} \text{subject to} \quad & u(t+\delta, x+h) \in G(u(t+\delta,x), u(t,x+h), u(t,x), t, x)), \\ & u(t+\delta, 0) \in G_1(u(t,0),t); \quad u(0,x+h) \in G_2(u(0,x),x), \quad u(0,0) = u_0 \\ & t = 0, \delta, \ldots, 1-\delta; \quad x = 0, h, \ldots, 1-h. \end{aligned} \tag{6.92}$$

Now, our main problem is to get necessary and sufficient conditions for optimality for the problem in Eq. (6.91) or (6.92). We need the following supplementary formulas:

$$H_G(p,q,u,v^*,t,x) = \delta h H_F(u,v^*,t,x) + \langle p+q-u, v^* \rangle,$$

$$H_{G_1}(u, v^*, t) = \delta H_{F_1}(u, v^*, t) + \langle u, v^* \rangle, \tag{6.93}$$

$$H_{G_2}(u, v^*, x) = hH_{F_2}(u, v^*, x) + \langle u, v^* \rangle, \quad v^* \in \mathbb{R}^n.$$

Then for convex multivalued functions G, F and F_i $(i = 1,2)$, elementary calculations of subdifferentials of Hamiltonian functions give us

$$\begin{aligned}\partial_{(p,q,u)} H_G(p, q, u, v^*, t, x) &= (0, 0) \times \delta h \partial_u H_F(u, v^*, t, x) + (v^*, v^*, -v^*)\\ &= (v^*, v^*) \times \{\delta h \partial_u H_F(u, v^*, t, x) - v^*\},\end{aligned}$$

$$\partial_u H_{G_1}(u, v_1^*, t) = \delta\, \partial H_{F_1}(u, v_1^*, t) + v_1^*; \quad \partial_u H_{G_2}(u, v_2^*, x) = h\, \partial H_{F_2}(u, v^*, x) + v_2^*,$$

whence by Theorem 2.1, it is immediately shown that

$$G^*(v^*, (p, q, u, v), t, x) = (v^*, v^*) \times \{\delta h F^*(v^*, (u, v), t, x) - v^*\}, \quad v \in G(p, q, u, v^*, t, x),$$

$$\begin{aligned}G_1^*(v_1^*; (u, v_1), t) &= \delta\, F_1^*\left(v_1^*; \left(u, \frac{v_1 - u}{\delta}\right), t\right) + v_1^*; \quad v_1 \in G_1(u, v_1, t),\\ G_2^*(v_2^*; (u, v_2), x) &= h\, F_2^*\left(v_2^*; \left(u, \frac{v_2 - u}{h}\right), x\right) + v_2^*; \quad v_2 \in G_2(u, v_2, x).\end{aligned} \tag{6.94}$$

Obviously, the first relationship in Eq. (6.94) means that the following inclusions are equivalent:

$$(p^*, q^*, u^*) \in G^*(v^*; (p, q, u, v), t, x), \quad v \in G((p, q, u); v^*, t, x),$$

$$\begin{aligned}&\frac{u^* + v^*}{\delta h} \in F^*\left(v^*; \left(u, \frac{v - p - q + u}{\delta h}\right), t, x\right),\\ &\frac{v - p - q + u}{\delta h} \in F(u; v^*, t, x), \quad p^* = q^* = v^*.\end{aligned} \tag{6.95}$$

And the remaining relationships can be rewritten as follows:

$$u^* \in G_1^*(v_1^*; (u, v_1), t); \quad v_1 \in G_1(u; v_1^*, t),$$

$$\frac{u^* - v_1^*}{\delta} \in F_1^*\left(v_1^*; \left(u, \frac{v_1 - u}{\delta}\right), t\right); \quad \frac{v_1 - u}{\delta} \in F_1(u, v_1^*, t), \tag{6.96}$$

$$u^* \in G_2^*(v_2^*; (u, v_2), x); \quad v_2 \in G_2(u, v_2^*, x),$$

$$\frac{u^* - v_2^*}{h} \in F_2^*\left(v_2^*; \left(\frac{v_2 - u}{h}\right), x\right); \qquad \frac{v_2 - u}{h} \in F_2(u, v_2^*, x),$$

respectively. In more general cases, i.e., in the nonconvex cases and existences of local tents, the equivalence of the inclusions in Eqs. (6.95) and (6.96) can be proved in a way similar to that in Theorem 6.23 and Remark 6.9.

Now using the equivalence of Eqs. (6.95) and (6.96), we can write necessary and sufficient conditions for the problem in Eq. (6.91). At first we use Theorem 6.15 for the discrete-approximation problem in Eq. (6.82). By this theorem, there exist vectors $\{\eta^*(t,x)\},\{\varphi^*(t,x)\}$, $\{u^*(t,x)\}$ and a number $\lambda = \lambda_{\delta h} \in \{0,1\}$, not all zero, such that

$$\begin{gathered}(\varphi^*(t+\delta, x+h), \eta^*(t+\delta, x+h), u^*(t,x) - \varphi^*(t, x+h) - \eta^*(t+\delta, x)) \\ \in G^*(u^*(t+\delta, x+h); (\tilde{u}(t+\delta, x), \tilde{u}(t, x+h), \tilde{u}(t+\delta, x+h)), t, x) \\ (0,0) \times \{-\lambda_{\delta h}\delta h \partial g(\tilde{u}(t,x), t, x)\},\end{gathered} \tag{6.97}$$

$$\begin{aligned}&\eta^*(t+\delta, 0) \in G_1^*(u^*(t+\delta, 0), (\tilde{u}(t,0), \tilde{u}(t+\delta, 0)), t), \\ &\varphi^*(0, x+h) \in G_2^*(u^*(0, x+h), (\tilde{u}(0,x), \tilde{u}(0, x+h)), x),\end{aligned} \tag{6.98}$$

$$\begin{aligned}&\varphi^*(1, x+h) - u^*(1,x) \in \lambda_{\delta h} h \partial g_1(\tilde{u}(1,x), x), \\ &\eta^*(t+\delta, 1) - u^*(t,1) \in \lambda_{\delta h}\delta \partial g_2(\tilde{u}(t,1), t),\end{aligned} \tag{6.99}$$

$$u^*(0,0) = u^*(1+\delta, 1) = u^*(1, 1+h) = 0.$$

Using Eq. (6.95), the condition in Eq. (6.97) can be rewritten as follows:

$$\begin{aligned}&Au^*(t+\delta, x+h) \in F^*(u^*(t+\delta, x+h); (\tilde{u}(t,x), A\tilde{u}(t+\delta, x+h)), t, x) \\ &-\lambda_{\delta h}\partial g(\tilde{u}(t,x), t, x).\end{aligned} \tag{6.100}$$

Here, we have taken into account (see Eq. (6.95)) that

$$u^*(t,x) = \eta^*(t,x) = \varphi^*(t,x). \tag{6.101}$$

Furthermore, Eq. (6.98) implies that

$$\begin{aligned}&\frac{\eta^*(t+\delta, 0) - u^*(t+\delta, 0)}{\delta} \in F_1^*(u^*(t+\delta, 0), (\tilde{u}(t,0), \tilde{u}(t+\delta, 0)), t), \\ &\frac{\varphi^*(0, x+h) - u^*(0, x+h)}{h} \in F_2^*(u^*(0, x+h), (\tilde{u}(0,x), \tilde{u}(0, x+h)), x).\end{aligned} \tag{6.102}$$

Let us rewrite Eq. (6.99) in the form

$$\begin{aligned}&\frac{\varphi^*(1,x+h) - u^*(1,x+h)}{h} \in \lambda_{\delta h}\partial g_1(\tilde{u}_\delta(1,x)x) - \frac{u^*(1,x+h) - u^*(1,x)}{h}, \\ &\frac{\eta^*(t+\delta,1) - u^*(t+\delta,1)}{\delta} \in \lambda_{\delta h}\partial g_2(\tilde{u}(t,1),t) - \frac{u^*(t+\delta,1) - u^*(t,1)}{h}.\end{aligned} \tag{6.103}$$

By Eq. (6.101), the left-hand sides of Eqs. (6.102) and (6.103) are equal to zero. Then taking into account Theorem 2.1 and the relations in Eqs. (6.101) and (6.102), it is easy to see that

$$0 \in \partial_u H_{F_1}(\tilde{u}(t,0), u^*(t+\delta,0)), \quad 0 \in \partial_u H_{F_2}(\tilde{u}(0,x), u^*(0,x+h)),$$

or, equivalently,

$$\begin{aligned} &H_{F_1}(\tilde{u}(t,0), u^*(t+\delta,0)) = \langle \Delta_\delta \tilde{u}(t,0), u^*(t+\delta,0)\rangle, \\ &H_{F_2}(\tilde{u}(0,x), u^*(0,x+h)) = \langle \Delta_h \tilde{u}(0,x), u^*(0,x+h)\rangle; \end{aligned} \tag{6.104}$$

i.e., at the points $\tilde{u}(t,0), \tilde{u}(0,x)$, the Hamiltonian functions H_{F_1} and H_{F_2}assume their maximums.

On the other hand, because of Eq. (6.101), it follows from Eq. (6.103) that

$$\Delta_h u^*(1,x) \in \lambda_{\delta h} \partial g_1(\tilde{u}(1,x),x), \quad \Delta_\delta u^*(t,1) \in \lambda_{\delta h} \partial g_2(\tilde{u}(t,1),t). \tag{6.105}$$

This proves Theorem 6.17.

Theorem 6.17. For optimality of the solution $\{\tilde{u}(t,x)\}$ in the convex discrete-approximation problem in Eqs. (6.91)−(6.93) with set-valued boundary conditions, it is necessary that there exist a number $\lambda_{\delta h} \in \{0,1\}$ and vectors $\{u^*(t,x)\}$ ($u^*(0,0) = u^*(1+\delta,1) = u^*(1,1+h) = 0$), not all equal to zero, satisfying Eqs. (6.100), (6.104), and (6.105). Also, under the regularity condition, the conditions in Eqs. (6.100), (6.104), and (6.105) are sufficient for the optimality of $\{\tilde{u}(t,x)\}$.

3 Optimization of Problem (P_C)

Setting $\lambda_{\delta h} = 1$ and passing to the formal limit in Eqs. (6.100), (6.104), and (6.105) as δ and h approach zero, we find that

i. $u^*_{tx}(t,x) \in F^*(u^*(t,x), (\tilde{u}(t,x), \tilde{u}_{tx}(t,x)), t, x) - \partial g(\tilde{u}(t,x), t, x),$

ii. $u^*_x(1,x) \in \partial g_1(\tilde{u}(1,x),x); \quad u^*_t(t,1) \in \partial g_2(\tilde{u}(t,1),t),$
$u^*(0,0) = u^*(1,1) = 0,$

iii. $\tilde{u}_{tx}(t,x) \in F(\tilde{u}(t,x), u^*(t,x), t, x),$
$\tilde{u}_t(t,0) \in F_1(\tilde{u}(t,0), u^*(t,0), t), \quad \tilde{u}_x(0,x) \in F_2(\tilde{u}(0,x), u^*(0,x), x).$

iv. $H_{F_1}(\tilde{u}(t,0), u^*(t,0)) = \langle \tilde{u}_t(t,0), u^*(t,0)\rangle, \ H_{F_2}(\tilde{u}(0,x), u^*(0,x)) = \langle \tilde{u}_x(0,x), u^*(0,x)\rangle.$

Theorem 6.18. Let g, g_1, g_2 be continuous and convex functions on the first argument u and suppose that the multivalued functions F, F_1, F_2 are convex. Then for the optimality of a feasible solution $\tilde{u}(t,x)$, it is sufficient that there exists an absolutely continuous function $u^*(t,x)$ on Q with an integrable mixed derivative $u^*_{tx}(t,x)$ such that conditions (i)−(iv) hold almost everywhere.

□ By Theorem 2.1

$$F^*(u^*(t,x), (\tilde{u}(t,x), \tilde{u}_{tx}(t,x), t,x)) = \partial_u H_F(\tilde{u}(t,x), u^*(t,x), t, x),$$
$$\tilde{u}_{tx}(t,x) \in F(\tilde{u}(t,x), u^*(t,x), t, x).$$

Then using the Theorem 1.29 (Moreau−Rockafellar), from condition (i), we have

$$u_{tx}^*(t,x)\in\partial_u[H_F(\tilde{u}(t,x),u^*(t,x),t,x)-g(\tilde{u}(t,x),t,x)].$$

By definition of a subdifferential,

$$\begin{aligned}&H_F(u(t,x),u^*(t,x),t,x)-H_F(\tilde{u}(t,x),u^*(t,x),t,x)-g(u(t,x),t,x)\\&\qquad+g(\tilde{u}(t,x),t,x)\le\langle u_{tx}^*(t,x),u(t,x)-\tilde{u}(t,x)\rangle.\end{aligned}$$

Using the definition of Hamiltonian H_F, by integration of this inequality we establish that

$$\begin{aligned}\iint\limits_Q[g(u(t,x),t,x)-g(\tilde{u}(t,x),t,x)]\mathrm{d}t\,\mathrm{d}x&\ge\iint\limits_Q\langle u_{tx}(t,x)-\tilde{u}_{tx}(t,x),u^*(t,x),u^*(t,x)\rangle\mathrm{d}t\,\mathrm{d}x\\&-\iint\limits_Q\langle u_{tx}^*(t,x),u(t,x)-\tilde{u}(t,x)\rangle\mathrm{d}t\,\mathrm{d}x.\end{aligned}\tag{6.106}$$

Let us rewrite the double integrals on the right-hand side as follows:

$$\begin{aligned}&\iint\limits_Q\langle u_{tx}(t,x)-\tilde{u}_{tx}(t,x),u^*(t,x)\rangle\mathrm{d}t\,\mathrm{d}x=\int_0^1\langle u_x(1,x)-\tilde{u}_x(1,x),u^*(1,x)\rangle\mathrm{d}x\\&-\int_0^1\langle u_x(0,x)-\tilde{u}_x(0,x),u^*(0,x)\rangle\mathrm{d}x-\iint\limits_Q\langle u_x(t,x)-\tilde{u}_x(t,x),u_t^*(t,x)\rangle\mathrm{d}t\mathrm{d}x\end{aligned}\tag{6.107}$$

$$\begin{aligned}&\iint\limits_Q\langle u_{tx}^*(t,x),u(t,x)-\tilde{u}(t,x)\rangle\mathrm{d}t\,\mathrm{d}x=\int_0^1\langle u_t^*(t,1),u(t,1)-\tilde{u}(t,1)\rangle\mathrm{d}t\\&-\int_0^1\langle u_t^*(t,0),u(t,0)-\tilde{u}(t,0)\rangle\mathrm{d}t-\iint\limits_Q\langle u_t^*(t,x),u_x(t,x)-\tilde{u}_x(t,x)\rangle\mathrm{d}t\,\mathrm{d}x.\end{aligned}\tag{6.108}$$

Moreover, the second and third condition of (iii) mean that

$$\langle u_t(t,0)-\tilde{u}_t(t,0),u^*(t,0)\rangle\le0,\quad\langle u_x(0,x)-\tilde{u}_x(0,x),u^*(0,x)\rangle\le0.\tag{6.109}$$

Subtracting Eqs. (6.107) and (6.108) and using Eq. (6.109), we have from Eq. (6.106)

$$\begin{aligned}\iint\limits_Q[g(u(t,x),t,x)-g(\tilde{u}(t,x),t,x)]\mathrm{d}t\,\mathrm{d}x&\ge\int_0^1\langle u_x(1,x)-\tilde{u}_x(1,x),u^*(1,x)\rangle\mathrm{d}x\\-\int_0^1\langle u_x(0,x)-\tilde{u}_x(0,x),u^*(0,x)\rangle\mathrm{d}x&-\int_0^1\langle u_t^*(t,1),u(t,1)-\tilde{u}(t,1)\rangle\mathrm{d}t\end{aligned}$$

$$+\int_0^1 \langle u_t^*(t,0), u(t,0)-\tilde{u}(t,0)\rangle \mathrm{d}t + \int_0^1 \langle u_t(t,0)-\tilde{u}_t(t,0), u^*(t,0)\rangle \mathrm{d}t$$
$$+\int_0^1 \langle u_x(0,x)-\tilde{u}_x(0,x), u^*(0,x)\rangle \mathrm{d}x. \tag{6.110}$$

On the other hand, condition (iii) implies that

$$\begin{aligned} &g_1(u(1,x),x)-g_1(\tilde{u}(1,x),x) \geq \langle u_x^*(1,x), u(1,x)-\tilde{u}(1,x)\rangle, \\ &g_2(u(t,1),t)-g_2(\tilde{u}(t,1),t) \geq \langle u_t^*(t,1), u(t,1)-\tilde{u}(t,1)\rangle. \end{aligned} \tag{6.111}$$

Integrating Eq. (6.111) and adding Eq. (6.110), we have

$$\begin{aligned} J(u(\cdot,\cdot))-J(\tilde{u}(\cdot,\cdot)) &\geq \int_0^1 d_x\langle u^*(1,x), u(1,x)-\tilde{u}(1,x)\rangle + \int_0^1 d_t\langle u^*(t,0), u(t,0)-\tilde{u}(t,0)\rangle \\ &= \langle u^*(1,1), u(1,1)-\tilde{u}(1,1)\rangle - \langle u^*(1,0), u(1,0)-\tilde{u}(1,0)\rangle + \langle u^*(1,0), u(1,0)-\tilde{u}(1,0)\rangle \\ &- \langle u^*(0,0), u(0,0)-\tilde{u}(0,0)\rangle = 0. \end{aligned}$$

Thus, $J[u(\cdot,\cdot)] \geq J[\tilde{u}(\cdot,\cdot)]$. The proof of the theorem is completed. ■

Theorem 6.19. Let $\tilde{u}(t,x)$ be some feasible solution of a nonconvex problem (P_C) and the LAM to a nonconvex multivalued function F at a point $(u,v) \in \operatorname{gph} F$ be defined as

$$F^*(v^*,(u,v)) = \{u^* : H_F(u_1,v^*) - H_F(u,v^*) \leq \langle u^*, u_1-u\rangle \quad \forall u_1 \in \mathbb{R}^n\}, \ v \in F(u,v^*).$$

Moreover, let $u^*(t,x)$ be an absolutely continuous function satisfying almost everywhere on Q and for arbitrary u the following conditions:

a. $u_{tx}^*(t,x) + u^*(t,x) \in F^*(u^*(t,x); (\tilde{u}(t,x), \tilde{u}_{tx}(t,x), t, x))$,
b. $g(u,t,x) - g(\tilde{u}(t,x),t,x) \geq \langle u^*(t,x), u-\tilde{u}(t,x)\rangle$,
c. $g_1(u,x) - g_1(\tilde{u}(1,x),x) \geq \langle u^*(1,x), u-\tilde{u}(1,x)\rangle$,
$g_2(u,t) - g_2(\tilde{u}(t,1),t) \geq \langle u^*(t,1), u-\tilde{u}(t,1)\rangle$
d. $\langle \tilde{u}_{tx}(t,x), u^*(t,x)\rangle \in H_F(\tilde{u}(t,x), u^*(t,x), t, x)$,
e. $\langle \tilde{u}_t(t,0), u^*(t,0)\rangle \in H_{F_1}(\tilde{u}(t,0), u^*(t,0), t)$,
$\langle \tilde{u}_x(0,x), u^*(0,x)\rangle \in H_{F_2}(\tilde{u}(0,x), u^*(0,x), x)$.

Then $\tilde{u}(t,x)$ is optimal.

□ As in Section 6.2 (see Remark 6.4), it is easy to see that from (a) and (b) for all arbitrary feasible solutions $u(t,x)$, the inequality

$$g(u(t,x),t,x) - g(\tilde{u}(t,x),t,x) \geq \langle u_{tx}(t,x) - \tilde{u}(t,x), u^*(t,x)\rangle - \langle u_{tx}^*(t,x), u(t,x) - \tilde{u}(t,x)\rangle$$

holds. If we integrate this inequality on Q, we get Eq. (6.106). Then starting from Eq. (6.106) and using conditions (c)–(e), we can prove that $\tilde{u}(t,x)$ is optimal. ■

Let $F(u_1,u_2,u_3) = \{f(u_1,u_2,u_3)\}$, $f: \mathbb{R}^{3n} \to \mathbb{R}^n$ be a single-valued smooth mapping with components f^i, $i = 1, \ldots, n$. Then by Eq. (3.69) (see Corollary 3.11),

$$F^*(v^*; (u_1^0, u_2^0, u_3^0, v^0)) = \{f_{u_1}^{'*}(u_1^0, u_2^0, u_3^0)v^*, f_{u_2}^{'*}(u_1^0, u_2^0, u_3^0)v^*, f_{u_3}^{'*}(u_1^0, u_2^0, u_3^0)v^*\},$$
$$v^0 = f(u_1^0, u_2^0, u_3^0),$$

where the derivative of $f(u_1,u_2,u_3)$ is, by definition, a linear operator $\{f_{u_1}^{'*}(u_1^0, u_2^0, u_3^0), f_{u_2}^{'*}(u_1^0,u_2^0,u_3^0), f_{u_3}^{'*}(u_1^0,u_2^0,u_3^0)\}$ with components $(\partial f^i/u_1^j),(\partial f^i/u_2^j),(\partial f^i/u_3^j)$, $i,j=1,\ldots,n$ and $f_{u_i}^{'*}$ is the adjoint operator to $f_{u_i}^{'}$, $i=1,2,3$.

Let us consider the following problem:

$$\inf \sum_{(t,x)\in H_0 \times L_0} g_{t,x}(u_{t,x}) \tag{6.112}$$

$$\text{subject to } u_{t,x} = f(u_{t,x-1}, u_{t-1,x}, u_{t-1,x-1}), \quad (t,x)\in H_1 \times L_1 \tag{6.113}$$

$$\begin{aligned} &u_{t,0} = A_1 u_{t-1,0} + B_1 w_{t-1,0}, \quad w_{t-1,0} \in U_1, \;\; t\in H_1, \\ &u_{0,x} = A_2 u_{0,x-1} + B_2 w_{0,x-1}, \quad w_{0,x-1} \in U_2, \;\; x\in L_1 \end{aligned} \tag{6.114}$$

where $f: \mathbb{R}^{3n} \to \mathbb{R}^n$ is a smooth mapping, $g_{t,x}$ are continuously differentiable functions on u,A_i,B_i, $i = 1,2$ are $n \times n$ and $n \times r$ matrices, respectively, and $U_i \subset \mathbb{R}^n$, $i = 1,2$. It is now required to find a pair of controlling parameters $\{\tilde{w}_{t-1,0}, \tilde{w}_{0,x-1}\}$, $(t,x)\in H_1 \times L_1$ such that the corresponding solution $\{\tilde{u}_{t,x}\}$ minimizes Eq. (6.112).

Taking into account conditions (1)–(3) of Theorem 6.16, we have

$$\begin{aligned} &\varphi_{t,x}^* = f_{u_1}^{'*}(\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1})u_{t,x}^*, \quad \eta_{t,x}^* = f_{u_2}^{'*}(\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1})u_{t,x}^*, \\ &u_{t-1,x-1}^* - \varphi_{t-1,x}^* - \eta_{t,x-1}^* = f_{u_3}^{'*}(\tilde{u}_{t,x-1}, \tilde{u}_{t-1,x}, \tilde{u}_{t-1,x-1})u_{t,x}^* - \lambda g_{t-1,x-1}'(\tilde{u}_{t-1,x-1}). \end{aligned} \tag{6.115}$$

Using the calculation technique of dual cones, it is easy to see that

$$K_{\text{gph } F_{t,0}}^*(u^0, v^0) = \{(-A_1^* v^*, v^*) : \langle \overline{w}, B^* v^* \rangle \geq 0, \overline{w} \in K_{U_1}(w^0)\}.$$

So, by definition of LAM

$$F_{t,0}^*(u^0, v^0) = \{u^* : (u^* - v^*) \in K_{\text{gph } F_{t,0}}^*(u^0, v^0)\} = \{A_1^* v^* : -B^* v^* \in K_{U_1}^*(w^0)\},$$
$$v^0 = A_1 u^0 + B_1 w^0,$$

where A_1^*, B^* are the adjoint (transposed) matrices of A_1 and B, respectively. Therefore, using Eq. (6.115), we have

$$\begin{aligned} &\eta_{t,0}^* = A_1^* u_{t,0}^*, \quad -B_1^* u_{t,0}^* \in K_{u_1}^*(\tilde{w}_{t-1,0}), \\ &\varphi_{0,x}^* = A_2^* u_{0,x}^*, \quad -B_2^* u_{0,x}^* \in K_{u_2}^*(\tilde{w}_{0,x-1}), \;\; (t,x)\in H_1 \times L_1, \\ &\varphi_{T,x+1}^* - u_{T,x}^* = \lambda g_{T,x}'(\tilde{u}_{T,x}), \;\; x\in L_0, \quad \eta_{t+1}^* - u_{t,L}^* = \lambda g_{t,L}'(\tilde{u}_{t,L}), \;\; t\in H_0. \end{aligned} \tag{6.116}$$

Then we see that Eq. (6.116) can be expressed only in terms of $u^*_{t,x}$:

$$u^*_{t,x} = f'^*_{u_1}(\tilde{u}_{t,x}, \tilde{u}_{t-1,x+1}, \tilde{u}_{t-1,x})u^*_{t,x+1} + f'^*_{u_2}(\tilde{u}_{t+1,x-1}, \tilde{u}_{t,x}, \tilde{u}_{t-1,x-1})u^*_{t+1,x} + f'^*_{u_3}(\tilde{u}_{t+1,x}, \tilde{u}_{t,x+1}, \tilde{u}_{t,x})u^*_{t+1,x+1} - \lambda g'_{t,x}(\tilde{u}_{t,x}),\ t = 0, \ldots, T-1,\ x = 0, \ldots, L-1, \tag{6.117}$$

$$\begin{gathered} f'^*_{u_1}(\tilde{u}_{T,x}, \tilde{u}_{T-1,x+1}, \tilde{u}_{T-1,x})u^*_{T,x+1} - u^*_{T,x} = \lambda g'_{T,x}(\tilde{u}_{T,x}),\ x \in L_0, \\ f'^*_{u_2}(\tilde{u}_{t+1,L-1}, \tilde{u}_{t,L}, \tilde{u}_{t,L-1})u^*_{t+1,L} - u^*_{t,L} = \lambda g_{t,L}(\tilde{u}_{t,L}),\ t \in H_0, \\ \varphi^*_{0,x} = A^*_2 u^*_{0,x},\ \ \eta^*_{t,0} = A^*_1 u^*_{t,0},\ \ (t,x) \in H_1 \times L_1, \end{gathered} \tag{6.118}$$

$$\langle B_1 \tilde{w}_{t-1,0}, u^*_{t,0} \rangle = W_{U_1}(B^*_1 u^*_{t,0}), \quad \langle B_2 \tilde{w}_{0,x-1}, u^*_{0,x} \rangle = W_{U_2}(B^*_2 u^*_{0,x}). \tag{6.119}$$

Theorem 6.20. If the solution $\{\tilde{u}_{t,x}\}_{H_0 \times L_0}$ corresponding to a pair of controlling parameters $\{\tilde{w}_{t-1,0}, \tilde{w}_{0,x-1}\}_{H_1 \times L_1}$ for the problem in Eqs. (6.112)–(6.114) with Darboux-type hyperbolic discrete inclusions is optimal, then there exist adjoint variables $\{u^*_{t,x}\}_{H_0 \times L_0}$, not all zero, and a number $\lambda \in \{0,1\}$ satisfying the discrete adjoint equation (6.117) and the conditions in Eqs. (6.118) and (6.119). In addition, if $\lambda = 1$, these conditions are sufficient for the optimality of $\{\tilde{u}_{t,x}\}_{H_0 \times L_0}$.

Now let us consider the problem with a Darboux-type differential equation:

$$\inf \quad J[u(\cdot,\cdot)] = \iint_Q g(u(t,x), t, x)\mathrm{d}t\, \mathrm{d}x + \int_0^1 g_1(u(1,x), x)\mathrm{d}x + \int_0^1 g_2(u(t,1), t)\mathrm{d}t, \tag{6.120}$$

$$\text{subject to } \ u_{tx}(t,x) = f(u(t,x), t, x), \tag{6.121}$$

$$\begin{aligned} &u_t(t,0) = A_1 u(t,0) + B_1 w_1(t), \quad w_1(t) \in U_1,\ \ t \in [0,1], \\ &u_x(0,x) = A_2 u(0,x) + B_2 w_2(x), \quad w_2(x) \in U_2,\ \ x \in [0,1], \\ &u(0,0) = u_0, \end{aligned} \tag{6.122}$$

where A_i $(i = 1,2)$ are $(n \times n)$ matrices, B_i $(i = 1,2)$ are $(n \times r)$matrices, U_1, U_2 are convex sets in $\mathbb{R}^r$, and $f(\cdot,t,x) : \mathbb{R}^n \to \mathbb{R}^n$; $g(\cdot,t,x) : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions. It is required to find a pair of controlling parameters $\{\tilde{w}_1(t), \tilde{w}_2(x)\}$ such that the corresponding solution $\tilde{u}(t,x)$minimizes $J[u(\cdot,\cdot)]$.

In the present case, we have

$$F^*(v^*, (u,v), t, x) = \{f'^*(u,t,x)v^*\} \tag{6.123}$$

and it is not hard to see that the relations $0 \in F^*_1(v^*_1, (u_1, v_1))$, $0 \in F^*_2(v^*_2, (u_2, v_2))$, where $F_1(u) = A_1 u + B_1 U_1$, $F_2(u) = A_2 u + B_2 U_2$, mean that $A^*_1 v^*_1 = 0$, $A^*_2 v^*_2 = 0$ (see the formula preceding Eq. (6.116)). Then taking into account Eq. (6.123), we see that conditions (a), (c), and (d) of Theorem 6.19 consist of the following:

$$\begin{gathered} u_{tx}^{*}(t,x)+u^{*}(t,x)=f'^{*}(\tilde{u}(t,x),t,x))u^{*}(t,x), \\ A_1^{*}u^{*}(t,0)=0,\ \ t\in[0,1],\quad A_2^{*}u^{*}(0,x)=0,\ \ x\in[0,1], \\ \langle B_1\tilde{w}_1(t),u^{*}(t,0)\rangle=W_{U_1}(B_1^{*}u^{*}(t,0)),\quad \langle B_2\tilde{w}_2(x),u^{*}(0,x)\rangle=W_{U_2}(B_2^{*}u^{*}(0,x)). \end{gathered} \tag{6.124}$$

Let us summarize the obtained result.

Theorem 6.21. Assume that $\tilde{u}(t,x)$ is a feasible solution corresponding to the pair of controlling parameters $\{\tilde{w}_1(t),\tilde{w}_2(x)\}$ in the problem in Eqs. (6.120)–(6.122). Then for the optimality of $\tilde{u}(t,x)$, it is sufficient that there exists an absolutely continuous function $u^{*}(t,x)$ on Q satisfying the condition (6.124) and hypotheses (b) and (e) of Theorem 6.19.

Now, let us consider the nonconvex problem in Eqs. (6.80)–(6.82), where the right-hand side of the differential inclusion in Eq. (6.81) depends on both $u(\cdot,\cdot)$ and its first partial derivatives with respect to t and x:

$$u_{tx}(t,x)\in F(u_t(t,x),u_x(t,x),u(t,x),t,x),\quad (t,x)\in Q. \tag{6.125}$$

If we define

$$\tilde{G}(p,q,u,t,x)=p+q-u+\delta hF\left(\frac{p-u}{\delta},\frac{q-u}{h},u,t,x\right), \tag{6.126}$$

then the discrete-approximation problem associated with the problem in Eqs. (6.80), (6.82), and (6.125) is Eq. (6.92), where G is replaced by $\tilde{G}$.

Lemma 6.2. Let $\tilde{G}(\cdot,\cdot,\cdot,t,x):\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to P(\mathbb{R}^n)$ be a convex multivalued mapping defined by Eq. (6.126). Then

$$H_{\tilde{G}(\cdot,\cdot,\cdot,t,x)}(p,q,u,v^{*})=\delta hH_{F(\cdot,\cdot,\cdot,t,x)}\left(\frac{p-u}{\delta},\frac{q-u}{h},u,v^{*}\right)+\langle p+q-u,v^{*}\rangle.$$

□ Indeed, by definition (6.126), of multivalued function $\tilde{G}$, we can write

$$\begin{aligned} H_{\tilde{G}(\cdot,\cdot,\cdot,t,x)}(p,q,u,v^{*}) &= \sup_{v}\{\langle v,v^{*}\rangle : v\in\tilde{G}(p,q,u,t,x)\} \\ &= \sup_{v_1}\left\{\langle v^{*},p+q-u+\delta hv_1\rangle : v_1\in F\left(\frac{p-u}{\delta},\frac{q-u}{h},u,t,x\right)\right\}.\blacksquare \\ &= \langle p+q-u,v^{*}\rangle+\delta hH_{F(\cdot,\cdot,\cdot,t,x)}\left(\frac{p-u}{\delta},\frac{q-u}{h},u,v^{*}\right). \end{aligned}$$

Let $\varphi_i : \mathbb{R}^{3n} \to \mathbb{R}^n$; $i = 1, \ldots, m$ be Frechet differentiable functions at a point $a = (p_0, q_0, u_0)$ and $g : \mathbb{R}^{mn} \to \mathbb{R}$ be continuous convex at $(\varphi_1(a), \ldots, \varphi_m(a))$. Then for a function

$f(p,q,u) = g(\varphi_1(p,q,u), \ldots, \varphi_m(p,q,u))$, applying Theorem 2 in Section 4.5 of Ref. [111] at a point $a = (p_0, q_0, u_0)$, we have

$$\partial f(a) = \Lambda^* \partial g(\varphi_1(a), ..., \varphi_m(a)) \tag{6.127}$$

where Λ^* is the transpose of

$$\Lambda = \begin{pmatrix} \dfrac{\partial \varphi_1(a)}{\partial p} & \dfrac{\partial \varphi_1(a)}{\partial q} & \dfrac{\partial \varphi_1(a)}{\partial u} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial \varphi_m(a)}{\partial p} & \dfrac{\partial \varphi_m(a)}{\partial q} & \dfrac{\partial \varphi_m(a)}{\partial u} \end{pmatrix} \tag{6.128}$$

and $(\partial\varphi_i/\partial p)$, $(\partial\varphi_i/\partial q)$, $(\partial\varphi_i/\partial u)$, $i = 1, \ldots, m$ are the Jacobi matrices. Now, taking

$$\varphi_1(p,q,u) \equiv \frac{p-u}{\delta}, \quad \varphi_2(p,q,u) \equiv \frac{q-u}{h}, \quad \varphi_3(p,q,u) \equiv u$$

in the case of $m = 3$ from Eq. (6.128), we derive that

$$\Lambda = \begin{pmatrix} E/\delta & 0 & -E/\delta \\ 0 & E/h & -E/h \\ 0 & 0 & E \end{pmatrix},$$

where E and 0 are $n \times n$ identity and zero matrices, respectively. Thus, applying Lemma 6.2, from formula (6.127), we have the result in Theorem 6.22.

Theorem 6.22. Let $\tilde{G}(\cdot,\cdot,\cdot,t,x) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ be convex multivalued mapping. Then

$$\partial H_{\tilde{G}(\cdot,\cdot,\cdot,t,x)}(p,q,u,v^*) = \delta h \Lambda^* \partial H_{F(\cdot,\cdot,\cdot,t,x)}\left(\frac{p-u}{\delta}, \frac{q-u}{h}, u, v^*\right) + \{v^*\} \times \{v^*\} \times \{-v^*\}.$$

Observe that $\delta h \Lambda^*$ is a nonsingular partitioned matrix. Taking partitioned matrices

$$A_1 = \begin{pmatrix} hE & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \delta E \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}, \quad A_3 = (-hE \ \ \vdots \ \ -\delta E), \quad A_4 = \delta h E,$$

it follows from the Frobenius formula [175,176] for the inverse of a partitioned matrix that

$$(\delta h \Lambda^*)^{-1} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}^{-1} = \begin{pmatrix} E/h & 0 & 0 \\ 0 & E/\delta & 0 \\ E/\delta h & E/\delta h & E/\delta h \end{pmatrix}.$$

On the other hand, it can be easily checked that for the mapping in Eq. (6.126),

$$\frac{v-p-q+u}{\delta h} \in F\left(\frac{p-u}{\delta}, \frac{q-u}{h}, u, t, x\right), \quad \text{if } v \in \tilde{G}(p, q, u; v^*, t, x).$$

By applying Theorems 2.1 and 6.22, we have proved Theorem 6.23.

Theorem 6.23. The following inclusions are equivalent:

1. $(p^*, q^*, u^*) \in \tilde{G}^*(v^*; (p, q, u, v), t, x)$,
2. $\left(\frac{p^*-v^*}{h}, \frac{q^*-v^*}{\delta}, \frac{p^*+q^*+u^*-v^*}{\delta h}\right) \in F^*\left(v^*; \left(\frac{p-u}{\delta}, \frac{q-u}{h}, u, \frac{v-p-q+u}{\delta h}\right), t, x\right)$.

Remark 6.9. In accordance with Theorem 6.10, it can be pointed out that in the more general cases, i.e., in the nonconvex case and the existence of local tents $K_{\text{gph } F(\cdot,\cdot,\cdot,t,x)}(p, q, x, v)$, $K_{\text{gph } F_1(\cdot,t)}\left(u, (v_1-u)/\delta\right)$, and $K_{\text{gph } F_2(\cdot,x)}\left(u, (v_2-u)/h\right)$, the equivalence of the inclusions in Eq. (6.96) and conditions (1) and (2) of Theorem 6.23 can be proved in a similar way. We prove this fact for the mapping $\tilde{G}$ (this can be proved a similar way for the mappings F_1 and F_2). Indeed, by the definition of LAM, $(p^*, q^*, u^*, -v^*) \in K^*_{\text{gph } F(\cdot,\cdot,\cdot,t,x)}(p, q, x, v)$ means that

$$\langle \overline{p}, p^* \rangle + \langle \overline{q}, q^* \rangle + \langle \overline{u}, u^* \rangle - \langle \overline{v}, v^* \rangle \geq 0, \ (\overline{p}, \overline{q}, \overline{u}, \overline{v}) \in K_{\text{gph } F(\cdot,\cdot,\cdot,t,x)}(p, q, u, v).$$

Then it is not hard to show that this inequality can be reduced to the equivalent form

$$\left\langle \frac{\overline{p}-\overline{u}}{\delta}, \frac{p^*-v^*}{h} \right\rangle + \left\langle \frac{\overline{q}-\overline{u}}{h}, \frac{q^*-v^*}{\delta} \right\rangle + \left\langle \overline{u}, \frac{p^*+q^*+u^*-v^*}{\delta h} \right\rangle - \left\langle \frac{\overline{v}-\overline{p}-\overline{q}+\overline{u}}{\delta h}, v^* \right\rangle \geq 0,$$
$$\left(\frac{\overline{p}-\overline{u}}{\delta}, \frac{\overline{q}-\overline{u}}{h}, \overline{u}, \frac{\overline{v}-\overline{p}-\overline{q}+\overline{u}}{\delta h}\right) \in K_{\text{gph } F(\cdot,\cdot,\cdot,t,x)}\left(\frac{p-u}{\delta}, \frac{q-u}{h}, u, \frac{v-p-q+u}{\delta h}\right).$$

Therefore, by using Theorem 6.23, it is not hard to see that the adjoint discrete inclusion in Eq. (6.97) for the problem in Eqs. (6.80), (6.82), and (6.125) has the form

$$\begin{gathered}(-w^*(t+\delta, x+h), \quad -\psi^*(t+\delta, x+h), Au^*(t+\delta, x+h)) \\ \in F^*(u^*(t+\delta, x+h); (B\tilde{u}(t+\delta, x), C\tilde{u}(t, x+h), \tilde{u}(t, x), Au(t+\delta, x+h)), t, x) \\ (0,0) \times \{Bw^*(t+\delta, x+h) + C\psi^*(t+\delta, x+h) - \lambda\partial g(\tilde{u}(t, x), t, x)\},\end{gathered}$$

where

$$w^*(t+\delta,x+h) \equiv \frac{u^*(t+\delta,x+h)-\varphi^*(t+\delta,x+h)}{h},$$

$$\psi^*(t+\delta,x+h) \equiv \frac{u^*(t+\delta,x+h)-\eta^*(t+\delta,x+h)}{\delta}.$$

In conclusion, using the techniques of the proof of Theorem 6.18 and the latter adjoint inclusion, we have the result in Theorem 6.24.

Theorem 6.24. Let Eqs. (6.80), (6.82), and (6.125) be a convex problem. Then for the optimality of the solution $\tilde{u}(\cdot,\cdot)$, it is sufficient that there exist absolutely continuous functions $u^*(\cdot,\cdot)$, $v^*(\cdot,\cdot)$, and $w^*(\cdot,\cdot)$ on Q satisfying the following conditions:

1. $(-w^*(t,x),\ -\psi^*(t,x),\ u^*_{tx}(t,x)) \in F^*(u^*(t,x);(\tilde{u}_t(t,x),\tilde{u}_x(t,x),\tilde{u}(t,\cdot),\tilde{u}_{tx}(t,x)),t,x))$ $+(0,0)\times\{w^*_t(t,x)+\varphi^*_x(t,x)-\partial g(\tilde{u}(t,x),t,x)\}$,
2. $-\psi^*(t,0)\in F_1^*(u^*(t,0);((\tilde{u}(t,0),\tilde{u}_t(t,0)),t),\ \ -w^*(0,x)\in F_2^*(u^*(0,x);((\tilde{u}(0,x),\tilde{u}_x(0,x)),x)$,
3. $-w^*(1,x)\in\partial g_1(\tilde{u}(1,x),x)-u^*_x(1,x),\qquad -\psi^*(t,1)\in\partial g_2(\tilde{u}(t,1),t)-u^*_t(t,1)$,
4. $\tilde{u}_{tx}(t,x)\in F((\tilde{u}_t(t,x),\tilde{u}_x(t,x),\tilde{u}(t,x));u^*(t,x),t,x)$,
5. $\tilde{u}_t(t,0)\in F_1(\tilde{u}(t,0);u^*(t,0),t),\ \ \tilde{u}_x(0,x)\in F_2(\tilde{u}(0,x);u^*(0,x),x)$ $u^*(0,0)=u^*(1,1)=0$.

Remark 6.10. For the nonconvex problem in Eqs. (6.80), (6.82), and (6.125), the analogous conditions (a)–(e) of Theorem 6.19 and condition (2) of Theorem 6.24 are satisfied, where the condition (a) has a form

$$\begin{aligned}(-w^*(t,x),\quad -\psi^*(t,x),\quad & u^*_{tx}(t,x)\\ & +u^*(t,x))\in F^*(u^*(t,x);(\tilde{u}_t(t,x),\tilde{u}_x(t,x),\tilde{u}(t,\cdot),\tilde{u}_{tx}(t,x)),t,x),\end{aligned}$$

where the LAMs F_i^* $(i=1,2)$ are defined as follows:

$$F_i^*(v_i^*,(u,v_i))=\{u^*:H_F(u_1,v_i^*)-H_F(u,v_i^*)\le\langle u^*,u_1-u\rangle,\quad\forall u_1\in\mathbb{R}^n\},$$
$$v\in F(u,v_i^*),\quad i=1,2.$$

Let us consider the problem

$$\inf\quad J[u(\cdot,\cdot)]=\iint_Q g(u(t,x),t,x)\mathrm{d}t\,\mathrm{d}x+\int_0^1 g_1(u(1,x),x)\mathrm{d}x+\int_0^1 g_2(u(t,1),t)\mathrm{d}t \tag{6.129}$$

$$\text{subject to}\ \ u_{tx}(t,x)=Au_t(t,x)+Bu_x(t,x)+Cu(t,x),\quad (t,x)\in Q, \tag{6.130}$$

$$u_t(t,0)\in F_1(u(t,0)),\ F_1(u)=A_1u+U_1;\quad u_x(0,x)\in F_2(u(0,x)),\ F_2(u)=A_2u+U_2, \tag{6.131}$$

where A, A_i $(i=1,2)$, B, C are $n\times n$ matrices; $U_i\subset \mathrm{R}^n$ $(i=1,2)$ are convex sets, and $g(\cdot,t,x)$, $g_1(\cdot,x)$, $g_2(\cdot,t)$ are convex continuously differentiable functions. Obviously, to every admissible pair of controlling parameters $u_1(t)\in U_1$, $u_2(x)\in U_2$, there corresponds some feasible solution of the hyperbolic Eq. (6.130)—say $u_1(\cdot)$, $u_2(\cdot)$ from the class of bounded measurable functions. It is necessary to find a feasible pair of controlling parameters $\{\tilde{u}_1(\cdot),\ \tilde{u}_2(\cdot)\}$ such that the corresponding solution $\tilde{u}(\cdot,\cdot)$ minimizes $J[u(\cdot,\cdot)]$. In a way similar to that described for the problem in Eqs. (6.120)–(6.122), it is easy to see that conditions (1)–(5) of Theorem 6.24 consist of the following:

$$\begin{gathered}
-w^*(t,x)=A^*u^*(t,x), \qquad -\psi^*(t,x)=B^*u^*(t,x),\\
u^*_{tx}(t,x)=C^*u^*(t,x)+w^*_t(t,x)+\psi^*_x(t,x)-g'_u(\tilde{u}(t,x),t,x),\\
-\psi^*(t,0)=A_1^*u^*(t,0), \qquad -w^*(0,x)=A_2^*u^*(0,x),
\end{gathered}$$

$$-w^*(1,x)=g_1'(\tilde{u}(1,x),x)-u^*_x(1,x), \qquad -\psi^*(t,1)=g_2'(\tilde{u}(t,1),t)-u^*_t(t,1),$$

$$\begin{aligned}
&\langle \tilde{u}_{tx}(t,x),u^*(t,x)\rangle=H_F(\hat{u}_t(t,x),\hat{u}_x(t,x),\tilde{u}(t,x),u^*(t,x),t,x),\\
&\langle \tilde{u}_t(t,0),u^*(t,0)\rangle=H_{F_1}(\tilde{u}(t,0),u^*(t,0),t), \quad \langle \tilde{u}_x(0,x),u^*(0,x)\rangle=H_{F_2}(\tilde{u}(t,0),u^*(0,x),x).
\end{aligned}$$

Obviously these conditions can be rewritten as follows:

$$\begin{gathered}
u^*_{tx}(t,x)=-A^*u^*_t(t,x)-B^*u^*_x(t,x)+C^*u^*(t,x)-g'_u(\tilde{u}(t,x),t,x),\\
(A_1^*-B^*)u^*(t,0)=0, \quad (A_2^*-A^*)u^*(0,x)=0,\\
u^*_x(1,x)=g_1'(\tilde{u}(1,x),x)-A^*u^*(1,x),\ u^*_t(t,1)=g_2'(\tilde{u}(t,1),t)-B^*u^*(t,1),
\end{gathered} \tag{6.132}$$

$$\langle \tilde{u}_t(t,0),u^*(t,0)\rangle=\sup_{u_1\in U_1}\langle u_1,u^*(t,0)\rangle, \quad \langle \tilde{u}_x(0,x),u^*(0,x)\rangle=\sup_{u_2\in U_2}\langle u_2,u^*(0,x)\rangle. \tag{6.133}$$

Theorem 6.25. If the conditions in Eqs. (6.132) and (6.133) are satisfied, then the solution $\tilde{u}(\cdot,\cdot)$ corresponding to the optimal control pair $\{\tilde{u}_1(\cdot),\tilde{u}_2(\cdot)\}$ minimizes $J[u(\cdot,\cdot)]$ in the problem in Eqs. (6.129)–(6.131), where $u^*(0,0)=0, u^*(1,1)=0$.

Example 6.3. Consider the following example:

$$\inf \quad J(u(\cdot,\cdot))=\int_0^1 u(1,x)\mathrm{d}x-\int_0^1 u(t,1)\mathrm{d}t,$$

$$\begin{aligned}
\text{subject to}\quad u_{tx}=v,\quad &|v(t,x)|\le 1,\ (t,x)\in Q,\\
&u(t,0)=0,\ \ t\in[0,1],\\
&u(0,x)=0,\ \ x\in[0,1].
\end{aligned} \tag{6.134}$$

It is assumed that $v(\cdot,\cdot)$ is a piecewise continuous function (control) with a finite number of discontinuity lines. In our notations, the problem in Eq. (6.134) has the form

$$
\begin{aligned}
&\inf \quad J(x(\cdot,\cdot)) = \int_0^1 u(1,x)\mathrm{d}x - \int_0^1 u(t,1)\mathrm{d}t,\\
&\text{subject to} \quad u_{tx} \in F(u), \quad F(u) = U = [-1,\ +1],\ (t,x) \in Q,\\
&\qquad\qquad\qquad\qquad\quad u(t,0) = 0,\ \ t \in [0,1],\\
&\qquad\qquad\qquad\qquad\quad u(0,x) = 0,\ \ x \in [0,1].
\end{aligned}
\tag{6.135}
$$

For the mapping $F(u) = U$ (see Section 2.4), we have

$$
F^*(y^*; z_0) = \begin{cases} 0, & \text{if } \ -y^* \in [\mathrm{cone}(U - v_0)]*, \\ \varnothing, & \text{if } \ -y^* \notin [\mathrm{cone}(U - v_0)]* \end{cases} z_0 = (u_0, v_0).
$$

Thus, the adjoint hyperbolic DFI is $u^*_{tx} = 0$. Since in the problem in Eq. (6.135), $g \equiv 0$, $g_1(u(1,x),x) = u(1,x)$, $g_2(u(t,1),t) = -u(t,1)$, conditions (i)–(iii) of Theorem 6.18 imply that

$$
u_x^{*'}(1,x) = -1, \quad u_t^{*'}(t,1) = 1, u^*(1,1) = u^*(0,0) = 0, \quad \partial g(\tilde{u}(t,x),t,x) = \{0\}.
$$

It is easy to verify that $u^*(t,x) = t - x$ being a solution of the DFI $u^*_{tx} = 0$ satisfies the latter conditions. We show that

$$
\tilde{u}(t,x) = \begin{cases} tx - 2t^2 + p(t), & \text{if } \ 0 \le t < x \le 1, \\ -tx + p(x), & \text{if } \ 0 \le x < t \le 1 \end{cases}
\tag{6.136}
$$

is an optimal solution of problem (6.135) and hence of Eq. (6.134), where $p(p(0) = 0)$ is an arbitrary smooth function. Indeed from condition

$$
H_F(\tilde{v}(t,x), u^*(t,x)) = \tilde{v}(t,x) \cdot u^*(t,x)
$$

it follows that $\tilde{v}(t,x) = +1$ if $u^*(t,x) > 0$ or $t > x$. Otherwise, if $u^*(t,x) < 0$ (or $t < x$), then $\tilde{v}(t,x) = -1$. Thus,

$$
\tilde{v}(t,x) = \begin{cases} +1, & \text{if } \ 0 \le x < t \le 1, \\ -1, & \text{if } \ 0 \le t < x \le 1 \end{cases}
$$

and the admissible pair $(\tilde{u}(t,x), \tilde{v}(t,x))$ and the function $u^*(t,x) = t - x$ satisfy the hypotheses of Theorem 6.18. Here the least value of the cost functional J does not depend on the function p. Indeed,

$$
J(\tilde{u}(t,x)) = \int_0^1 (-x + p(x))\mathrm{d}x - \int_0^1 (t - 2t^2 + p(t))\mathrm{d}t = -\frac{1}{3}.
$$

Note that as is seen from Eq. (6.136), the function $\tilde{u}(t,x)$ is not defined uniquely because of the arbitrariness of the function p. The reason is that the line $x = t$ is a line of discontinuity (of the first kind). Obviously, by means of the choice of a function p it can always be arranged that the solution $\tilde{u}(t,x)$, as well as its first-order partial derivatives, be continuous on the line $x = 1$.

4 On Duality in the Convex Problem (P_C)

Consider the convex Darboux problem for a hyperbolic DFI with state constraint

$$\begin{aligned} &\inf \quad J[u(\cdot,\cdot)] = \iint_Q g(u(t,x),t,x)\mathrm{d}t\,\mathrm{d}x, \\ &\text{subject to} \quad u_{tx}(t,x)\in F(u(t,x)), \quad (t,x)\in Q=[0,1]\times[0,1], \\ &\qquad u(t,x)\in\Phi(t,x), \\ &\qquad u(t,0)=0, \quad u(0,x)=0, \end{aligned} \tag{6.137}$$

where $\Phi : Q\to P(\mathbb{R}^n)$ is a convex-valued mapping.

Theorem 6.26. Let g be continuous and convex with respect to u, and let F be a closed convex multivalued function. Moreover, let Φ be a convex-valued mapping. Then for the optimality of the solution $\tilde{u}(t,x)$ among all feasible solutions of the problem in Eq. (6.137), it is sufficient that there exists an absolutely continuous function $u^*(t,x)$ on Q with an integrable mixed derivative $u^*_{tx}(t,x)$ such that the following conditions hold almost everywhere:

a. $u^*_{tx}(t,x)\in F^*(u^*(t,x);(\tilde{u}(t,x),\tilde{u}^*_{tx}(t,x)) + K^*_{\Phi(t,x)}(\tilde{u}(t,x))$,

b. $u^*_t(t,1)\in K^*_{\Phi(t,1)}(\tilde{u}(t,1)),\ u^*_x(1,x)\in K^*_{\Phi(1,x)}(\tilde{u}(1,x))$,
$u^*(0,0)=u^*(1,1)=0$,

c. $\tilde{u}_{tx}(t,x)\in F(\tilde{u}(t,x);u^*(t,x))$.

☐ The proof is similar to the proof of Theorem 6.18. ■

The maximization problem

$$\sup_{\substack{u^*(t,x),\varphi^*(t,x),z^*(t,x),\\ u^*(0,0)=u^*(1,1)}} J_*[u^*(t,x),\varphi^*(t,x),z^*(t,x)] \tag{6.138}$$

where

$$J_*[u^*(t,x),\varphi^*(t,x),z^*(t,x)] = \iint_Q [M(u^*_{tx}(t,x)-\varphi^*(t,x)+z^*(t,x),u^*(t,x))$$
$$-W_{\Phi(t,x)}(-\varphi^*(t,x)) - g^*(z^*(t,x),t,x)]\mathrm{d}t\mathrm{d}x - \int_0^1 W_{\Phi(t,1)}(u^*_t(t,1))\mathrm{d}t - \int_0^1 W_{\Phi(1,x)}(u^*_x(1,x))\mathrm{d}x$$

is called the dual problem to the primary convex problem in Eq. (6.137). It is assumed that $\varphi^*(\cdot,\cdot)$, $z^*(\cdot,\cdot)$ are absolutely continuous functions on Q and that $u^*(\cdot,\cdot)$ is an absolutely continuous function on Q having an integrable mixed partial derivative $u^*_{tx}(\cdot,\cdot)$.

Theorem 6.27. The inequality

$$J(\tilde{u}(t,x)) \geq J_*[x^*(t,x), \varphi^*(t,x), z^*(t,x)]$$

holds for all feasible solutions $u(t,x)$ and $\{u^*(t,x), \varphi^*(t,x), z^*(t,x)\}$ of the primary problem in Eq. (6.137) and the dual problem in Eq. (6.138), respectively.

□ It is clear from the definitions of the functions M, W_Φ, and conjugate g^* that

$$\begin{gathered}
M(u^*_{tx}(t,x) - \varphi^*(t,x) + z^*(t,x), u^*(t,x)) \\
\leq \langle u^*_{tx}(t,x) - \varphi^*(t,x) + z^*(t,x), u(t,x)\rangle - \langle u^*(t,x), u_{tx}(t,x)\rangle, \\
W_{\Phi(t,x)}(\varphi^*(t,x)) \geq \langle \varphi^*(t,x), u^*(t,x)\rangle, \\
W_{\Phi(t,1)}(u^*_t(t,1)) \geq \langle u^*_t(t,1), u(t,1)\rangle, \quad t \in [0,1], \\
W_{\Phi(1,x)}(u^*_x(1,x)) \geq \langle u^*_x(1,x), u(1,x)\rangle, \quad x \in [0,1].
\end{gathered}$$

Using these inequalities, we deduce

$$\begin{aligned}
&J_*[u^*(t,x), \varphi^*(t,x), z^*(t,x)] \\
&\quad \leq \iint_Q [\langle u^*_{tx}(t,x), u(t,x)\rangle - \langle u^*(t,x), u_{tx}(t,x)\rangle + g(u(t,x),t,x)]\mathrm{d}t\,\mathrm{d}x \\
&\quad - \int_0^1 \langle u^*_t(t,1), u(t,1)\rangle \mathrm{d}t - \int_0^1 \langle u^*_x(1,x), u(1,x)\rangle \mathrm{d}x
\end{aligned} \tag{6.139}$$

But since

$$\begin{aligned}
&\iint_Q \langle u^*(t,x), u_{tx}(t,x)\rangle \mathrm{d}t\,\mathrm{d}x = \int_0^1 \langle u^*(1,x), u_x(1,x)\rangle \mathrm{d}x \\
&\quad - \int_0^1 \langle u^*(0,x), u_x(0,x)\rangle \mathrm{d}x - \iint_Q \langle u^*_t(t,x), u_x(t,x)\rangle \mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\iint_Q \langle u(t,x), u^*_{tx}(t,x)\rangle \mathrm{d}t\,\mathrm{d}x = \int_0^1 \langle u(t,1), u^*_t(t,1)\rangle \mathrm{d}t \\
&\quad - \int_0^1 \langle u(t,0), u_t(t,0)\rangle \mathrm{d}t - \iint_Q \langle u^*_t(t,x), u_x(t,x)\rangle \mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

We obtain from the inequality in Eq. (6.139) that

$$\begin{aligned}J_*[u^*(t,x),\varphi^*(t,x),z^*(t,x)] &\leq -\int_0^1 d_x\langle u^*(1,x),u(1,x)\rangle \mathrm{d}x \\ &+J[u(t,x)]+\int_0^1\langle u^*(0,x),u_x(0,x)\rangle \mathrm{d}x-\int_0^1\langle u(t,0),u_t(t,0)\rangle \mathrm{d}t\end{aligned} \tag{6.140}$$

The solutions $u(t,x)$ and $\{u^*(t,x),\varphi^*(t,x),z^*(t,x)\}$ are feasible solutions and $u(t,0)=u(0,x)=u^*(1,1)=u^*(0,0)=0$. So it is not hard to show that the right-hand side of the latter inequality equals zero. Indeed,

$$\begin{aligned}&\int_0^1\langle u^*(0,x),u_x(0,x)\rangle \mathrm{d}x-\int_0^1 d_x\langle u^*(1,x),u(1,x)\rangle \mathrm{d}x=\int_0^1 d_x\langle u^*(0,x),u(0,x)\rangle \mathrm{d}x \\ &\quad-\int_0^1\langle u_x^*(0,x),u(0,x)\rangle \mathrm{d}x-\int_0^1 d_x\langle u^*(1,x),u(1,x)\rangle \mathrm{d}x \\ &\quad=\langle u^*(1,0),u(1,0)\rangle-\langle u^*(1,1),u(1,1)\rangle=0.\end{aligned}$$

That is why for all feasible solutions $J(\tilde{u}(t,x))\leq J_*[u^*(t,x),\varphi^*(t,x),z^*(t,x)]$. ■

Theorem 6.28. If the functions $\tilde{u}(t,x)$ and $\{u^*(t,x),\varphi^*(t,x),z^*(t,x)\}$, where $z^*(t,x)\in\partial g(\tilde{u}(t,x),t,x)$, satisfy conditions (a)–(c) of Theorem 6.26, then they are optimal solutions of the primary and dual problems, respectively, and their values are equal.

□ Actually, by Theorem 6.26, $\tilde{u}(t,x)$ is a solution of the primary problem. We study the remaining assertions. By the definition of LAM, condition (a) implies that

$$\begin{aligned}&\langle u_{tx}^*(t,x)-\varphi^*(t,x)+z^*(t,x),u-\tilde{u}(t,x)\rangle-\langle u^*(t,x),v-\tilde{u}_{tx}(t,x)\rangle\geq 0,\\ &(u,v)\in \operatorname{gph} F;\quad W_{\Phi(t,x)}(\varphi^*(t,x))=\langle\varphi^*(t,x),\tilde{u}(t,x)\rangle.\end{aligned}$$

This means that

$$\begin{aligned}&(u_{tx}^*(t,x)-\varphi^*(t,x)+z^*(t,x),u^*(t,x))\in \operatorname{dom} M,\\ &\operatorname{dom} M=\{(u^*,-v^*):M(u^*,v^*)>-\infty\}.\end{aligned}$$

On the other hand, it is known that $\partial g(u,t,x)\subset \operatorname{dom}\, g^*(\cdot,t,x)$ and so $z^*(t,x)\in \operatorname{dom}\, g^*(\cdot,t,x)$. Thus, the triplet $\{u^*(t,x),\varphi^*(t,x),z^*(t,x)\}$ is a feasible solution. By

Lemma 2.6, it is also clear that

$$\begin{aligned} &M(u^*_{tx}(t,x) - \varphi^*(t,x) + z^*(t,x), u^*(t,x)) \\ &\quad = \langle u^*_{tx}(t,x) - \varphi^*(t,x) + z^*(t,x), \tilde{u}(t,x)\rangle - H_F(\tilde{u}(t,x), u^*(t,x)). \end{aligned} \tag{6.141}$$

By conditions (b) and (c) of Theorem 6.26, we have

$$\begin{aligned} W_{\Phi(t,1)}(u^*_t(t,1)) &= \langle u^*_t(t,1), \tilde{u}(t,1)\rangle, \quad t\in[0,1], \\ W_{\Phi(1,x)}(u^*_x(1,x)) &= \langle u^*_x(1,x), \tilde{u}(1,x)\rangle, \quad x\in[0,1], \\ H_F(\tilde{u}(t,x), u^*(t,x)) &= \langle \tilde{u}^*_{tx}(t,x), u^*(t,x)\rangle, (t,x)\in Q. \end{aligned} \tag{6.142}$$

Since $z^*(t,x)\in\partial g(\tilde{u}(t,x),t,x)$, we can write (Theorem 1.26)

$$\langle \tilde{u}(t,x), z^*(t,x)\rangle - g(\tilde{u}(t,x),t,x) = g^*(z^*(t,x),t,x). \tag{6.143}$$

Then, in view of Eqs. (6.141) and (6.142), having instead of the inequalities in Eqs. (6.139) and (6.140) the corresponding equalities and following the proof of Theorem 6.27, it is easy to establish that

$$J(\tilde{u}(t,x)) = J_*[u^*(t,x), \varphi^*(t,x), z^*(t,x)]$$

and so $\{u^*(t,x),\varphi^*(t,x),z^*(t,x)\}$ is an optimal solution. ■

6.5 Optimal Control of the Elliptic-Type Discrete and Differential Inclusions with Dirichlet and Neumann Boundary Conditions via Approximation

This section deals for the first time with the Dirichlet problem for discrete (P_D) inclusion, discrete-approximation problems on a uniform grid, and differential (P_C) inclusions of elliptic type. Necessary and sufficient conditions for optimality in the form of a Euler−Lagrange inclusion are derived for the problems under consideration on the basis of LAM. The results obtained are generalized to the multidimensional case with a second-order elliptic operator.

It must be pointed out that in elliptic differential inclusions, the solution is taken in the space of classical solutions. However, as will be seen from the context, the definition below of the concept of a solution in this or that sense is introduced only for simplicity and does not in any way restrict the class of problems under consideration. Therefore, at the end of the section, we indicate general ways of extending the results to the case of generalized solutions [246].

Let us mention the notations:

$$H(u,v^*) = \sup_v\{\langle v,v^*\rangle : v\in F(u)\}, \quad F(u,v^*) = \{v\in F(u) : \langle v,v^*\rangle = H(u,v^*)\}, \ v^*\in\mathbb{R}^n,$$

where $F : \mathbb{R}^{4n} \to P(\mathbb{R}^n)$.

For a convex mapping F, by definition of LAM, a multivalued mapping from $\mathbb{R}^n$ into $\mathbb{R}^{4n}$

$$F^*(v^*,(u,v)) = \{u^* : (u^*, -v^*) \in K^*_{\mathrm{gph}\ F}(u,v)\},$$

where $K^*_{\mathrm{gph}\ F}(u,v)$ is the dual to the cone $K_{\mathrm{gph}\ F}(u,v)$.

And the following multivalued mapping defined by

$$F^*(v^*,(u^0,v^0)) = \{u^* : H(u,v^*) - H(u^0,v^*) \le \langle u^*, u-u^0\rangle \ \ \forall (u,v)\in \mathrm{gph}\ F\},$$
$$v^0 \in F(u^0,v^*)$$

is the LAM to the nonconvex mapping F at a point $(u^0,v^0)\in \mathrm{gph}\ F$.

As before, we denote

$$M(u^*,v^*) = \inf\{\langle u,u^*\rangle - \langle v,v^*\rangle : (u,v)\in \mathrm{gph}\ F\}.$$

We consider the following optimization problem:

$$\inf \sum_{x_1=1,\ldots,T-1,x_2=1,\ldots,L-1} g_{x_1,x_2}(u_{x_1,x_2}) \tag{6.144}$$

$$\text{subject to } \ u_{x_1+1,x_2} \in F_{x_1,x_2}(u_{x_1-1,x_2}, u_{x_1,x_2-1}, u_{x_1,x_2}, u_{x_1,x_2+1}), \tag{6.145}$$

$$\begin{aligned} &u_{x_1,0} = \alpha_{0x_1}, \quad u_{x_1,L} = \alpha_{Lx_1}, \quad u_{0,x_2} = \beta_{0x_2}, \quad u_{T,x_2} = \beta_{Tx_2} \\ &x_1 = 1,\ldots,T-1; \quad x_2 = 1,\ldots,L-1, \end{aligned} \tag{6.146}$$

where $g_{x_1,x_2} : \mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$ are functions taking values on the extended line, F_{x_1,x_2} are multivalued mappings, $F_{x_1,x_2} : \mathbb{R}^{4n} \to P(\mathbb{R}^n)$, α_{0x_1}, α_{Lx_1}, β_{0x_2}, β_{Tx_2} are fixed vectors, and T, L are natural numbers. We label this problem (P_D) and call it the Dirichlet problem for a discrete inclusion of elliptic type.

Let us denote $D = \{(x_1,x_2) : x_1 = 0, \ldots, T; x_2 = 0, \ldots, L, (x_1,x_2) \ne (0,0),(0,L),(T,0),(T,L)\}$. Then a set of points $\{u_{x_1,x_2}\}_D = \{u_{x_1,x_2} : (x_1,x_2)\in D\}$ is called a feasible solution for the problem (P_D) if it satisfies the inclusions in Eq. (6.146). It is easy to see that for each fixed T and L, the boundary condition in Eq. (6.146) enables us to choose some feasible solution, and the number of points to be determined and discrete inclusions are equal. In this sense, the name *elliptic discrete inclusions* is justified. The following condition is assumed below for the functions g_{x_1,x_2} and the mappings $F_{x_1,x_2}(x_1 = 1,\ldots,T-1;\ \ x_2 = 1,\ldots,L-1)$.

Hypothesis H1. Assume that in the problem (P_D), the mappings F_{x_1,x_2} are such that the cone of tangent directions $K_{\mathrm{gph}\ F_{x_1,x_2}}(\tilde{u}_{x_1-1,x_2}, \tilde{u}_{x_1,x_2-1}, \tilde{u}_{x_1,x_2}, \tilde{u}_{x_1,x_2+1}, \tilde{u}_{x_1+1,x_2})$ are local tents, where $\tilde{u}_{x_1,x_2}$ are the points of the optimal solution $\{\tilde{u}_{x_1,x_2}\}_D$. Assume, moreover, that the functions g_{x_1,x_2} admit a CUA $h_{x_1,x_2}(\overline{u},\tilde{u}_{x_1,x_2})$ at the points $\tilde{u}_{x_1,x_2}$ that is continuous with respect to $\overline{u}$. This implies that the subdifferentials $\partial g_{x_1,x_2}(\tilde{u}_{x_1,x_2}) :\ = \partial h_{x_1,x_2}(0,\tilde{u}_{x_1,x_2})$ are defined.

The problem (P_D) is said to be convex if the mappings F_{x_1,x_2} are convex and g_{x_1,x_2} are convex proper functions.

Hypothesis H2. Suppose the problem (P_D) is convex and $\{u^0_{x_1,x_2}\}_D$ is some feasible solution for it. Then suppose that

$$(u^0_{x_1-1,x_2}, u^0_{x_1,x_2-1}, u^0_{x_1,x_2}, u^0_{x_1,x_2+1}, u^0_{x_1+1,x_2}) \in \text{ri gph} F_{x_1,x_2}$$
$$u^0_{x_1,x_2} \in \text{ri dom } g_{x_1,x_2}, \quad x_1 = 1,\ldots,T-1, \;\; x_2 = 1,\ldots,L-1$$

In Subsection 3, we study the following problem for an elliptic differential inclusion:

$$\inf \quad J(u(\cdot)) := \iint_{\mathrm{R}} g(u(x),x)\mathrm{d}x \tag{6.147}$$

$$\text{subject to } \Delta u(x) \in F(u(x),x), \quad x \in \mathrm{R} \tag{6.148}$$

$$u(x) = \beta(x), \quad x \in B \tag{6.149}$$

where Δ is Laplace's operator: $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, $\quad F(\cdot,x) : \mathbb{R}^n \to P(\mathbb{R}^n)$ is a multivalued mapping for all $x = (x_1,x_2)$ in the bounded region $\mathrm{R} \subset \mathbb{R}^2$, which has a closed piecewise-smooth simple curve B as its boundary, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous convex function on u, β is continuous, and $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2$.

We label this continuous problem (P_C) and call it the Dirichlet problem for elliptic differential inclusions. The problem is to find a solution $\tilde{u}(x)$ of the boundary value problem in Eqs. (6.148) and (6.149) that minimizes $J(u(\cdot))$. Here, for simplicity of the exposition, a feasible solution is understood to be a classical solution. At the end of Subsection 4, we introduce the concept of a generalized solution and show that it is possible to carry over the results to this case.

Consider the following multidimensional optimal control problem (P_M) for elliptic differential inclusions:

$$\inf J(u(\cdot)) := \iint_G g(u(x),x)\mathrm{d}x \tag{6.150}$$

$$\text{subject to } Lu(x)) \in F(u(x),x), \quad x \in G \tag{6.151}$$

$$u(x) = \alpha(x), \quad x \in S \tag{6.152}$$

where $F(\cdot,x) : \mathbb{R} \to P(\mathbb{R})$ is a convex closed multivalued mapping for all n-dimensional vectors $x = (x_1,\ldots,x_n)$ in the bounded set $G \subset \mathbb{R}^n$, which has a closed piecewise-smooth surface S as its boundary, $g : \mathbb{R} \times G \to \mathbb{R}$ is a continuous function convex on u, α is continuous, $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n$, and L is the second-order elliptic operator:

$$Lu := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial u_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$
$$a_{ij}(x) \in C^1(\overline{G}), \;\; b_i(x) \in C^1(\overline{G}), \;\; c(x) \in C(\overline{G}),$$

where $||a_{ij}(x)||$ is a positive definite matrix, and $C(\overline{G})$ and $C^1(\overline{G})$ are the spaces of continuous functions and functions having a continuous derivative in G, respectively.

A function $u(x)$ in $C^2(G) \cap C(\overline{G})$, that satisfies the inclusion in Eq. (6.151) in G and the boundary condition in Eq. (6.152) on S, we call a classical solution of the problem posed, where $C^2(G)$ is the space of functions $u(x)$ having continuous second-order derivatives $(\partial^2 u/\partial x_i \partial x_j)$, $i, j = 1, \ldots, n$. It is required to find a classical solution $\tilde{u}(x)$ of the boundary value problem (P_M) that minimizes $J(u(\cdot))$.

1 Necessary and Sufficient Conditions for the Dirichlet Problem of Elliptic Discrete Inclusions

At first we consider the convex problem (P_D). In order to use convex programming results, we form the $m = 2n(L-1) + n(T-1)(L+1)$-dimensional vector $w = (u_0, u_1, \ldots, u_T)$, where for $x_1 = 1, \ldots, T-1$, $u_{x_1} = (u_{x_1,0}, u_{x_1,1}, \ldots, u_{x_1,L}) \in \mathbb{R}^{n(L+1)}$ is an $n(L+1)$-dimensional vector, and $u_0 = (u_{0,1}, \ldots, u_{0,L-1}) \in \mathbb{R}^{n(L-1)}$, $u_T = (u_{T,1}, \ldots, u_{T,L-1}) \in \mathbb{R}^{n(L-1)}$. Let us consider the following convex sets defined in the space $\mathbb{R}^m$:

$$M_{x_1,x_2} = \{w = (u_0, u_1, \ldots, u_T) : (u_{x_1-1,x_2}, u_{x_1,x_2-1}, u_{x_1,x_2}, u_{x_1,x_2+1}, u_{x_1+1,x_2}) \in \operatorname{gph} F_{x_1,x_2}\},$$
$$x_1 = 1, \ldots, T-1, \quad x_2 = 1, \ldots, L-1,$$

$$\begin{aligned}
H_1 &= \{w = (u_0, \ldots, u_T) : u_{x_1,0} = \alpha_{0x_1}, x_1 = 1, \ldots, T-1\},\\
H_2 &= \{w = (u_0, \ldots, u_T) : u_{0,x_2} = \beta_{0x_2}, x_2 = 1, \ldots, L-1\},\\
H_L &= \{w = (u_0, \ldots, u_T) : u_{x_1,L} = \alpha_{Lx_1}, x_1 = 1, \ldots, T-1\},\\
H_T &= \{w = (u_0, \ldots, u_T) : u_{T,x_2} = \beta_{Tx_2}, x_2 = 1, \ldots, L-1\}.
\end{aligned}$$

Now setting

$$g(w) = \sum_{x_1 = 1,\ldots,T-1; x_2 = 1,\ldots,L-1} g_{x_1,x_2}(u_{x_1,x_2})$$

we reduce the convex problem (P_D) to the following convex minimization problem in the space $\mathbb{R}^m$:

$$\inf \quad g(w) \ \text{ subject to } \ w \in N = \left(\bigcap_{\substack{x_1 = 1,\ldots,T-1\\ x_2 = 1,\ldots,L-1}} M_{x_1,x_2} \right) \cap H_1 \cap H_2 \cap H_L \cap H_T. \tag{6.153}$$

We apply Theorem 3.4 to the convex minimization problem in Eq. (6.153). For this, it is necessary to calculate the dual cones $K^*_{M_{x_1,x_2}}(w)$, $K^*_{H_1}(w)$, $K^*_{H_2}(w)$, $K^*_{H_L}(w)$, $K^*_{H_T}(w)$, $\quad w \in N$.

Lemma 6.3. $K^*_{M_{x_1,x_2}}(w)=\{w^*=(u^*_0,\ldots,u^*_T:(u^*_{x_1-1,x_2},u^*_{x_1,x_2-1},u^*_{x_1,x_2},u^*_{x_1,x_2+1},u^*_{x_1+1,x_2})$ $\in K^*_{\text{gph}Fx_1,x_2}(u_{x_1-1,x_2},u_{x_1,x_2-1},u_{x_1,x_2},\ u_{x_1,x_2+1},u_{x_1+1,x_2}),\ u^*_{i,j}=0,(i,j)\neq(x_1-1,x_2),$ $(x_1,x_2-1),$

$$(x_1,x_2),(x_1,x_2+1),(x_1+1,x_2)\},\quad x_1=1,\ldots,T-1,\ \ x_2=1,\ldots,L-1.$$

□ Let $\overline{w}\in K^*_{M_{x_1,x_2}}(w),\ \ w\in N$. This means that $w+\lambda\overline{w}\in M_{x_1,x_2}$ for sufficiently small $\lambda>0$, which is the same as

$$(u_{x_1-1,x_2}+\lambda\overline{u}_{x_1-1,x_2},u_{x_1,x_2-1}+\lambda\overline{u}_{x_1,x_2-1},u_{x_1,x_2}+\lambda\overline{u}_{x_1,x_2},u_{x_1,x_2+1}$$
$$+\lambda\overline{u}_{x_1,x_2+1},u_{x_1+1,x_2}+\lambda\overline{u}_{x_1+1,x_2},)\in\text{gph}\,F_{x_1,x_2}.$$

Thus,

$$K_{M_{x_1,x_2}}(w)=\{\overline{w}=(\overline{u}_0,\ldots,\overline{u}_T):(\overline{u}_{x_1-1,x_2},\overline{u}_{x_1,x_2-1},\overline{u}_{x_1,x_2},\overline{u}_{x_1,x_2+1},\overline{u}_{x_1+1,x_2})$$
$$\in K_{F_{x_1,x_2}}(u_{x_1-1,x_2},u_{x_1,x_2-1},u_{x_1,x_2},u_{x_1,x_2+1},u_{x_1+1,x_2}).\tag{6.154}$$

On the other hand, $w^*\in K^*_{M_{x_1,x_2}}(w)$ is equivalent to the condition

$$\langle\overline{w},w^*\rangle=\sum_{\substack{x_1=1,\ldots,T-1\\ x_2=1,\ldots,L-1}}\langle\overline{u}_{i,j},u^*_{i,j}\rangle\geq0,\quad\overline{w}\in K_{M_{x_1,x_2}}(w),$$

where the components $\overline{u}_{i,j}$ of the vector $\overline{w}$ (see Eq. (6.154)) are arbitrary. Therefore, the last relation is valid only for $u^*_{i,j}=0$, $(i,j)\neq(x_1-1,x_2)$, (x_1,x_2-1), (x_1,x_2), (x_1,x_2+1), (x_1+1,x_2). This ends the proof of the lemma. ■

It is also easy to show that

$$\begin{aligned}&K^*_{H_1}(w)=\{w^*=(u^*_0,\ldots,u^*_T):u^*_{x_1,x_2}=0,x_1=1,\ldots,T-1,x_2\neq0,u^*_0=u^*_T=0\},\\&K^*_{H_2}(w)=\{w^*=(u^*_0,\ldots,u^*_T):u^*_{x_1}=0,x_1=1,\ldots,T\},\\&K^*_{H_L}(w)=\{w^*=(u^*_0,\ldots,u^*_T):u^*_{x_1,x_2}=0,x_1=1,\ldots,T-1,x_2\neq L,u^*_0=u^*_T=0\},\\&K^*_{H_T}(w)=\{w^*=(u^*_0,\ldots,u^*_T):u^*_0=0,u^*_{x_1}=0,x_1=1,\ldots,T-1\}.\end{aligned}\tag{6.155}$$

Theorem 6.29. Assume that F_{x_1,x_2}, $x_1=1,\ldots,T-1$; $x_2=1,\ldots,L-1$ are convex multivalued mappings, and g_{x_1,x_2} are convex proper functions continuous at the points of some feasible solution $\{u^0_{x_1,x_2}\}_D$. Then for the $\{\tilde{u}_{x_1,x_2}\}_D$ to be an optimal solution of the problem (P_D), it is necessary that there exist a number $\lambda=0$ or 1 and vectors $\{\psi^*_{x_1,x_2}\},\{\eta^*_{x_1,x_2}\},\{\xi^*_{x_1,x_2}\},\{u^*_{x_1,x_2}\}$, not all equal to zero, such that:

i. $(\psi^*_{x_1,x_2},\xi^*_{x_1,x_2},u^*_{x_1,x_2},\eta^*_{x_1,x_2})\in F^*_{x_1x_2}(u^*_{x_1,x_2},(\tilde{u}_{x_1-1,x_2},\tilde{u}_{x_1,x_2-1},\tilde{u}_{x_1,x_2},\tilde{u}_{x_1,x_2+1},\tilde{u}_{x_1+1,x_2}))+\{0\}\times\{0\}\times$ $\{\psi^*_{x_1+1,x_2}+\xi^*_{x_1,x_2+1}+\eta^*_{x_1,x_2-1}-\lambda\partial g_{x_1,x_2}(\tilde{u}_{x_1,x_2})\}\times\{0\}$

ii. $\begin{aligned}&\psi^*_{0,x_2}=0,\quad u^*_{T,x_2}=0,\ x_2=1,\ldots,L-1;\\&\eta^*_{x_1,0}=0,\quad\xi^*_{x_1,L}=0,\ x_1=1,\ldots,T-1.\end{aligned}$

Under Hypothesis H2, the conditions (i) and (ii) are also sufficient for the optimality of $\{\tilde{u}_{x_1,x_2}\}_D$.

□ By the hypothesis of the theorem, $\tilde{w} = (\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_T)$is a solution of the convex minimization problem in Eq. (6.153) and $g(w)$ is continuous at the point $w^0 = (u_0^0, \ldots, u_T^0)$. Then we can assert the existence of vectors

$w^*(x_1,x_2) \in K^*_{M_{x_1,x_2}}(\tilde{w})$, $\tilde{w}^* \in K^*_{H_1}(\tilde{w})$, $\hat{w}^* \in K^*_{H_2}(\tilde{w})$, $w^{L^*} \in K^*_{H_L}(\tilde{w})$, $w^{T^*} \in K^*_{H_T}(\tilde{w})$, $w^{0*} \in \partial_w g(\tilde{w})$ and of a number $\lambda = 0$ or 1, not all equal to zero, such that

$$\sum_{\substack{x_1 = 1,\ldots,T-1\\ x_2 = 1,\ldots,L-1}} w^*(x_1,x_2) + \overline{w}^* + \hat{w}^* + w^{L^*} + w^{T^*} = \lambda w^{0*}. \tag{6.156}$$

Let $[w^*]_{x_1,x_2}$ denote the components of the vector w^* for the given pair (x_1,x_2). Then using Lemma 6.3 and the relations in Eq. (6.155), we get

$$\left[\sum_{\substack{x_1 = 1,\ldots,T-1\\ x_2 = 1,\ldots,L-1}} w^*(x_1,x_2) + \overline{w}^* + \hat{w}^* + w^{L^*} + w^{T^*}\right]_{x_1,x_2} \tag{6.157}$$

$$= \begin{cases} u^*_{0,x_2}(1,x_2) + \hat{u}^*_{0,x_2}, & \text{if } x_1 = 0, x_2 = 1,\ldots,L-1, \\ u^*_{T,x_2}(T-1,x_2) + u^{T^*}_{T,x_2}, & \text{if } x_1 = T, x_2 = 1,\ldots,L-1, \\ u^*_{x_1,0}(x_1,1) + \overline{u}^*_{x_1,0}, & \text{if } x_1 = 1,\ldots,T-1, x_2 = 0, \\ u^*_{x_1,L}(x_1,L-1) + u^{L^*}_{x_1,L}, & \text{if } x_1 = 1,\ldots,T-1, x_2 = L, \end{cases}$$

where it is taken into account that

$$[\hat{w}^*]_{0,x_2} = \hat{u}^*_{0,x_2}, \quad [w^{T^*}]_{T,x_2} = u^{T^*}_{T,x_2}, \quad [\overline{w}^*]_{x_1,0} = \overline{u}^*_{x_1,0}, \quad [w^{L^*}]_{x_1,L} = u^{L^*}_{x_1,L}.$$

Because of the arbitrariness of the vectors $\hat{u}^*_{0,x_2}$, $u^{T^*}_{T,x_2}$, $x_2 = 1,\ldots,L-1$, $\overline{u}^*_{x_1,0}$, $u^{L^*}_{x_1,L}$, $x_1 = 1,\ldots,T-1$, it follows from Eqs. (6.156) and (6.157) that

$$\begin{aligned} &u^*_{0,x_2}(1,x_2) + \hat{u}^*_{0,x_2} = 0, \quad u^*_{T,x_2}(T-1,x_2) + u^{T^*}_{T,x_2} = 0, \\ &u^*_{x_1,0}(x_1,1) + \overline{u}^*_{x_1,0} = 0, \quad u^*_{x_1,L}(x_1,L) + u^{L^*}_{x_1,L} = 0, \ x_1 = 1,\ldots,T-1, \ x_2 = 1,\ldots,L-1. \end{aligned}$$

Thus, Eq. (6.156) implies that

$$\begin{aligned} &u^*_{x_1,x_2}(x_1+1,x_2) + u^*_{x_1,x_2}(x_1,x_2+1) + (u^*_{x_1,x_2}(x_1,x_2) + u^*_{x_1,x_2}(x_1,x_2-1) + u^*_{x_1,x_2}(x_1-1,x_2) \\ &\quad = \lambda u^{0*}_{x_1,x_2}, \ u^*_{1,x_2}(0,x_2) = 0, \ u^*_{x_1,1}(x_1,0) = 0, \ u^*_{x_1,L-1}(x_1,L) = 0, \ u^*_{T-1,x_2}(T,x_2) = 0, \\ &[w^{0*}]_{x_1,x_2} = u^{0*}_{x_1,x_2}, \ x_1 = 1,\ldots,T-1, \ x_2 = 1,\ldots,L-1. \end{aligned} \tag{6.158}$$

Using Lemma 6.3 and the definition of an LAM, it can be concluded that

$$\begin{aligned}&(u^*_{x_1-1,x_2}(x_1,x_2), u^*_{x_1,x_2-1}(x_1,x_2),\ \ u^*_{x_1,x_2}(x_1,x_2),\ \ u^*_{x_1,x_2+1}(x_1,x_2))\\ &\in F^*_{x_1,x_2}(-u^*_{x_1+1,x_2}(x_1,x_2),\ \ (\tilde{u}_{x_1-1,x_2},\tilde{u}_{x_1,x_2-1},\tilde{u}_{x_1,x_2},\tilde{u}_{x_1,x_2+1},\tilde{u}_{x_1+1,x_2}))\\ &\qquad x_1=1,\ldots,T-1,\ \ x_2=1,\ldots,L-1.\end{aligned} \tag{6.159}$$

Then, introducing the new notations $u^*_{x_1-1,x_2}(x_1,x_2)=\psi^*_{x_1,x_2}$, $u^*_{x_1,x_2-1}(x_1,x_2)=\xi^*_{x_1,x_2}$, $u^*_{x_1,x_2+1}(x_1,x_2)=\eta^*_{x_1,x_2}$, $-u^*_{x_1+1,x_2}(x_1,x_2)=u^*_{x_1,x_2}$, we see from Eqs. (6.158) and (6.159) that the first part of the theorem is valid. On the other hand, it follows from Hypothesis H2 that Eq. (6.156) holds with $\lambda=1$ for the point $w^{0*}\in\partial_w g(\tilde{w})\cap K^*_N(\tilde{w})$. Hence, conditions (i) and (ii) are sufficient for the optimality of $\{\tilde{u}_{x_1,x_2}\}_D$. This completes the proof of the theorem. ■

Now let us write the result of Theorem 6.29 in a more symmetrical form. Taking into account Theorem 2.1,we obtain the result in Corollary 6.2.

Corollary 6.2. Suppose that the conditions of the Theorem 6.29 are satisfied and in addition $F(u_1, u_2, u_3, u_4)$ is a closed set for every (u_1, u_2, u_3, u_4). Then for the optimality of $\{\tilde{u}_{x_1,x_2}\}_D$ it is necessary that there exist a number $\lambda=0$ or 1 and vectors $\{\psi^*_{x_1,x_2}\},\{\eta^*_{x_1,x_2}\},\{\xi^*_{x_1,x_2}\},\{u^*_{x_1,x_2}\}$ not all equal to zero, such that

$$\begin{aligned}&u^*_{x_1,x_2}\in\partial_{v^*}H_{x_1,x_2}(\tilde{u}_{x_1-1,x_2},\tilde{u}_{x_1,x_2-1},\tilde{u}_{x_1,x_2},\tilde{u}_{x_1,x_2+1},u^*_{x_1,x_2}),\\ &(\psi^*_{x_1,x_2},\xi^*_{x_1,x_2},u^*_{x_1-1,x_2},\eta^*_{x_1,x_2})\in\partial_u H_{x_1,x_2}(\tilde{u}_{x_1-1,x_2},\tilde{u}_{x_1,x_2-1},\tilde{u}_{x_1,x_2},\tilde{u}_{x_1,x_2+1},u^*_{x_1,x_2})\\ &\quad+\{0\}\times\{0\}\times\{\psi^*_{x_1+1,x_2}+\xi^*_{x_1,x_2+1}+\eta^*_{x_1,x_2-1}-\lambda\partial g_{x_1,x_2}(\tilde{u}_{x_1,x_2})\}\times\{0\},\\ &\psi^*_{0,x_2}=0,\ \ u^*_{T,x_2}=0,\ \ x_2=1,\ldots,L-1;\ \ \eta^*_{x_1,0}=0,\ \ \xi^*_{x_1,L}=0,\ \ x_1=1,\ldots,T-1.\end{aligned}$$

If Hypothesis H2 is fulfilled, conditions (i) and (ii) are sufficient for optimality.

Theorem 6.30. Assume Hypothesis H1 for the nonconvex problem (P_D). Then for $\{\tilde{u}_{x_1,x_2}\}_D$ to be an optimal solution of this nonconvex problem (P_D) it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors $\{\psi^*_{x_1,x_2}\},\{\eta^*_{x_1,x_2}\},\{\xi^*_{x_1,x_2}\},\{u^*_{x_1,x_2}\}$, not all equal to zero, satisfying conditions (i) and (ii) of Theorem 6.29.

□ In this case Hypothesis H1 ensures the relation in Eq. (6.156) for the nonconvex problem (P_D) or Eq. (6.153). Therefore, we get the necessary condition as in Theorem 6.29 by starting from the relation in Eq. (6.156), written out for the nonconvex problem (P_D). ■

Remark 6.11. Suppose D_1 is a set of pairs (x_1,x_2) consisting of integer numbers x_1 and x_2. Then we denote by D the set of interior points of D_1—i.e., those for which the points of the form $(x_1\pm 1, x_2)$ and $(x_1, x_2\pm 1)$ belong to this set. Let D have the connectivity property, that all points of D can be connected with some zigzag whose segments are parallel either to the $0x_1$-axis or the $0x_2$-axis. Moreover, assume that Γ is the set of boundary points of D so that $D_1=D\cup\Gamma$. Now, instead of Eq. (6.146), we consider the following condition

$$u_{x_1,x_2} = \alpha_{x_1,x_2}, \quad (x_1,x_2)\in\Gamma \tag{6.160}$$

where α_{x_1,x_2} are a fixed vectors for every (x_1, x_2). It is understood that for every point belonging to Γ there exists some interior point $(x_1, x_2)\in D$ for which the given boundary point is one of the form $(x_1, x_2 \pm 1)$, $(x_1 \pm 1, x_2)$. In this case, the set of points of the form $(x_1 + 1, x_2)$, $(x_1 - 1, x_2)$, $(x_1, x_2 + 1)$, and $(x_1, x_2 - 1)$ we call right, left, upper, and lower sets respectively, and denote them by Q_{r}, Q_{le}, Q_{u}, and Q_{lo}. Obviously, $\Gamma = Q_{\mathrm{r}}\cup Q_{\mathrm{le}}\cup Q_{\mathrm{u}}\cup Q_{\mathrm{lo}}$.

Then by analogy it can be shown that for the problem in Eqs. (6.144), (6.145), and (6.160), boundary condition (ii) of Theorem 6.29 consists of the following:

$$\psi^*_{x_1,x_2} = 0, \;\; (x_1,x_2)\in Q_{\mathrm{le}}; \quad \eta^*_{x_1,x_2} = 0, \;\; (x_1,x_2)\in Q_{\mathrm{lo}};$$
$$\xi^*_{x_1,x_2} = 0, \;\; (x_1,x_2)\in Q_{\mathrm{u}}; \quad u^*_{x_1,x_2} = 0, \;\; (x_1,x_2)\in Q_{\mathrm{r}}.$$

2 Approximation of the Continuous Problem and Necessary Condition for the Discrete-Approximation Problem

In this section we use difference operators to approximate the problem (P_C) and formulate a necessary (and sufficient in the convex case) condition for it with the help of Theorems 6.29 and 6.30. We choose steps δ and h on the x_1-axis and x_2-axis, respectively, using the grid functions $u_{x_1,x_2} = u_{\delta h}(x_1, x_2)$ on a uniform grid on R.

Let $\Delta u = A_1u + A_2u$, where $A_iu = \partial^2 u/\partial x_i^2$ $(i = 1, 2)$. We introduce the following difference operators, defined on the three-point models [178,246]; i.e., each of the operators A_1u, A_2u are approximated by the $\tilde{A}_1u$ and $\tilde{A}_2u$:

$$\tilde{A}_1u(x) := \frac{u(x_1+\delta, x_2) - 2u(x_1, x_2) + u(x_1-\delta, x_2)}{\delta^2},$$
$$\tilde{A}_2u(x) := \frac{u(x_1, x_2+h) - 2u(x_1, x_2) + u(x_1, x_2-h)}{h^2}.$$

The point (x_1, x_2) is called regular [31] if the four points $(x_1 \pm \delta, x_2)$, $x_1, x_2 \pm h$ belong to $\overline{\mathrm{R}} = \mathrm{R}\cup B$. Otherwise, the point (x_1, x_2) is called irregular.

The set of regular knot points are denoted by $\omega^o_{\delta h}$ and irregular points by $\omega^*_{\delta h}$. The set of intersections of lines $x_1 = i\delta$, $x_2 = jh$, $i, j = 0, \pm 1, \pm 2, \pm 3, \ldots$ and arc B are called boundary knot points and denoted by $\gamma_{\delta h}$. Thus, according to the set $\overline{\mathrm{R}}$ we have grid $\overline{\omega}_{\delta h} = \omega^o_{\delta h} \cup \omega^*_{\delta h} \cup \gamma_{\delta h}$. Assume that $\overline{\omega}_{\delta h}$ is a connected set. According to Eq. (6.149), we have $u_{\delta h}(x_1, x_2) = \beta(x_1, x_2)$, $(x_1,x_2)\in\gamma_{\delta h}$.

For irregular knot points, there are different conditions. For such points, we use the value $\beta(\bar{x})$ of the function β, where $\bar{x} \in \gamma_{\delta h}$ is a closest knot point for a given irregular point

$$u(x) = u_{\delta h}(x) = \beta(\bar{x}), \quad x \in \omega^*_{\delta h}.$$

Now, with respect to the problem (P_C), we associate the following difference boundary value problem approximating it:

$$(P_A) \quad \begin{aligned} &\inf \quad J_{\delta h}(u(x_1,x_2)) := \sum_{(x_1,x_2)\in\overline{\omega}_{\delta h}} \delta h g(u(x_1,x_2),x_1,x_2) \\ &\text{subject to } \tilde{A}_1 u(x) + \tilde{A}_2 u(x) \in F(u(x),x), x = (x_1,x_2)\in\overline{\omega}_{\delta h}, \\ &\qquad u(x) = \beta(x), \quad x \in \gamma_{\delta h}. \end{aligned}$$

First, for simplicity assume that (P_A) is a discrete-approximation problem for problem (P_C), where $R = (0,1)\times(0,1)$ so that

$$\overline{\omega}_{\delta h} = \{(x_1,x_2) : x_1 = 0,\delta,\ldots,1; x_2 = 0,h,\ldots,1,(x_1,x_2) \neq (0,0),(0,1),(1,0),(1,1)\}.$$

Now we reduce the problem (P_A) to a problem of the form (P_D) and introduce a new mapping $Q(\cdot,x) : \mathbb{R}^{4n} \to P(\mathbb{R}^n)$:

$$Q(u_1,u_2,u_3,u_4,x) := 2(1+\theta)u_3 - u_1 - \theta(u_4+u_2) + \delta^2 F(u_3,x), \quad \theta = \frac{\delta^2}{h^2} \tag{6.161}$$

and we rewrite the problem (P_A) as follows:

$$\inf \quad J_{\delta h}(u(\cdot,\cdot)), \tag{6.162}$$

$$\begin{aligned} &\text{subject to } u(x_1+\delta,x_2) \in Q(u(x_1-\delta,x_2),u(x_1,x_2-h),u(x_1,x_2),u(x_1,x_2+h),x_1,x_2) \\ &(x_1,x_2)\in\overline{\omega}_{\delta h},\ u(x_1,x_2) = \beta(x_1,x_2),(x_1,x_2)\in\gamma_{\delta h}. \end{aligned} \tag{6.163}$$

By Theorem 6.29 for optimality of the feasible solution $\{\tilde{u}(x_1,x_2)\}$, $(x_1,x_2)\in\overline{\omega}_{\delta h}$, in the problem in Eqs. (6.162) and (6.163), it is necessary that there exist vectors $\{u^*(x_1,x_2)\}$, $\{\psi^*(x_1,x_2)\}$, $\{\xi^*(x_1,x_2)\}$, $\{\eta^*(x_1,x_2)\}$ and a number $\lambda = \lambda_{\delta h} \in \{0,1\}$, not all zero, such that

$$\begin{aligned} &(\psi^*(x_1,x_2),\xi^*(x_1,x_2),u^*(x_1-\delta,x_2),\eta^*(x_1,x_2)) \\ &\in Q^*(u^*(x_1,x_2);(\tilde{u}(x_1-\delta,x_2),\tilde{u}(x_1,x_2-h),\tilde{u}(x_1,x_2),\tilde{u}(x_1,x_2+h),\tilde{u}(x_1+\delta,x_2),x_1,x_2) \\ &\quad + \{0\}\times\{0\}\times\left\{\psi^*(x_1+\delta,x_2)+\xi^*(x_1,x_2+h)+\eta^*(x_1,x_2-h)\right. \\ &\quad - \lambda\delta g(\tilde{u}(x_1,x_2),x_1,x_2\}\times\{0\}, \\ &\psi^*(0,x_2)=0,\ u^*(1,x_2)=0,\ x_2=1,\ldots,1-h, \\ &\eta^*(x_1,0)=0,\ \xi^*(x_1,1)=0,\ x_1=1,\ldots,1-\delta. \end{aligned} \tag{6.164}$$

In the problem in Eq. (6.164), we express the LAM Q^* in terms of F^*.

Theorem 6.31. Let $Q(\cdot,x)$ be a multivalued mapping such that the cone of tangent directions $K_{\text{gph } Q(\cdot,x)}(u_1,u_2,u_3,u_4,v)$, $(u_1,u_2,u_3,u_4,v)\in\text{gph } Q(\cdot,x)$ is a local tent. Then

$$K_{\text{gph } Q(\cdot,x)}\left(u_3,\frac{v+u_1+\theta(u_2+u_4)-2(1+\theta)u_3}{\delta^2}\right)$$

is a local tent to gph $F(\cdot,x)$ and the following inclusions are equivalent:

a. $(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)\in K_{\text{gph } Q(\cdot,x)}(u_1,u_2,u_3,u_4,v)$,

b. $\left(\overline{u}_3,\frac{\overline{v}+\overline{u}_1+\theta(\overline{u}_2+\overline{u}_4)-2(1+\theta)\overline{u}_3}{\delta^2}\right)\in K_{\text{gph } F(\cdot,x)}\left(u_3,\frac{v+u_1+\theta(u_2+u_4)-2(1+\theta)u_3}{\delta^2}\right)$.

□ By the definition of a local tent, there exist functions $r_i(\overline{z})$, $i=0,1,2,3,4$ $\overline{z}=(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4,\overline{v})$ such that $r_i(\overline{z})\|\overline{z}\|^{-1}\to 0$ as $\overline{z}\to 0$ and

$$\begin{aligned}
&v+\overline{v}+r_0(\overline{z})\in 2(1+\theta)(u_3+\overline{u}_3+r_3(\overline{z}))-u_1-\overline{u}_1-r_1(\overline{z})\\
&-\theta(u_4+u_2+\overline{u}_4+\overline{u}_2+r_4(\overline{z})+r_2(\overline{z}))+\delta^2F(u_3+\overline{u}_3+r_3(\overline{z}),x)
\end{aligned}$$

for sufficiently small $\overline{z}\in K$, where $K\subseteq\text{ri } K_{\text{gph } Q(\cdot,x)}(z)$ is a convex cone.

Transforming this inclusion, we get

$$\begin{aligned}
&\frac{v-2(1+\theta)u_3+u_1+\theta(u_2+u_4)}{\delta^2}+\frac{\overline{v}-2(1+\theta)\overline{u}_3+\overline{u}_1+\theta(\overline{u}_2+\overline{u}_4)}{\delta^2}\\
&+\frac{r_0(\overline{z})-2(1+\theta)r_3(\overline{z})+r_1(\overline{z})+\theta(r_4(\overline{z})+r_2(\overline{z}))}{\delta^2}\in F(u_3+\overline{u}_3+r_3(\overline{z}),x).
\end{aligned}$$

From this relation it is clear that

$$\begin{aligned}
&\left(\overline{u}_3,\frac{\overline{v}+\overline{u}_1+\theta(\overline{u}_2+\overline{u}_4)-2(1+\theta)\overline{u}_3}{\delta^2}\right)\\
&\in K_{\text{gph } F(\cdot,x)}\left(u_3,\frac{v+u_1+\theta(u_2+u_4)-2(1+\theta)u_3}{\delta^2}\right).
\end{aligned}\tag{6.165}$$

By going in the reverse direction, it is also not hard to see from Eq. (6.165) that

$$(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)\in K_{\text{gph } Q(.,x)}(u_1,u_2,u_3,u_4,v).\tag{6.166}$$

Therefore, Eqs. (6.165) and (6.166) are equivalent. ■

In what follows, Theorem 6.32 is very important.

Theorem 6.32. Assume that the mapping $Q(\cdot,x)$ is such that the cones of tangent directions $K_{\text{gph } Q(\cdot,x)}$ (u_1,u_2,u_3,u_4,v) determine a local tent. Then the following inclusions are equivalent under the conditions that $v^*+u_1^*=0,\ u_2^*=u_4^*=-\theta v^*$:

a. $(u_1^*,u_2^*,u_3^*,u_4^*)\in Q^*(v^*,(u_1,u_2,u_3,u_4,v),x)$,

b. $\frac{u_3^*-2(1+\theta)v^*}{\delta^2}\in F^*\left(v^*,\left(u_3,\frac{v+u_1+\theta(u_2+u_4)-2(1+\theta)u_3}{\delta^2}\right),x\right)$.

□ Suppose that condition (a) is fulfilled. By the definition of LAM, this means that in the case of Eq. (6.166)

$$\langle \overline{u}_1, u_1^* \rangle + \langle \overline{u}_2, u_2^* \rangle + \langle \overline{u}_3, u_3^* \rangle + \langle \overline{u}_4, u_4^* \rangle - \langle \overline{v}, v^* \rangle \geq 0. \tag{6.167}$$

Let us rewrite the inequality in Eq. (6.166) in the form

$$\langle \overline{u}_3, \psi_3^* \rangle - \left\langle \frac{\overline{v} + \overline{u}_1 + \theta(\overline{u}_2 + \overline{u}_4) - 2(1+\theta)\overline{u}_3}{\delta^2}, \psi_3^* \right\rangle \geq 0, \tag{6.168}$$

where it is taken into account that the inclusions in Eqs. (6.165) and (6.166) are equivalent. Here ψ_3^* and ψ^* are to be determined. Carrying out the necessary transformations in Eq. (6.168) and comparing it with Eq. (6.167), it is not hard to see that

$$\psi^* = v^*, \quad -\psi^* = u_1^*, \quad -\theta\psi^* = u_2^*, \quad -\theta\psi^* = u_4^*, \quad \delta^2\psi_3^* + 2(1+\theta)\psi^* = u_3^*.$$

These equalities imply that $v^* + u_1^* = 0$, $u_2^* = u_4^* = -\theta v^*$, and $\psi_3^* = (u_3^* - 2(1+\theta)v^*)/\delta^2$. Then from Theorem 6.31, we obtain condition (b); i.e., a ⇒ b. By analogy, it can be shown that b ⇒ a.■

Let us return to the inclusion in Eq. (6.164). By Theorem 6.32, this condition has the form

$$\begin{aligned}&\frac{u^*(x_1-\delta,x_2) - \psi^*(x_1+\delta,x_2) - \xi^*(x_1,x_2+h) - \eta^*(x_1,x_2-h) - 2(1+\theta)u^*(x_1,x_2)}{\delta^2}\\ &\in F^*(u^*(x_1,x_2),(\tilde{u}(x_1,x_2),\tilde{A}_1\tilde{u}(x_1,x_2)+\tilde{A}_2\tilde{u}(x_1,x_2)),x_1,x_2) - \lambda\,\partial g(\tilde{u}(x_1,x_2),x_1,x_2)\end{aligned} \tag{6.169}$$

$$\begin{aligned}&u^*(x_1,x_2) = -\psi^*(x_1,x_2),\ \ \xi^*(x_1,x_2) = \eta^*(x_1,x_2) = -\theta u^*(x_1,x_2)\\ &\theta = \frac{\delta^2}{h^2},\quad x_1 = \delta, 2\delta, \ldots, 1-\delta;\ \ x_2 = h, 2h, \ldots, 1-h.\end{aligned} \tag{6.170}$$

Note for further convenience that because of the positive homogeneity of LAM F^*, the coefficient δh in front of $\partial g(\tilde{u}(x_1,x_2),x_1,x_2)$ in Eq. (6.169) is removed.

Further, using Eq. (6.170), it is not hard to verify that the left side of Eq. (6.169) has the form

$$\begin{aligned}&\frac{1}{\delta^2}[u^*(x_1-\delta,x_2) + u^*(x_1+\delta,x_2) + \theta(u^*(x_1,x_2+h) + u^*(x_1,x_2-h)) - 2(1+\theta)u^*(x_1,x_2)]\\ &\quad = \frac{u^*(x_1+\delta,x_2) - 2u^*(x_1,x_2) + u^*(x_1-\delta,x_2)}{\delta^2}\\ &\qquad + \frac{u^*(x_1,x_2+h) - 2u^*(x_1,x_2) + u^*(x_1,x_2-h)}{h^2}.\end{aligned} \tag{6.171}$$

On the other hand, from the boundary conditions in Eq. (6.164) and from Eq. (6.170), we obtain

$$\begin{aligned} &u^*(x_1,0)=0, \quad u^*(x_1,1)=0, \ \ x_1=\delta,\ldots,1-\delta; \\ &u^*(0,x_2)=0, \quad u^*(1,x_2)=0, \ \ x_2=h,2h,\ldots,1-h. \end{aligned} \tag{6.172}$$

Taking into account the relations in Eqs. (6.169), (6.171), and (6.172), we have proved the result in Theorem 6.33 for (P_A).

Theorem 6.33. Suppose $g(\cdot,x)$ is a convex proper function and is continuous at the points of some feasible solution $\{u^0(x)\}$, $x\in\overline{\omega}_{\delta h}$. Then for the optimality of the solution $\{\tilde{u}(x)\}$ in the convex problem (P_A), it is necessary that there exist a number $\lambda=\lambda_{\delta h}\in\{0,1\}$ and grid functions $\{u^*(x)\}$, $x\in\overline{\omega}_{\delta h}$, not all equal to zero, such that

i. $\tilde{A}_1u^*(x)+\tilde{A}_2u^*(x)\in F^*(u^*(x),(\tilde{u}(x),\tilde{A}_1\tilde{u}(x)+\tilde{A}_2\tilde{u}(x)),x)-\lambda\,\partial g(\tilde{u}(x),x)$.
ii. $u^*(x_1,0)=u^*(x_1,1)=0$, $x_1=\delta,\ldots,1-\delta$,
$u^*(0,x_2)=u^*(1,x_2)$, $x_2=h,\ldots,1-h$.

Under the condition of Hypothesis H2, these conditions are also sufficient for the optimality of $\{\tilde{u}(x)\}$, $x\in\overline{\omega}_{\delta h}$.

Remark 6.12. As in Theorem 6.29, conditions (i) and (ii) of Theorem 6.33 are necessary for optimality in the nonconvex case of the problem (P_A) under Hypothesis H1.

Remark 6.13. Observe that for problem (P_C) with nonsquare region R, boundary condition (ii) of Theorem 6.33 for boundary points consists of the following: $u^*(x)=0$, $x\in\gamma_{\delta h}\subset B$.

3 *Sufficient Conditions for Optimality for Differential Inclusions of Elliptic Type*

Using the results in Subsection 2, we now obtain a sufficient condition of optimality of the continuous problem (P_C) . Let us pass to the formal limit in condition (i) of Theorem 6.33 and in the boundary condition (see Remark 6.13) as δ, $h\to 0$ and set $\lambda=1$. Then we have

a. $\Delta u^*(x)\in F^*(u^*(x),(\tilde{u}(x),\Delta\tilde{u}(x)),x)-\partial g(\tilde{u}(x),x)$, $\quad x=(x_1,x_2)\in$ R.
b. $u^*(x)=0$, $x\in B$.
Along with this, we get condition (c), which ensures that the LAM $F^*(\cdot,\cdot,x)$ is nonempty for every fixed $x\in$ R (see Theorem 2.1).
c. $\Delta\tilde{u}(x)\in F(\tilde{u}(x),u^*(x),x)$.

The arguments in Subsection 2 guarantee the sufficiency of conditions (a)−(c) for optimality. It turns out that the assertion in Theorem 6.34 is true.

Theorem 6.34. Assume that a continuous function g is convex with respect to u and that $F(\cdot,x)$ is a convex mapping for all fixed x. Then for the optimality of the solution $\tilde{u}(x)$, among all feasible solutions in (P_C), it is sufficient that there exists a classical solution $u^*(x)$ such that conditions (a)−(c) hold.

□ By Theorem 2.1, $F^*(v^*, (u,v), x) = \partial_u H(u,v^*,x)$, $v \in F(u;v^*,x)$. Then applying the Moreau−Rockafellar theorem (Theorem 1.28) and the fact that $-\partial g(\cdot,x) = \partial(-g(\cdot,x))$, from condition (a), we obtain

$$\Delta u^*(x) \in \partial_u[H(\tilde{u}(x), u^*(x), x) - g(\tilde{u}(x), x)], \quad x \in \mathrm{R}$$

or

$$H(u(x), u^*(x), x) - H(\tilde{u}(x), u^*(x), x) - g(u(x), x) + g(\tilde{u}(x), x) \leq \langle \Delta u^*(x), u(x) - \tilde{u}(x) \rangle.$$

Now, taking into account condition (c) of Theorem 6.34, the definition of the function H, and integrating both sides of the latter inequality over the domain R, we get

$$\iint_{\mathrm{R}} [g(u(x), x) - g(\tilde{u}(x), x)]\mathrm{d}x - \iint_{\mathrm{R}} \langle \Delta(u(x) - \tilde{u}(x)), u^*(x) \rangle \mathrm{d}x + \iint_{\mathrm{R}} \langle u(x) - \tilde{u}(x), \Delta u^*(x) \rangle \mathrm{d}x \geq 0. \tag{6.173}$$

On the other hand, by Green's theorem [127,128,180], we have

$$\iint_{\mathrm{R}} [\langle u(x) - \tilde{u}(x), \Delta u^*(x) \rangle - \langle \Delta(u(x) - \tilde{u}(x)), u^*(x) \rangle]\mathrm{d}x = \iint_{B} \left[\left\langle u(x) - \tilde{u}(x), \frac{\partial u^*(x)}{\partial n} \right\rangle - \left\langle \frac{\partial(u(x) - \tilde{u}(x))}{\partial n}, u^*(x) \right\rangle \right] \mathrm{d}s, \tag{6.174}$$

where ds is a symbolic arc length element and n is the outer normal for the curve B.

Since $u(x)$ and $\tilde{u}(x)$ are feasible solutions—i.e., $u(x) = \tilde{u}(x) = \beta(x)$, $x \in B$ and condition (b) of theorem is fulfilled—the integral in Eq. (6.174) is equal to zero. Therefore, from the inequality in Eq. (6.173), it follows that

$$\iint_{\mathrm{R}} g(u(x), x)\mathrm{d}x \geq \iint_{\mathrm{R}} g(\tilde{u}(x), x)\mathrm{d}x$$

for arbitrary feasible solutions $u(x)$, $x \in \mathrm{R}$. ■

Corollary 6.3. In addition to the assumptions of Theorem 6.34, suppose that $F(\cdot,x)$ is a closed mapping. Then conditions (a) and (c) of Theorem 6.34 can be rewritten as follows:

i. $\Delta u^*(x) \in \partial_u H(\tilde{u}(x), u^*(x), x) - \partial g(\tilde{u}(x), x)$,
ii. $\Delta \tilde{u}(x) \in \partial_{v^*} H(\tilde{u}(x), u^*(x), x)$.

□ In fact, on the one hand, by Theorem 2.1 the following equality is correct:

$$F^*(v^*, (u, v), x) = \partial_u H(u, v^*, x), \quad v \in F(u; v^*, x).$$

On the other hand, $\partial_{v^*} H(u, v^*, x) = F(u; v^*, x)$. Therefore, (i) and (ii) are equivalent to conditions (a) and (c) of Theorem 6.34.■

Remark 6.14. It follows from condition (ii) of Corollary 6.3 and condition (c) of Theorem 6.34 that

$$\langle u^*(x), \Delta \tilde{u}(x)\rangle = H(\tilde{u}(x), u^*(x), x).$$

So, in particular, if $F(\cdot,x)$ is a quasisuperlinear mapping and $H(\cdot,v^*,x)$ is a convex proper function, then by Theorem 2.20 this equality can be written as follows:

$$\langle u^*(x), \Delta \tilde{u}(x)\rangle = \inf_{\Delta u^*(x) \in F^*(u^*(x),x)} \{\langle u^*(x), \Delta \tilde{u}(x)\rangle - M(\Delta u^*(x), u^*(x), x)\}.$$

Theorem 6.35. Let Hypothesis H1 be fulfilled for the nonconvex problem (P_C). Moreover, let $\tilde{u}(x)$, $x \in R$ be some feasible solution of this problem and suppose that $u^*(x)$ is a classical solution satisfying the following conditions:

i. $\Delta u^*(x) + u^*(x) \in F^*(u^*(x),\ (\tilde{u}(x), \Delta \tilde{u}(x)), x)$,
ii. $g(u, x) - g(\tilde{u}(x), x) \geq \langle u^*(x), u - \tilde{u}(x)\rangle$ for all u,
iii. $\langle u^*(x), \Delta \tilde{u}(x)\rangle = H(\tilde{u}(x), u^*(x), x)\}$,

where $F^*(\cdot,\cdot,x)$ is given by the second definition of LAM—i.e., by the Hamiltonian function. Then the feasible solution $\tilde{u}(x)$ is optimal.
□ Taking into account the definition of LAM by the Hamiltonian function, it follows from condition (i) that for all feasible solution $u(x)$,

$$H(u(x), u^*(x), x) - H(\tilde{u}(x), u^*(x), x) \leq \langle \Delta u^*(x) + u^*(x), u(x) - \tilde{u}(x)\rangle, \quad x \in R.$$

Then using condition (iii), we have

$$\langle \Delta(u(x) - \tilde{u}(x)), u^*(x)\rangle \leq \langle \Delta u^*(x) + u^*(x), u(x) - \tilde{u}(x)\rangle. \tag{6.175}$$

Now, from condition (ii) for an arbitrary feasible solution $u = u(x)$ and from the inequality in Eq. (6.175), it is easy to see that

$$g(u(x),x) - g(\tilde{u}(x),x) - \langle \Delta(u(x) - \tilde{u}(x)), u^*(x)\rangle + \langle u(x) - \tilde{u}(x), \Delta u^*(x)\rangle \geq 0, \quad x \in R.$$

Then by integrating this inequality over the domain R, we see that the obtained inequality takes the form of Eq. (6.173). Thus, in view of Eq. (6.173), it is easy to show, as in the proof of Theorem 6.34, that $\tilde{u}(x)$ is optimal. ■

Let us consider the following example:

$$\inf J(u(x)) = \iint_{\mathrm{R}} g(u(x), x)\mathrm{d}x, \text{ subject to } \Delta u(x) = Au(x) + Bw(x), \quad w(x) \in V, \tag{6.176}$$

where A is an $n \times n$ matrix, B is a rectangular $n \times r$ matrix, $V \subset \mathbb{R}^r$ is a closed convex set, and g is a continuously differentiable function of x. It is required to find a controlling parameter $w(x) \in V$ such that the feasible solution corresponding to it minimizes $J(u(\cdot))$.

Let us introduce the convex mapping $F(u) = Au + BV$. By elementary calculations, it can be shown that

$$F^*(v^*, (u, v)) = \begin{cases} A^* v^*, & \text{if } -B^* v^* \in K_V^*(w), \\ \varnothing, & \text{if } -B^* v^* \notin K_V^*(w). \end{cases}$$

Here, $v = Au + Bw$ and $K_V^*(w)$ is the cone dual to the cone of tangent directions $K_V(w)$ at a point $w \in V$. Then using Theorem 6.34, we get the relations

$$\Delta u^*(x) = A^* u^*(x) - g'(\tilde{u}(x), x), \quad x \in \mathrm{R}$$

$$u^*(x) = 0, \quad x \in B, \tag{6.177}$$

$$\langle B\tilde{u}(x), u^*(x) \rangle = \sup_{w \in V} \langle Bu, u^*(x) \rangle.$$

Thus, we have obtained the result as in Theorem 6.36.

Theorem 6.36. The feasible solution $\tilde{u}(x)$ corresponding to the control $\tilde{w}(x)$ minimizes $J(u(\cdot))$ in the problem in Eq. (6.176) if there exists a classical solution $u^*(x)$ satisfying the conditions in Eq. (6.177).

4 Multidimensional Optimal Control Problems for Elliptic Differential Inclusions

In this section, we study the following problem (P_M) with elliptic operator L considered in Subsection 1:

$$\inf J(u(\cdot)) = \int_G g(u(x), x)\mathrm{d}x,$$

$$(P_M) \text{ subject to } Lu(x) \in F(u(x), x)$$

$$u(x) = \alpha(x), \quad x \in S$$

Theorem 6.37. Suppose that g is a continuous function convex with respect to u, and $F(\cdot, x)$ is a convex closed mapping for every fixed $x \in G$. Then a solution $\tilde{u}(x)$ minimizes the functional $J(u(\cdot))$ among all feasible solutions of the problem (P_M) if there exists a classical solution of the following boundary value problem:

i. $L^* u^*(x) \in F^*(u^*(x); (\tilde{u}(x), L\tilde{u}(x)), x) - \partial g(\tilde{u}(x), x), \ u^*(x) = 0, \ x \in S,$
ii. $L\tilde{u}(x) \in F(\tilde{u}(x), u^*(x), x),$

where L^* is the operator adjoint to L.
□ By arguments analogous to those in the proof of Theorem 6.34 and by condition (i), it is easy to see that

$$H(u(x), u^*(x), x) - H(\tilde{u}(x), u^*(x), x) \le g(u(x), x) - g(\tilde{u}(x), x) + \langle L^* u^*(x), u(x) - \tilde{u}(x) \rangle,$$

where because of condition (ii),

$$H(\tilde{u}(x), u^*(x), x) = \langle u^*(x), L\tilde{u}(x) \rangle.$$

so

$$\int_G [g(u(x), x) - g(\tilde{u}(x), x)] \mathrm{d}x \ge \int_G u^*(x) L[u(x) - \tilde{u}(x)] \mathrm{d}x - \int_G L^* u^*(x)[u(x) - \tilde{u}(x)] \mathrm{d}x. \tag{6.178}$$

Then using the boundary conditions of (i) and the fact that the functions $u(x)$ are feasible solutions—i.e., $u(x) = \tilde{u}(x) = \alpha(x), \ x \in S$—we get from Green's formula that the right-hand side of Eq. (6.178) is equal to zero. This means that $J(u(x)) \ge J(\tilde{u}(x))$ for all feasible solutions in problem (P_M). ■

Remark 6.15. If, in addition to the assumptions of Theorem 6.37, we assume that $F(\cdot, x)$ is a closed mapping, then conditions (i) and (ii) of Theorem 6.37 can be rewritten as follows (see Corollary 6.3):

i. $L^* u^*(x) \in \partial_u H(\tilde{u}(x), u^*(x), x) - \partial g(\tilde{u}(x), x),$
ii. $L\tilde{u}(x) \in \partial_{v^*} H(\tilde{u}(x), u^*(x), x)$

Replacing the Laplacian Δ with an elliptic operator L and extending the proof of Theorem 6.35 to the problem (P_M) in the nonconvex case, it is not hard to get Theorem 6.38.

Theorem 6.38. Suppose that $\tilde{u}(x)$ is some feasible solution of the nonconvex problem (P_M) and that $u^*(x)$ is a classical solution satisfying the following conditions:

i. $L^* u^*(x) + u^*(x) \in F^*(u^*(x), \ (\tilde{u}(x), L\tilde{u}(x)), x),$
$u^*(x) = 0, \ x \in S,$
ii. $\langle u^*(x), L\tilde{u}(x) \rangle = H(\tilde{u}(x), u^*(x), x),$
iii. $g(u, x) - g(\tilde{u}(x), x) \ge \langle u^*(x), u - \tilde{u}(x) \rangle$ for all u,

where the LAM $F^*(\cdot, \cdot, x)$ is given by the Hamiltonian function. Then the feasible solution $\tilde{u}(x)$ is optimal.

In conclusion, we discuss the possibility of passing to more general function spaces of solutions in these problems. It is known that for the theory of partial differential equations, the concept of a generalized solution is important both from the theoretical and from the practical point of view [180,246]. The definition of such solutions associates with a given equation a certain integral identity that uses, in turn, the class of generalized derivatives.

Therefore, following this path, the most natural approach to elliptic differential inclusions is apparently the use of single-valued branches (selections) of a multivalued mapping [111].

Thus, suppose that we have the problem (P_M) with homogeneous boundary conditions and let $H^1(G)$ be the Hilbert space consisting of the elements $u(x) \in L_2(G)$ having square-integrable generalized derivatives on G, where the inner product and norm are defined by the expressions

$$\langle u_1, u_2 \rangle_{H^1(G)} = \int\limits_G (u_1 u_2 + u'_{x1} u'_{x2}) \mathrm{d}x, \quad \|u\|_{H^1(G)} = \sqrt{\langle u, u \rangle_{H^1(G)}}.$$

By analogy with the classical theory of the Dirichlet problem for elliptic equations [180], we call a function $u(x) \in H(G)$ a generalized solution of our problem if it satisfies the integral identity

$$\int\limits_G \sum_{i,j=1}^{n} a_{ij} u'_{x_i} \eta'_{x_j} \mathrm{d}x + \int\limits_G u \left[\sum_{i=1}^{n} b_i \eta'_{x_i} + \left(\sum_{i=1}^{n} b'_{ix_i} - c \right) \eta \right] \mathrm{d}x = -\int\limits_G g\eta \, \mathrm{d}x$$

for all $\eta(x) \in H^{1^\circ}(G)$ (for a more detailed study see, for example, Refs. [92,180,246]) of the multivalued mapping $F(u,x)$ a generalized solution is defined analogously for the adjoint boundary value problem.

We now remind that for all the results obtained here we have used the formula of integration by parts and it is easy to see that the Green and Gauss–Ostrogradsky formulae follow from it. The latter can be used, getting the indicated classes of generalized solutions. Therefore, it is not difficult to verify the validity of all the assertions in this general case.

5 Sufficient Conditions for Generalized Dirichlet and Neumann Problems in Dynamic Optimization

First in this section, we will investigate a generalized Dirichlet problem of the form

$$\inf \quad J[u(\cdot)] = \iint\limits_Q g(u(x), x) \mathrm{d}x \tag{6.179}$$

$$\text{subject to } \Delta u(x) \in F(u_x(x), u(x), x), \quad x \in Q, \tag{6.180}$$

$$u(x) = \beta_1(x), \quad x \in B, \tag{6.181}$$

where Δ is Laplace's operator, $u_x = (u_{x_1}, u_{x_2})$, $F(\cdot,\cdot,\cdot,x) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$, Q is a bounded region of $\mathbb{R}^2$ with a closed piecewise-smooth simple curve boundary B, $\beta_1(\cdot)$ is a continuous function, and $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2 (x=(x_1,x_2))$. We label this continuous Dirichlet problem in Eqs. (6.179)–(6.181) as (P_D). Another problem is an optimization problem consisting of Eqs. (6.179) and (6.180) and the boundary condition

$$\left.\frac{\partial u(x)}{\partial n}\right|_B = \beta_2(x), \tag{6.182}$$

where n is the normal to a curve B, and $\beta_2(\cdot)$ is a continuous function. We call this problem a second boundary value problem, or a Neumann problem (P_N).

The main subject will be the following pair of problems, one is the multidimensional Dirichlet problem (P_{MD}) and the other is the Neumann problem (P_{MN}):

$$\inf \quad J[u(\cdot)] = \int_G g(u(x), x)\mathrm{d}x, \tag{6.183}$$

$$\text{subject to } Lu(x) \in F(u_x(x), u(x), x), \quad x \in G, \tag{6.184}$$

and either, for the Dirichlet problem,

$$u(x) = \alpha_1(x), \quad x \in S, \tag{6.185}$$

or, for the Neumann problem,

$$\left.\frac{\partial u(x)}{\partial n}\right|_S = \alpha_2(x), \tag{6.186}$$

where $F(\cdot,\cdot,x) : \mathbb{R}^{n+1} \to P(\mathbb{R})$, $x = (x_1, \ldots, x_n)$, $u_x = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ is the gradient vector, G is a bounded set of $\mathbb{R}^n$ with a closed piecewise-smooth surface boundary S, $g : \mathbb{R} \times G \to \mathbb{R}$ is continuous on u function, $\alpha_1(\cdot)$,$\alpha_2(\cdot)$ are continuous functions, and L is a second-order elliptic operator.

It is required to find a classical solution $\tilde{u}(\cdot)$ of the boundary value problem (P_M) that minimizes $J[u(\cdot)]$. Clearly, such a class of functions is endowed with the corresponding norms form a Banach space. Moreover, by introducing a generalized solution, it is possible to carry over the results obtained in this section.

At once, we emphasize that in contrast to the problem considered in Ref. [166], the right-hand side of the differential inclusions contained in the problems (P_D), (P_N), and (P_M), depend also on the partial derivatives of the required function, too. Of course, such problems are an important generalization of the problems investigated in Ref. [166] and considerably complicate the posed problem. That is why these problems are a separate object for study. In fact, because of the presence of the gradient function, in the adjoint inclusion the divergence operation has arisen.

Note that such gradient–divergence connections appear in the duality relations for some variational problems of Ekeland and Temam [76].

Observe that since $F:\mathbb{R}^{4n}\to P(\mathbb{R}^n)$, in the next statements, the LAM F^* is a multivalued function from $\mathbb{R}^n$ into $\mathbb{R}^{4n}$.

In Theorems 6.39–6.41, we formulate sufficient conditions first for the problem (P_D) and then for (P_N).

Theorem 6.39. Suppose that a continuous function g is convex with respect to u, and $F(\cdot,\cdot,\cdot,x)$ is a convex multivalued function for all fixed x. Then for optimality of the solution $\tilde{u}(\cdot)$ in the convex problem (P_D), it is sufficient that there exists a triplet of classical solutions $\{\varphi^*(\cdot),\ p^*(\cdot), u^*(\cdot)\}$ such that conditions (a_D) to (c_D), below, hold.

$$\begin{aligned}
&(a_D)\quad (-\varphi^*(x),\ -p^*(x), \Delta u^*(x))\in F^*(u^*(x);(\tilde{u}_x(x),\tilde{u}(x),\Delta\tilde{u}(x)),x)\\
&\qquad\quad +(0,0)\times\{\varphi^*_{x_1}(x)+p^*_{x_2}(x)-\partial g(\tilde{u}(x),x)\},\quad x=(x_1,x_2)\in \mathrm{R},\\
&(b_D)\quad u^*(x)=0, x\in B,\\
&(c_D)\quad \Delta\tilde{u}(x)\in F(\tilde{u}_x(x),\tilde{u}(x),u^*(x),x).
\end{aligned}$$

□ In the convex case, by Theorem 2.1,

$$F^*(v^*,(u,v),x)=\partial_u H(u;v^*,x),\quad v\in F(u;v^*,x),\ \ u=(u_1,u_2,u_3).$$

Then from condition (a_D) of Theorem 6.39 and the Moreau–Rockafellar theorem (Theorem 1.29), we obtain

$$\begin{aligned}
&(-\varphi^*(x),\ -p^*(x),\Delta u^*(x)-\varphi^*_{x_1}(x),p^*_{x_2}(x))\\
&\in\partial_{(u_1,u_2,u_3)}[H(\tilde{u}_{x_1}(x),\tilde{u}_{x_2}(x),\tilde{u}(x),x)-g(\tilde{u}(x),x)],\quad x\in \mathrm{R}.
\end{aligned}$$

By definition of subdifferential this means that

$$\begin{aligned}
&H(u_x(x),u(x),u^*(x),x)-H(\tilde{u}_x(x),\tilde{u}(x),u^*(x),x)-g(u(x),x)+g(\tilde{u}(x),x)\\
&\le -\langle\varphi^*(x),u_{x_1}(x)-\tilde{u}_{x_1}(x)\rangle-\langle p^*(x),u_{x_2}(x)-\tilde{u}_{x_2}(x)\rangle+\langle\Delta u^*(x),u(x)-\tilde{u}(x)\rangle\\
&-\langle\varphi^*_{x_1}(x),u(x)-\tilde{u}(x)\rangle-\langle p^*_{x_2}(x),u(x)-\tilde{u}(x)\rangle.
\end{aligned}$$

By virtue of condition (c_D) and the definition of the Hamiltonian function H, the last inequality gains the form

$$\begin{aligned}
&\langle\Delta(u(x)-\tilde{u}(x)),u^*(x)\rangle-g(u(x),x)+g(\tilde{u}(x),x)\\
&\le\langle\Delta u^*(x),u(x)-\tilde{u}(x)\rangle-\frac{\partial}{\partial x_1}\langle\varphi^*(x),u(x)-\tilde{u}(x)\rangle-\frac{\partial}{\partial x_2}\langle p^*(x),u(x)-\tilde{u}(x)\rangle.
\end{aligned}$$

Then by integrating both sides of this inequality over the domain Q, we have

$$\iint_Q [g(u(x),x) - g(\tilde{u}(x),x]\mathrm{d}x \geq \iint_Q [\langle \Delta(u(x)-\tilde{u}(x)), u^*(x)\rangle - \langle \Delta u^*(x), u(x)-\tilde{u}(x)\rangle]\mathrm{d}x$$
$$+ \iint_Q \left[\frac{\partial}{\partial x_1}\langle \varphi^*(x), u(x)-\tilde{u}(x)\rangle + \frac{\partial}{\partial x_2}\langle p^*(x), u(x)-\tilde{u}(x)\rangle \right]\mathrm{d}x. \tag{6.187}$$

Since $u(\cdot)$ and $\tilde{u}(\cdot)$ are feasible solutions, $u(x)=\tilde{u}(x)=\beta_1(x)$, $x\in B$ (Green's theorem [91,180,246]) yields

$$\begin{aligned}&\iint_Q \left[\frac{\partial}{\partial x_1}\langle \varphi^*(x), u(x)-\tilde{u}(x)\rangle - \frac{\partial}{\partial x}\langle p^*(x), \tilde{u}(x)-u(x)\rangle \right]\mathrm{d}x \\ &= \int_B \langle p^*(x), \tilde{u}(x)-u(x)\rangle \mathrm{d}\underset{1}{x} + \int_B \langle \varphi^*(x), u(x)-\tilde{u}(x)\rangle \mathrm{d}x_2 = 0\end{aligned} \tag{6.188}$$

and

$$\begin{aligned}&\iint_Q [\langle \Delta(u(x)-\tilde{u}(x)), u^*(x)\rangle - \langle \Delta(u(x)-\tilde{u}(x)), u^*(x)\rangle]\mathrm{d}x \\ &= \int_B \left[\left\langle u(x)-\tilde{u}(x), \frac{\partial u^*(x)}{\partial n}\right\rangle - \left\langle \frac{\partial(u(x)-\tilde{u}(x))}{\partial n}, u^*(x)\right\rangle \right]\mathrm{d}s = 0,\end{aligned} \tag{6.189}$$

where ds is a symbolic arc length element and n is the outer normal for a curve B.

Theorem 6.40. Let $\tilde{u}(\cdot)$ be a feasible solution of the nonconvex problem (P_D) and suppose that the triplet $\{\varphi^*(\cdot), p^*(\cdot), u^*(\cdot)\}$ satisfies the following conditions:

- **i.** $(-\varphi^*(x), -p^*(x), -\Delta u^*(x) + u^*(x) - \varphi^*_{x_1}(x) - p^*_{x_2}(x)) \in F^*(u^*(x), (\tilde{u}_x(x), \tilde{u}(x), \Delta\tilde{u}(x)), x)$,
- **ii.** $g(u,x) - g(\tilde{u}(x),x) \geq \langle u^*(x), u - \tilde{u}(x)\rangle$ for all u,
- **iii.** $\langle u^*(x), \Delta\tilde{u}(x)\rangle = H(\tilde{u}_x(x), \tilde{u}(x), u^*(x), x)$,
- **iv.** $u^*(x) = 0$, $x\in B$,

where the LAM F^* is defined by the Hamiltonian function. Then the solution $\tilde{u}(\cdot)$ is optimal.

□ Using the definition of LAM in the nonconvex case, it follows from condition (i) that for all $u(\cdot)$,

$$\begin{aligned}&H(u_x(x), u(x), u^*(x), x) - H(\tilde{u}_x(x), \tilde{u}(x), x) \\ &\leq -\langle \varphi^*(x), u_{x_1}(x) - \tilde{u}_{x_1}(x)\rangle - \langle p^*(x), u_{x_2}(x) - \tilde{u}_{x_2}(x)\rangle \\ &+ \langle \Delta u^*(x) + u^*(x), u(x)-\tilde{u}(x)\rangle - \langle \varphi^*_{x_1}(x), u(x)-\tilde{u}(x)\rangle - \langle p^*_{x_2}(x), u(x)-\tilde{u}(x)\rangle.\end{aligned}$$

Then using the definition of function H, from the last inequality and condition (ii), we obtain for all feasible solutions $u(\cdot)$:

$$g(u(x),x) - g(\tilde{u}(x),x) \geq \langle \Delta(u(x) - \tilde{u}(x)), u^*(x)\rangle - \langle \Delta u^*(x), u(x) - \tilde{u}(x)\rangle \\ + \frac{\partial}{\partial x_1}\langle \varphi^*(x), u(x) - \tilde{u}(x)\rangle + \frac{\partial}{\partial x_2}\langle p^*(x), u(x) - \tilde{u}(x)\rangle.$$

If we integrate this inequality over the domain Q, we have the inequality in Eq. (6.187). Then beginning with the inequality in Eq. (6.187), we again are persuaded that the theorem is valid. ■

Now, let us consider the problem (P_N)—i.e., Eqs. (6.179)−(6.181).

Theorem 6.41. Assume that $\tilde{u}(\cdot)$ is a feasible solution of the nonconvex Neumann problem (P_N) and a classical solution satisfying conditions (i)−(iii) of Theorem 6.40 and boundary conditions:

$$\varphi^*(x) = 0, \quad p^*(x) = 0, \quad \frac{\partial u^*(x)}{\partial n} = 0, \;\; x \in B. \tag{6.190}$$

Then the solution $\tilde{u}(\cdot)$ is optimal.

□ The distinctive peculiarity in the proof is that in Eqs. (6.188) and (6.189), instead of the condition (b_D) we use the boundary conditions. ■

Remark 6.16. Obviously, for the convex Neumann problem (P_N), the sufficient conditions for the optimality of $\tilde{u}(\cdot)$ consist of conditions (a_D) and (c_D) of Theorem 6.39 and the boundary conditions in Eq. (6.190) of Theorem 6.41.

6 Optimization of the Multidimensional Problems (P_{MD}) *and* (P_{MN})

First, let us consider the Dirichlet problem with second-order elliptic operator L:

$$\text{minimize} \;\; J[u(\cdot)] = \int_G g(u(x),x)\mathrm{d}x$$

$$(P_{MD}) \quad \text{subject to} \quad \begin{array}{l} Lu(x) \in F(u_x(x), u(x), x), x \in G, \\ u(x) = \alpha_1(x), x \in S; \quad u_x = (u_{x_1}, \ldots, u_{x_n}). \end{array}$$

Theorem 6.42. Assume that $\tilde{u}(\cdot)$ is some feasible solution of the convex Dirichlet problem (P_{MD}) and that there is a pair $\{w^*(\cdot), u^*(\cdot)\}$ of classical solutions $w^*(x) = (w_1^*(x), \ldots, w_n^*(x))$ and $u^*(x)$ of the following boundary value problem:

i. $(-w^*(x), L^*u^*(x)) \in F^*(u^*(x), (\tilde{u}_x(x), \tilde{u}(x), L\tilde{u}(x)), x)$
$+ \{0\} \times \{\mathrm{div}\, w^*(x) - \partial g(\tilde{u}(x), x)\}, \; x \in G,$

ii. $u^*(x) = 0, \; x \in S,$

iii. $L\tilde{u}(x) \in F(\tilde{u}_x(x), \tilde{u}(x), u^*(x), x),$

where L^* is the operator adjoint to L and

$$\operatorname{div} w^*(x) = \frac{\partial w_1^*(x)}{\partial x_1} + \cdots + \frac{\partial w_n^*(x)}{\partial x_n}.$$

Then $\tilde{u}(\cdot)$ is an optimal solution of the Dirichlet problem (P_M).

□ By analogy with the proof of Theorem 6.39, it is easy to see that

$$\begin{aligned} &H(u_x(x), u(x), u^*(x), x) - H(\tilde{u}_x(x), \tilde{u}(x), u^*(x), x) - g(u(x), x) + g(\tilde{u}(x), x) \\ &\quad \le -\langle w^*(x), u_x(x) - \tilde{u}_x(x)\rangle + (u(x) - \tilde{u}(x))L^*u^*(x) - (u(x) - \tilde{u}(x))\operatorname{div} w^*(x). \end{aligned}$$

Then because of the definition of H and condition (iii), from the previous inequality, we have

$$\begin{aligned} g(u(x), x) - g(\tilde{u}(x), x) \ge\, & u^*(x)L(u(x) - \tilde{u}(x)) - (u(x) - \tilde{u}(x))L^*u^*(x) \\ & + \langle w^*(x), u_x(x) - \tilde{u}_x(x)\rangle + \operatorname{div} w^*(x)(u(x) - \tilde{u}(x)). \end{aligned} \tag{6.191}$$

It is not hard to see that

$$\begin{aligned} &\langle w^*(x), u_x(x) - \tilde{u}_x(x)\rangle + \operatorname{div} w^*(x)(u(x) - \tilde{u}(x)) \\ &\quad = \sum_{i=1}^{n} \left[w_i^*(x)\frac{\partial(u(x) - \tilde{u}(x))}{\partial x_i} + \frac{\partial w_i^*}{\partial x_i}(u(x) - \tilde{u}(x)) \right] \\ &\quad = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[w_i^*(x)(u(x) - \tilde{u}(x)) \right] = \operatorname{div}[w^*(x)(u(x) - \tilde{u}(x))]. \end{aligned} \tag{6.192}$$

Integrating Eq. (6.191) over the domain G and taking into account Eq. (6.192), we obtain

$$\begin{aligned} \int_G [g(u(x), x) - g(\tilde{u}(x), x)]\mathrm{d}x \ge & \int_G [u^*(x)L(u(x) - \tilde{u}(x)) - (u(x) - \tilde{u}(x))L^*u^*(x)]\mathrm{d}x \\ & + \int_G \operatorname{div} w^*(x)(u(x) - \tilde{u}(x))\mathrm{d}x. \end{aligned} \tag{6.193}$$

Then since $u(\cdot)$ is a feasible solution and in particular, $u(x) = \tilde{u}(x) = \alpha_1(x)$, $x \in S$, it follows from Green's formula in the multidimensional case that the first integral on the right-hand side of Eq. (6.193) is equal to zero. Furthermore, using Gauss's theorem, it is easy to see that

$$\int_G \operatorname{div} w^*(x)(u(x) - \tilde{u}(x))\mathrm{d}x = \int_S \langle w^*(x)(u(x) - \tilde{u}(x)), n\rangle \mathrm{d}s = 0$$

where $n = n(x)$ is the unit outer normal vector for the surface S. Thus,

$$\int_G [g(u(x), x) - g(\tilde{u}(x), x)]\mathrm{d}x \geq 0$$

for all feasible $u(\cdot)$, so $\tilde{u}(\cdot)$ is an optimal solution. ■

Formally replacing Δ with the second-order elliptic operator L in Theorem 6.40 and using the proof techniques of Theorem 6.42, in a similar way we get Theorem 6.43.

Theorem 6.43. Suppose that $\tilde{u}(\cdot)$ is a feasible solution of the multidimensional nonconvex Dirichlet problem (P_{MD}) and there exists a pair $\{w^*(x), u^*(x)\}$ of classical solutions $w^*(x) = (w_1^*(x), \ldots, w_n^*(x)), u^*(x)$ of the adjoint differential inclusion such that the following conditions are satisfied:

i. $(-w^*(x), L^*u^*(x) + u^*(x) - \operatorname{div} w^*(x)) \in F^*(u^*(x), (\tilde{u}_x(x), \tilde{u}(x), L\tilde{u}(x)), x)$,
ii. $g(u, x) - g(\tilde{u}(x), x) \geq \langle u^*(x), u - \tilde{u}(x)\rangle \quad \forall u$,
iii. $\langle u^*(x), L\tilde{u}(x)\rangle = M(\tilde{u}_x(x), \tilde{u}(x), u^*(x), x)$,
iv. $u^*(x) = 0,\ x \in S$.

Then the solution $\tilde{u}(\cdot)$ is optimal.

Remark 6.17. It is understood that in both the convex and the nonconvex multidimensional Neumann's problem (P_{MN}), the boundary conditions of the adjoint differential inclusions consist of the following $w^*(x) = 0$, $\partial u^*(x)/(\partial n) = 0$, $x \in S$.

Remark 6.18. As shown in Refs. [92,180,246], the results obtained can be generalized to the case of more general function spaces—i.e., for generalized solutions of these problems. According to what has been said, it is necessary only to take care of how to define generalized solutions for classical partial differential equations of elliptic type for which the right-hand side is a branch (selection) of the considered multivalued function.

In conclusion, we consider the following problem:

$$\inf \quad J[u(\cdot)] = \iint_Q g(u(x), x)\mathrm{d}x, \tag{6.194}$$

$$\text{subject to } \Delta u(x) = A_1 u_{x_1}(x) + A_2 u_{x_2}(x) + A_3 u(x) + Bw(x), \quad w(x) \in V, \tag{6.195}$$

$$\left.\frac{\partial u(x)}{\partial n}\right|_B = \beta_2(x), \tag{6.196}$$

where A_i (i = 1,2,3) are $n \times n$ matrices, B is an $n \times r$ matrix, $V \subset \mathbb{R}^r$ is a convex set, and g is a continuously differentiable function on u. It is required to find a controlling parameter $\tilde{w}(x) \in V$ such that the solution $\tilde{u}(\cdot)$ corresponding to it minimizes $J[u(\cdot)]$.

The problem in Eqs. (6.194)–(6.196) can be converted to the Neumann problem in Eqs. (6.179)–(6.181) with the multivalued function

$$F(u_1, u_2, u_3, x) = A_1 u_1 + A_2 u_2 + A_3 u_3 + BV. \tag{6.197}$$

Therefore, by Remark 6.16, it is necessary to calculate the LAM F^* to write out a sufficient condition for optimality.

By an elementary calculation, it can be shown that

$$\begin{aligned} H(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, v^*, x) &= \sup_v \{\langle v, v^* \rangle : v \in F(u_1, u_2, u_3, x)\} \\ &= \langle u_1, A_1^* v^* \rangle + \langle u_2, A_2^* v^* \rangle + \langle u_3, A_3^* v^* \rangle + \sup_{w \in V} \langle Bw, v^* \rangle. \end{aligned}$$

So by Theorem 2.1,

$$F^*(v^*, (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{v}), x) = \partial_{(u_1,u_2,u_3)} M(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, v^*, x) = \{A_1^* v^*, A_2^* v^*, A_3^* v^*\}, \tag{6.198}$$

where A_i^* is the adjoint of A_i, $\tilde{v} = A_1 \tilde{u}_1 + A_2 \tilde{u}_2 + A_3 \tilde{u}_3 + B\tilde{w}$, $\tilde{w} \in V$.

Moreover, $F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3; v^*, x) = \{v \in F(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, x) : \langle B\tilde{w}, u^* \rangle = sup_{w \in V} \langle Bw, v^* \rangle\}$.

Now applying Remarks 6.16 and 6.17, we are persuaded that conditions (a_D) and (c_D) of Theorem 6.39 consist of the following:

$$\begin{aligned} &-\varphi^*(x) = A_1^* u^*(x), \quad -p^*(x) = A_2^* u^*(x), \\ &\Delta u^*(x) - \varphi^*_{x_1}(x) - p^*_{x_2}(x) = A_3^* u^*(x) - g'(\tilde{u}(x), x). \end{aligned} \tag{6.199}$$

Then taking into account the boundary conditions for problem (P_N) (Remark 6.16) and the conditions in Eq. (6.199), we have obtained Theorem 6.44.

Theorem 6.44. The feasible solution $\tilde{u}(\cdot)$ corresponding to the control function $\tilde{w}(\cdot) \in V$ minimizes the functional $J[u(\cdot)]$ in the Neumann problem in Eqs. (6.194)–(6.196) if there exists a classical solution $u^*(\cdot)$ satisfying the following conditions:

$$\begin{aligned} &\Delta u^*(x) = A_1^* u^*_{x_1}(x) + A_2^* u^*_{x_2}(x) + A_3^* u^*(x) - g'(\tilde{u}(x), x), \\ &A_1^* u^*(x) = 0, \quad A_2^* u^*(x) = 0, \quad x \in B, \\ &\left. \frac{\partial u^*(x)}{\partial n} \right|_B = 0, \\ &\langle B\tilde{w}(x), u^*(x) \rangle = \sup_{w \in V} \langle Bw, u^*(x) \rangle. \end{aligned}$$

Remark 6.19. As has been shown in these problems, in general cases the adjoint inclusion involves several (auxiliary and main) adjoint variables. Nevertheless, in concrete problems the same inclusion involves only the main variable—i.e., $u^*(\cdot)$ (see Theorem 6.44).

7 On Duality in Elliptic Differential Inclusions

According to the previous definition, let us denote

$$J_*(u^*(x), z^*(x)) : = \iint\limits_{R} [M(\Delta u^*(x) + z^*(x), u^*(x), x) - g^*(z^*(x), x]\mathrm{d}x,$$

where H is a Hamiltonian function and $g^*(z^*,x)$ is the conjugate function to $g(\cdot,x)$ for every fixed $x \in \mathbb{R}^2$. Then the maximization problem,

$$(P_{Dl}) \quad \sup_{\substack{u^*(x), z^*(x), x \in G, \\ u^*(x) = 0, x \in S}} J_*(u^*(x), z^*(x)),$$

is called the dual problem to the primary convex problem (P_C), where $\beta(x) \equiv 0$. It is assumed that $u^*(x) \in C^2(\mathrm{R}) \cap C(\overline{\mathrm{R}})$, $z^*(x) \in C(\mathrm{R})$.

Theorem 6.45. Assume that $u(x)$, $x \in \mathrm{R}$ is an arbitrary feasible solution of the primary problem (P_C) and $\{u^*(x), z^*(x)\}$ is a feasible solution of the dual problem (P_{Dl}) . Then $J(u(x)) \geq J^*(u^*(x))$.

☐ It is clear from the definitions of the functions M and g^* that

$$\begin{aligned} &M(\Delta u^*(x) + z^*(x), u^*(x), x) \leq \langle \Delta u^*(x) + z^*(x), u(x) \rangle - \langle u^*(x), \Delta u(x) \rangle, \\ &\quad g^*(z^*(x), x) \geq \langle z^*(x), u(x) \rangle - g(u(x), x). \end{aligned}$$

Therefore,

$$\begin{aligned} &M(\Delta u^*(x) + z^*(x), u^*(x), x) - g^*(z^*(x), x) \\ &\quad \leq \langle \Delta u^*(x), u(x) \rangle - \langle u^*(x), \Delta u(x) \rangle + g(u(x), x). \end{aligned} \tag{6.200}$$

Then, since $u^*(x) = 0$, $u(x) = 0$, $x \in B$, by Green's theorem we have

$$\begin{aligned} &\iint\limits_{R} [\langle \Delta u^*(x), u(x) \rangle - \langle u^*(x), \Delta u(x) \rangle] \mathrm{d}x \\ &\quad = \int\limits_{B} \left[\left\langle \frac{\partial u^*(x)}{\partial n}, u(x) \right\rangle - \left\langle u^*(x), \frac{\partial u(x)}{\partial n} \right\rangle \right] \mathrm{d}s = 0, \end{aligned} \tag{6.201}$$

where n is the outer normal to the curve B. Then integrating both sides of the inequality in Eq. (6.200), because of Eq. (6.201), we obtain the required inequality. ■

Theorem 6.46. If the feasible solutions $\tilde{u}(x)$ and $\{u^*(x), z^*(x)\}$, $z^*(x) \in \partial g(\tilde{u}(x), x)$ satisfy the conditions of Theorem 6.34, then they are optimal solutions of the primary (P_C) and dual (P_{Dl}) problems, respectively, and their values are equal.

□ To proceed, first note that by Theorem 6.34 $\tilde{u}(x)$ is a solution of the primary problem (P_C). We need to prove that the pair $\{u^*(x),z^*(x)\}$ is a solution to problem (P_{Dl}). By definition of a LAM, condition (a) of Theorem 6.34 is equivalent to the inequality $\langle \Delta u^*(x) + z^*(x), u - \tilde{u}(x)\rangle - \langle u^*(x), v - \Delta\tilde{u}(x)\rangle \geq 0, \quad (u,v) \in \text{gph}\, F(.,.,x)$.

The latter yields

$$(\Delta u^*(x) + z^*(x), u^*(x)) \in \text{dom}\, M(\cdot,\cdot,x) \tag{6.202}$$

where dom $M(\cdot,\cdot,x) := \{(u^*,v^*) : M(u^*,v^*,x) > -\infty\}$. Further, since $\partial g(u,x) \subset$ dom $g^*(\cdot,x)$ it is clear that

$$z^*(x) \in \text{dom}\, g^*(\cdot,x). \tag{6.203}$$

Consequently, it can be concluded from Eqs. (6.203) and (6.203) that the indicated pair of functions $\{u^*(x),z^*(x)\}$ is a pair of feasible solutions; i.e., the set of feasible solutions to (P_{Dl}) is nonempty. Now, let us justify the optimality of the solution $\{u^*(x),z^*(x)\}$ to problem (P_{Dl}). By Lemma 2.6,

$$F^*(v^*,(u,v),x) = \{u^* : M(u^*,v^*,x) = \langle u,u^*\rangle - H(u,v^*,x)\}.$$

Using this formula and condition (a) of Theorem 6.34, we get

$$M(\Delta u^*(x) + z^*(x), u^*(x), x) = \langle \tilde{u}(x), \Delta u^*(x) + z^*(x)\rangle - H(\tilde{u}(x), u^*(x), x).$$

Now based on condition (c) of Theorem 6.34, we have $\langle \Delta\tilde{u}(x), u^*(x)\rangle = H(\tilde{u}(x), u^*(x), x)$. Thus,

$$M(\Delta u^*(x) + z^*(x), u^*(x), x) = \langle \tilde{u}(x), \Delta u^*(x) + z^*(x)\rangle - \langle \Delta\tilde{u}(x), u^*(x)\rangle. \tag{6.204}$$

On the other hand, the inclusion $z^*(x) \in \partial g(\tilde{u}(x),x)$ is equivalent to the equality

$$g^*(z^*(x),x) = \langle \tilde{u}(x), z^*(x)\rangle - g(\tilde{u}(x),x). \tag{6.205}$$

Then, in view of Eqs. (6.203)–(6.205), as in the proof of Theorem 6.45, it is not hard to show that $J(\tilde{u}(x)) = J_*(u^*(x), z^*(x))$. This completes the proof of the theorem. ■

Now let us formulate the dual problem to the convex problem (P_M) with homogeneous boundary conditions. In this case, the duality problem consists of the following:

$$(P_{MDl}) \qquad \sup_{\substack{u^*(x),z^*(x),x\in G,\\ u^*(x)=0,x\in S.}} J_*(u^*(x),z^*(x)),$$

Here,

$$J_*(u^*(x),z^*(x)) = \int_G [M(L^*u^*(x) + z^*(x), u^*(x), x) - g^*(z^*(x),x)]\mathrm{d}x,$$

$$u^*(x) \in C^2(G) \cap C(\overline{G}), \quad z^*(x) \in C(G), \quad x = (x_1,\ldots,x_n).$$

Now by replacing Δ with the second-order elliptic operator L and using the idea suggested in the proofs of Theorems 6.45 and 6.46, it is easy to get Theorem 6.47.

Theorem 6.47. If $\tilde{u}(x)$ and the pair of functions $\{u^*(x),z^*(x)\}$ are feasible solutions to the primary convex problem (P_M) with homogeneous boundary value conditions and dual problem (P_{MDl}), respectively, then $J(\tilde{u}(x)) \geq J_*(u^*(x), z^*(x))$. In addition, if assertions (i) and (ii) of Theorem 6.37 for sufficiency of optimality are valid here and $z^*(x) \in \partial g(\tilde{u}(x), x)$, then the values of the cost functional are equal and $\{u^*(x),z^*(x)\}$ is a solution of the dual problem (P_{MDl}).

6.6 Optimization of Discrete-Approximation and Differential Inclusions of Parabolic Type and Duality

The present section is devoted to an investigation of first boundary value problems for partial differential inclusions of parabolic type. It can be divided conditionally into two parts.

In the second part of this section, we construct the dual problem to convex problems for partial differential inclusions of parabolic type. Note that a sufficient condition for an extremum is the extremal relation for the direct and dual problems. As in the preceding sections, it is known that for the theory of partial differential equations, the concept of a generalized solution is important from both the theoretical and the practical point of view.

We emphasize that the solution of the considered differential inclusions is taken in the space $C^{1,2}$ [92,178].

At the end of Subsection 2, we consider an optimal control problem described by the heat equation. This example shows that in known problems the conjugate inclusion coincides with the conjugate equation, which is traditionally obtained with the help of the Hamiltonian function.

A mapping $F^*(\cdot, (\tilde{x}, \tilde{v})) : \mathbb{R}^n \to P(\mathbb{R}^n), (\tilde{x}, \tilde{v}) \in \operatorname{gph} F$ defined by

$$F^*(v^*; (\tilde{x}, \tilde{v})) = \{x^* : H_F(x, v^*) - H_F(\tilde{x}, v^*) \leq \langle x^*, x - \tilde{x} \rangle \forall x \in \mathrm{R}^n\} \tag{6.206}$$

is the LAM for a nonconvex mapping $F : \mathbb{R}^n \to P(\mathbb{R}^n)$.

Remember that for a convex mapping F, the LAM is equal to $\partial H_F(x,v^*)$ (Theorem 2.1).

In Subsection 1, we study the problem for the so-called partial differential inclusions of parabolic type:

$$\inf \quad I(x) = \iint_Q g(x(t,\tau), t, \tau) \mathrm{d}t\, \mathrm{d}\tau \tag{6.207}$$

$$\text{subject to} \quad \frac{\partial^2 x(t,\tau)}{\partial \tau^2} - \frac{\partial x(t,\tau)}{\partial t} \in F(x(t,\tau), t, \tau), \quad 0 < t \leq 1, \;\; 0 < \tau < 1, \tag{6.208}$$

$$x(0,\tau)=\alpha(\tau),\quad x(t,0)=\beta_0(t),\quad x(t,1)=\beta_1(t),\quad Q=[0,1]\times[0,1]. \tag{6.209}$$

Here, $F(\cdot,t,\tau):\mathbb{R}^n\to P(\mathbb{R}^n)$ is a multivalued mapping, $g:\mathbb{R}^n\times Q\to\mathbb{R}\nabla$, and α and β_0,β_1 are continuous functions, $\alpha:[0,1]\to\mathbb{R}^n$, β_i:$[0,1]\to\mathbb{R}^n(i=0,1)$. The problem is to find a classical solution $\tilde{x}(t,\tau)$ of the so-called first boundary value problem in Eqs. (6.207) and (6.208) that minimizes I. Note that in Refs. [4,21,46, 164,166], the classical optimal control problems described by hyperbolic or elliptic-type differential equations are extended to the case of corresponding differential inclusions and the obtained problems, naturally, are called problems for hyperbolic and elliptic differential inclusions, respectively. Of course, in this sense the name parabolic differential inclusions for problems (6.207)−(6.209) is justified.

Here a solution is understood to be classical solution only for simplicity of the exposition. As will be seen at the end of Subsection 1, the results obtained can be extended to the case of a generalized solution.

The subject of the next investigation in Subsection 1 is the multidimensional optimal control problem for partial differential inclusions of parabolic type:

$$\inf \quad I(x)=\int_0^1\int_G g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau \tag{6.210}$$

$$\text{subject to }\ Lx(t,\tau)\in\frac{\partial x(t,\tau)}{\partial t}+F(x(t,\tau)), \tag{6.211}$$

$$x(0,\tau)=\alpha(\tau),\quad t\in G\subset\mathbb{R}^n, \tag{6.212}$$

$$x(t,\tau)=\beta(t,\tau),\quad (t,\tau)\in H, \tag{6.213}$$

where $F:\mathbb{R}^n\to P(\mathbb{R}^n)$ and G is the domain of change of arguments $\tau=(\tau_1,\ldots,\tau_n)$ in the differential inclusion in Eq. (6.211), with piecewise smooth boundary S. Thus, the domain in which Eq. (6.211) is given is a cylinder $D=\{\tau\in G,\ 0<t<1\}$ $(D\subset\mathbb{R}^{n+1})$ of height 1, base G,H is the lateral surface of D:$H=\{\tau\in S,\ 0<t<1\}$, and $G\times\{0\}$ and $G\times\{1\}$ are the lower and upper bases, respectively. L is a second-order elliptic operator.

A function $x(t,\tau)$ in $C^{1,2}(D)\cap C[D\cup H\cup(G\times\{0\}]$, that satisfies the inclusion in Eq. (6.211) in D, the initial condition in Eq. (6.212) on $G\times\{0\}$, and the boundary condition in Eq. (6.213) on H is called a classical solution of the problem posed, where $C^{1,2}$ (D) is the space of functions u, having continuous derivatives $\partial u/\partial t$, $\partial^2 u/\partial\tau_i\partial\tau_j$, $i,j=1,\ldots,n$ [164,178].

1 *Sufficient Conditions for Optimality for Differential Inclusions of Parabolic Type*

Theorem 6.48. Suppose that $g:\mathbb{R}^n\times Q\to\mathbb{R}$ is continuous and convex with respect to x and $F:\mathbb{R}^n\times Q\to P(\mathbb{R}^n)$ is a convex mapping. Then for the optimality of the solution $\tilde{x}(t,\tau)$ among all admissible solutions in the problem in Eqs. (6.207) and (6.208), it is sufficient that there exists a classical solution $x^*(t,\tau)$ of the following boundary value problem:

a. $\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} \in F^*(x^*(t,\tau); (\tilde{x}(t,\tau), \tilde{x}_{\tau^2}(t,\tau) - \tilde{x}_t(t,\tau)), t,\tau) - \partial g(\tilde{x}(t,\tau), t,\tau)$,
b. $x^*(1,\tau) = 0,\ x^*(t,0) = 0,\ x^*(t,1) = 0,\ (t,\tau)\in Q$,
c. $\frac{\partial^2 \tilde{x}(t,\tau)}{\partial\tau^2} - \frac{\partial \tilde{x}(t,\tau)}{\partial t} \in F(\tilde{x}(t,\tau); x^*(t,\tau), t,\tau)$.

□ For a convex multivalued mapping F, using the Moreau−Rockefeller theorem from condition (a), we obtain the inclusion

$$\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} \in \partial[H_F(\tilde{x}(t,\tau), x^*(t,\tau), t,\tau) - g(x(t,\tau), t,\tau], \quad (t,\tau)\in Q.$$

Using the definitions of subdifferential and condition (c) for all admissible solutions $x(t,\tau)$, we rewrite the last relation in the form

$$\left\langle \frac{\partial^2 x(t,\tau)}{\partial\tau^2} + \frac{\partial x(t,\tau)}{\partial t}, x^*(t,\tau)\right\rangle - \left\langle \frac{\partial^2 \tilde{x}(t,\tau)}{\partial\tau^2} + \frac{\partial \tilde{x}(t,\tau)}{\partial t}, x^*(t,\tau)\right\rangle$$
$$- g(x(t,\tau), t,\tau) + g(\tilde{x}(t,\tau), t,\tau) \le \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t}, x(t,\tau) - \tilde{x}(t,\tau)\right\rangle.$$

Integrating this inequality over Q, we get

$$\iint_Q [g(x(t,\tau),t,\tau) - g(\tilde{x}(t,\tau),t,\tau]\mathrm{d}t\,\mathrm{d}\tau \ge -\iint_Q \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t}, x(t,\tau) - \tilde{x}(t,\tau)\right\rangle \mathrm{d}t\,\mathrm{d}\tau$$
$$+ \iint_Q \left\langle \frac{\partial^2 (x(t,\tau) - \tilde{x}(t,\tau))}{\partial\tau^2} - \frac{\partial (x(t,\tau) - \tilde{x}(t,\tau))}{\partial t}, x^*(t,\tau)\right\rangle \mathrm{d}t\,\mathrm{d}\tau. \tag{6.214}$$

After simple transformations, we obtain that the right side of the inequality in Eq. (6.214) is equal to zero. Indeed for brevity of notation, we denote the right side of Eq. (6.214) by R. Then

$$\mathrm{R} = -\iint_Q \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}, x(t,\tau) - \tilde{x}(t,\tau)\right\rangle \mathrm{d}t\,\mathrm{d}\tau + \iint_Q \left\langle \frac{\partial^2 (x(t,\tau) - \tilde{x}(t,\tau))}{\partial\tau^2}, x^*(t,\tau)\right\rangle \mathrm{d}t\,\mathrm{d}\tau$$
$$- \iint_Q \frac{\partial}{\partial t}\langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau,$$

where, since by condition (b), $x^*(1,\tau) = 0$ and $x(t,\tau)$, and $\tilde{x}(t,\tau)$ are admissible solutions, i.e., $x(0,\tau) = \tilde{x}(0,\tau) = \alpha(\tau)$, we have

$$\iint_Q \frac{\partial}{\partial t}\langle x(t,\tau) - \tilde{x}(t,\tau), x^*(t,\tau)\rangle \mathrm{d}t\,\mathrm{d}\tau = \int_0^1 \langle x(1,\tau) - \tilde{x}(1,\tau), x^*(1,\tau)\rangle \mathrm{d}\tau$$
$$- \int_0^1 \langle x(0,\tau) - \tilde{x}(0,\tau), x^*(0,\tau)\rangle \mathrm{d}\tau = 0.$$

So it is clear that

$$\mathrm{R}=-\iint\limits_{Q}\frac{\partial}{\partial\tau}\left\langle\frac{\partial x^{*}(t,\tau)}{\partial\tau},x(t,\tau)-\tilde{x}(t,\tau)\right\rangle\mathrm{d}t\,\mathrm{d}\tau+\iint\limits_{Q}\frac{\partial}{\partial\tau}\left\langle x^{*}(t,\tau),\frac{\partial}{\partial\tau}(x(t,\tau)-\tilde{x}(t,\tau))\right\rangle\mathrm{d}t\,\mathrm{d}\tau$$

$$=-\int_{0}^{1}\left\langle\frac{\partial x^{*}(t,1)}{\partial\tau},x(t,1)-\tilde{x}(t,1)\right\rangle\mathrm{d}t+\int_{0}^{1}\left\langle\frac{\partial x^{*}(t,0)}{\partial\tau},x(t,0)-\tilde{x}(t,0)\right\rangle\mathrm{d}t$$

$$+\int_{0}^{1}\left\langle x^{*}(t,1),\frac{\partial}{\partial\tau}(x(t,1)-\tilde{x}(t,1))\right\rangle\mathrm{d}t-\int_{0}^{1}\left\langle x^{*}(t,0),\frac{\partial}{\partial\tau}(x(t,0)-\tilde{x}(t,0))\right\rangle\mathrm{d}t=0,$$

where it is taken into account that $x^{*}(t,0)=x^{*}(t,1)=0$ by condition (b) of the theorem. Thus, we have finally

$$\iint\limits_{Q}g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau\geq\iint\limits_{Q}g(\tilde{x}(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau.$$

The theorem is proved. ■

Theorem 6.49. Let us consider the nonconvex problem in Eqs. (6.207)–(6.209) and let $\tilde{x}(t,\tau)$ be an admissible solution of this problem. Then for the optimality of the solution $\tilde{x}(t,\tau)$ among all admissible solutions, it is sufficient that there exists a classical solution $x^{*}(t,\tau)$ of the following boundary value problem:

i. $\frac{\partial^2 x^{*}(t,\tau)}{\partial\tau^2}+\frac{\partial x^{*}(t,\tau)}{\partial t}\in F^{*}(x^{*}(t,\tau);(\tilde{x}(t,\tau),\tilde{x}_{\tau^2}(t,\tau)-\tilde{x}_t(t,\tau)),t,\tau)$,
ii. $g(x,t,\tau)-g(\tilde{x}(t,\tau),t,\tau)\geq\langle x^{*}(t,\tau),x-\tilde{x}(t,\tau)\rangle$ for all x,
iii. $x^{*}(1,\tau)=0$, $x^{*}(t,0)=0$, $x^{*}(t,1)=0$,
iv. $\frac{\partial^2\tilde{x}(t,\tau)}{\partial\tau^2}-\frac{\partial\tilde{x}(t,\tau)}{\partial t}\in F(\tilde{x}(t,\tau);x^{*}(t,\tau),t,\tau)$, $(t,\tau)\in Q$.

Here the LAM F^{*} is defined by Eq. (6.206).

□ By the definition of the LAM F^{*} and by conditions (i) and (ii), we have

$$\begin{aligned}&H_F(x,(t,\tau),x^{*}(t,\tau),t,\tau)-H_F(\tilde{x},(t,\tau),x^{*}(t,\tau),t,\tau)\\&\qquad\leq\left\langle\frac{\partial^2x^{*}(t,\tau)}{\partial\tau^2}+\frac{\partial x^{*}(t,\tau)}{\partial t},x(t,\tau)-\tilde{x}(t,\tau)\right\rangle,\\&g(x,(t,\tau),t,\tau)-g(\tilde{x},(t,\tau),t,\tau)\geq\langle x^{*}(t,\tau),x(t,\tau)-\tilde{x}(t,\tau)\rangle.\end{aligned}\tag{6.215}$$

Now, using condition (iv) and integrating the relations in Eq. (6.215) and adding them, we get the inequality in Eq. (6.214). Then starting from the inequality in Eq. (6.214) similarly as in the proof of Theorem 6.48, it can be shown that $\tilde{x}(t,\tau)$ is optimal.

We now apply the results of this section to the problem in Eqs. (6.210) and (6.213); here, the use of Theorem 6.48 plays a decisive role in the investigation of this problem.

Theorem 6.50. Suppose that g is a continuous function that is convex with respect to x, and F is a convex mapping. Then $\tilde{x}(t,\tau)$ minimizes the functional equation (6.210) among all admissible solutions of the problem in Eqs. (6.210)–(6.213) if there exists a classical solution $x^*(t,\tau)$ of the following boundary value problem:

$$\begin{gathered} L^*x^*(t,\tau)+x_t^*(t,\tau)\in F^*(x^*(t,\tau);(\tilde{x}(t,\tau),L\tilde{x}(t,\tau)-\tilde{x}_t(t,\tau)))-\partial g(\tilde{x}(t,\tau),t,\tau), \\ L\tilde{x}(t,\tau)-\tilde{x}_t(t,\tau)\in F(\tilde{x}(t,\tau);x^*(t,\tau)), \\ x^*(1,\tau)=0,\ \ \tau\in G;\quad x^*(t,\tau)=0,\ \ (t,\tau)\in H, \end{gathered} \tag{6.216}$$

where L^* is the operator adjoint to L.

□ By arguments analogous to those in the proof of Theorem 6.48, it is not hard to see that

$$\begin{aligned} \int_0^1\int_G[g(x(t,\tau),t,\tau)-g(\tilde{x}(t,\tau),t,\tau)]\mathrm{d}t\,\mathrm{d}\tau &\geq -\int_0^1\int_G(L^*x^*(t,\tau)+x_t^*(t,\tau))(x(t,\tau)-\tilde{x}(t,\tau))\mathrm{d}t\,\mathrm{d}\tau \\ &+\int_0^1\int_G\left[L(x(t,\tau)-\tilde{x}(t,\tau))-\frac{\partial}{\partial t}(x(t,\tau)-\tilde{x}(t,\tau))\right]x^*(t,\tau)\mathrm{d}t\,\mathrm{d}\tau. \end{aligned} \tag{6.217}$$

Using boundary condition (7) and the condition $x^*(t,\tau)=0$, $(t,\tau)\in H$, we get from Green's formula that

$$\int_G[x^*(t,\tau)L(x(t,\tau)-\tilde{x}(t,\tau))-(x(t,\tau)-\tilde{x}(t,\tau))L^*x^*(t,\tau)]\mathrm{d}\tau=0. \tag{6.218}$$

Moreover, by the initial condition in Eq. (6.212) and $x^*(1,\tau)=0$, we have

$$\int_0^1\int_G\frac{\partial}{\partial t}[x^*(t,\tau)(x(t,\tau)-\tilde{x}(t,\tau))]\mathrm{d}t\,\mathrm{d}\tau=\int_G x^*(1,\tau)(x(1,\tau)-\tilde{x}(1,\tau))\mathrm{d}\tau=0. \tag{6.219}$$

Then with the use of Eqs. (6.218) and (6.219), we conclude that the right side of Eq. (6.217) is equal to zero; i.e.,

$$\int_0^1\int_G g(x(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau\geq\int_0^1\int_G g(\tilde{x}(t,\tau),t,\tau)\mathrm{d}t\,\mathrm{d}\tau.$$

The theorem is proved. ■

Remark 6.20. If by analogy to the classical theory of parabolic equations, we take $x_t - Lx$ instead of $Lx - x_t$, then F^* must be replaced by $(-F)^*$. Therefore, for the computation of $\tilde{F}^* \neq \varnothing$, $\tilde{F}(x) = \alpha F(x)$ (α = constant) by Lemma 2.7, we have the formula

$$\tilde{F}^*(v^*; (x, \alpha v)) = |\alpha| F^*(v^* \text{sgn}\, \alpha; (x, v)),$$

which is easily obtained from the definition of LAM.

Remark 6.21. With the use of Theorem 6.49, there can be obtained sufficient conditions for the nonconvex problem in Eqs. (6.210) and (6.213).

Remark 6.22. Suppose that we have the problem in Eqs. (6.210)–(6.213) with homogeneous boundary conditions, where $\alpha(\tau) \in L_2(G)$, $H^{1,0}(D)$ is the Hilbert space (for a more detailed study, see, for example, Refs. [91,180]) consisting of the elements $x(t,\tau) \in L_2(D)$ having square-integrable generalized derivatives on D, where the inner product and the norm are defined by the respective expressions

$$\langle x_1, x_2 \rangle_{H^{1,0}(D)} = \int_D \left(x_1 x_2 + \frac{\partial x_1}{\partial \tau} \frac{\partial x_2}{\partial \tau} \right) \mathrm{d}t\, \mathrm{d}\tau, \quad \|x\|_{H^{1,0}(D)} = \sqrt{\langle x, x \rangle_{H^{1,0}(D)}}.$$

By analogy to the classical theory of the first boundary value problem for partial differential equations of parabolic type, a function $x(t,\tau) \in H^{1,0}(D)$ is called a generalized solution of the problem in Eqs. (6.210)–(6.213) if it satisfies the boundary conditions $x(t,\tau) = 0$, $(t,\tau) \in H$ and the identity

$$\int_D \left(x\zeta_t - \sum_{i,j=1}^{n} d_{ij} \frac{\partial x}{\partial \tau_j} \frac{\partial \zeta}{\partial \tau_i} - \sum_{i=1}^{n} b_i \frac{\partial x}{\partial \tau_i} \zeta - cx\zeta \right) \mathrm{d}t\, \mathrm{d}\tau = \int_G \alpha \zeta(0,\tau) \mathrm{d}\tau + \int_D f\zeta \mathrm{d}t\, \mathrm{d}\tau$$

for all $\zeta(t,\tau) \in H^1(D)$ [91,180,246] with the conditions $\zeta(1,\tau) = 0, \tau \in G$, $\zeta(t,\tau) = 0$, $(t,\tau) \in H$.

Here, $f = f(x)$ is an arbitrary measurable selection [111] of the multivalued mapping F.

It easy to see that the concept of a solution almost everywhere (a.e.) can be introduced in addition to the concepts of a classical solution and generalized solution.

A function $x(t,\tau) \in H^{2,1}(D)$ [25] is said to be a solution a.e. for the problem in Eqs. (6.210)–(6.213) with homogeneous boundary conditions if it satisfies for almost all $(t,\tau) \in D$ the inclusion in Eq. (6.211), the initial condition in Eq. (6.212), and the homogeneous boundary conditions. A generalized solution for the adjoint boundary value problem is defined analogously.

2 Duality in Differential Inclusions of Parabolic Type

Let us introduce the function

$$I_*(x^*,u^*)=\iint\limits_Q \left[M_F\left(\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t}+u^*(t,\tau),x^*(t,\tau)\right)-g^*(u^*(t,\tau),t,\tau)\right]\mathrm{d}t\,\mathrm{d}\tau$$
$$-\int_0^1\langle x^*(0,\tau),\alpha(\tau)\rangle\mathrm{d}\tau.$$

Then the problem of determining the supremum

$$\sup_{\substack{x^*(t,\tau),u^*(t,\tau),x^*(1,\tau)=0,\\ x^*(t,0)=x^*(t,1)=0}} I_*(x^*,u^*) \tag{6.220}$$

is called the dual problem to the convex problem in Eqs. (6.207)–(6.209) with homogenous boundary conditions ($\beta_0(t)=\beta_1(t)=0$). It is clear that the boundary conditions in the problem in Eqs. (6.207)–(6.209) can always be made homogeneous by a change of variables. We assume that $x^*\in C^{1,2}(Q)$ and u^* is in the class of continuous functions on Q in the problem in Eq. (6.220).

Theorem 6.51. The inequality $I(x)\geq I^*(x^*,u^*)$ is satisfied for all admissible solutions x and $\{x^*,u^*\}$ of the problem in Eqs. (6.206)–(6.209) with homogeneous boundary condition and all solutions of the dual problem in Eq. (6.220), respectively.

☐ The proof is similar to the one for duality theorems in the preceding sections. By definition of the functions M_F, it is clear that for all $x=x(t,\tau)$, $x^*=x^*(t,\tau)$, $u^*=u^*(t,\tau)$,

$$M_F\left(\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t}-u^*(t,\tau),x^*(t,\tau)\right)$$
$$\leq-\left\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t}-u^*(t,\tau),x^*(t,\tau)\right\rangle+\left\langle x^*(t,\tau),\frac{\partial^2 x(t,\tau)}{\partial\tau^2}-\frac{\partial x(t,\tau)}{\partial t}\right\rangle. \tag{6.221}$$

Moreover using the definition of $g^*$and $I^*(x^*,u^*)$ and integrating Eq. (6.221), we found

$$I_*(x^*(t,\tau),u^*(t,\tau))\leq-\iint\limits_Q\left\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2},x(t,\tau)\right\rangle\mathrm{d}t\,\mathrm{d}\tau+\iint\limits_Q\left\langle x^*(t,\tau),\frac{\partial^2 x(t,\tau)}{\partial\tau^2}\right\rangle\mathrm{d}t\,\mathrm{d}\tau$$
$$-\iint\limits_Q\frac{\partial}{\partial t}\langle x^*(t,\tau),x(t,\tau)\rangle\mathrm{d}t\,\mathrm{d}\tau-\int_0^1\langle x^*(0,\tau),\alpha(\tau)\rangle\mathrm{d}\tau+I(x(t,\tau)). \tag{6.222}$$

Similarly to the proof of Theorem 6.48, using the condition $x^*(1,\tau)=0$, it can be shown that the sum of the last two integrals on the right side of the equality in

Eq. (6.222) is equal to zero. On the other hand, taking into account the homogeneous boundary conditions $x(t,0)=0$ and $x\,(t,1)=0$, it is not hard to show that

$$\iint\limits_{Q}\left[\left\langle x^*(t,\tau),\frac{\partial^2 x(t,\tau)}{\partial\tau^2}\right\rangle-\left\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2},x(t,\tau)\right\rangle\right]\mathrm{d}t\,\mathrm{d}\tau$$

$$=\int\limits_0^1\left[\left\langle\frac{\partial x^*(t,\tau)}{\partial t},x(t,\tau)\right\rangle\right]_{\tau=0}^{\tau=1}\mathrm{d}t-\int\limits_0^1\left[\left\langle x^*(t,\tau),\frac{\partial x(t,\tau)}{\partial\tau}\right\rangle\right]_{\tau=0}^{\tau=1}\mathrm{d}t=0.$$

Thus, we have from the inequality in Eq. (6.222) that $I^*(x^*(t,\tau),u^*(t,\tau))\le I(x(t,\tau))$. ■

Theorem 6.52. If the solutions $\tilde{x}(t,\tau)$ and $\{x^*(t,\tau),\ u^*(t,\tau)\}$, $u^*(t,\tau)\in\partial g(\tilde{x}(t,\tau),t,\tau)$ satisfy conditions (a)–(c) of Theorem 6.48, then they are solutions of the direct and dual problems, and their values are equal.

□ The fact that $\tilde{x}(t,\tau)$is a solution of the direct problem was proved in Theorem 6.48. We prove the remaining assertions. By the definition of LAM, condition (a) of Theorem 6.48 is equivalent to the inequality

$$\left\langle u^*(t,\tau)+\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t},x-\tilde{x}(t,\tau)\right\rangle$$
$$-\left\langle x^*(t,\tau),v-\frac{\partial^2\tilde{x}(t,\tau)}{\partial\tau^2}+\frac{\partial\tilde{x}(t,\tau)}{\partial t}\right\rangle\ge 0,\quad (x,v)\in\operatorname{gph}F,$$

which means that

$$\left(u^*(t,\tau+\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t},x^*(t,\tau)\right)\in\operatorname{dom}M_F:=\left\{(x^*,v^*):M_F(x^*,v^*)>-\infty\right\}.$$

Moreover, from $\partial g(x,t,\tau)\subset\operatorname{dom}g^*(\,\cdot\,,t,\tau)$, it follows that $u^*(t,\tau)\in\operatorname{dom}g^*(\cdot,t,\tau)$.

Then taking into account these inclusions, it can be concluded that the solution $\{x^*,u^*\}$ is an admissible solution. Further, by Lemma 2.6, we get

$$M_F\left(\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}-\frac{\partial x^*(t,\tau)}{\partial t}-u^*(t,\tau),x^*(t,\tau)\right)$$
$$=\left\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t}+u^*(t,\tau),\tilde{x}(t,\tau)\right\rangle-H_F(\tilde{x}(t,\tau),x^*(t,\tau)),\tag{6.223}$$

and from condition (c), we have

$$H_F(\tilde{x}(t,\tau),x^*(t,\tau))=\left\langle\frac{\partial^2\tilde{x}(t,\tau)}{\partial\tau^2}-\frac{\partial\tilde{x}(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle.\tag{6.224}$$

Then, in view of Eqs. (6.223) and (6.223) and $u^*(t,\tau) \in \partial g(\tilde{x}(t,\tau),t,\tau)$, it is easy to establish that instead of the inequality in Eq. (6.222) in the proof of Theorem 2.1 (see also Theorem 6.46), we have equality; i.e., $I(\tilde{x}(t,\tau)) = I_*(x^*(t,\tau), u^*(t,\tau))$ and $\{x^*(t,\tau),\ u^*(t,\tau)\}$ is optimal. ■

We remark that the dual problem to the convex problem in Eqs. (6.210)–(6.212) with homogeneous boundary conditions consists of the following:

$$\sup_{\substack{x^*(t,\tau), u^*(t,\tau), x^*(1,\tau)=0,\\ x^*(t,\tau)=0, (t,\tau)\in H,}} I_*(x^*,u^*), \quad \tau \in G \tag{6.225}$$

where

$$I_*(x^*,u^*) = \int_0^1\int_G \left[M_F\left(L^* x^*(t,\tau) + \frac{\partial x^*(t,\tau)}{\partial t} + u^*(t,\tau), x^*(t,\tau) \right) - g^*(u^*(t,\tau),t,\tau) \right] \mathrm{d}t\,\mathrm{d}\tau$$
$$- \int_G x^*(0,\tau)\alpha(\tau)\mathrm{d}\tau. \tag{6.226}$$

By extending the proof of Theorems 6.51 and 6.52 to the case under consideration, it is not hard to get the following result.

Theorem 6.53. If $\tilde{x}(t,\tau)$ and $\{x^*(t,\tau),\ u^*(t,\tau)\}$, $u^*(t,\tau) \in \mathrm{dom}\ g^*(\cdot, t,\tau)$ are admissible solutions to the primary convex problem in Eqs. (6.210)–(6.213) with homogeneous boundary conditions and of the dual problem in Eqs. (6.225) and (6.226), respectively, then $I_*(\tilde{x}(t,\tau)) \geq I_*(x^*(t,\tau), u^*(t,\tau))$.

If the condition of Theorem 6.50 that suffices for optimality is valid here, then the strict equality holds, and $\{x^*(t,\tau),\ u^*(t,\tau)\}$ is a solution of the dual problem.

We now consider an example:

$$\inf \quad I(x)$$

$$\text{subject to } \frac{\partial^2 x(t,\tau)}{\partial \tau^2} - \frac{\partial x(t,\tau)}{\partial t} = Ax(t,\tau) + Bu(t,\tau), \quad u(t,\tau) \in U$$

$$x(0,\tau) = \alpha(\tau), \quad x(t,0) = \beta_0(t), \quad x(t,1) = \beta_1(t), \tag{6.227}$$

where A and B are $n \times n$ and $n \times r$ matrices, respectively, $U \subset \mathbb{R}^r$ is a convex closed set, and g is a continuously differentiable function of x. It is required to find a controlling parameter $\tilde{u}(t,\tau) \in U$ such that the solution $\tilde{x}(t,\tau)$ corresponding to it minimizes I.

Now applying Theorem 6.48 as in the example considered in Section 6.5, we get

$$\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} = A^* x^*(t,\tau) - g'(\tilde{x}(t,\tau), t, \tau),$$

$$x^*(1,\tau) = 0, \quad x^*(t,0), \quad x^*(t,1) = 0, \tag{6.228}$$

$$\langle B\tilde{u}(t,\tau), x^*(t,\tau)\rangle = \sup_{u\in U}\langle Bu, x^*(t,\tau)\rangle.$$

Therefore, we have obtained Theorem 6.54.

Theorem 6.54. The solution $\tilde{x}(t,\tau)$ corresponding to the control $\tilde{u}(t,\tau)$ minimizes $I(x)$ in the problem in Eq. (6.227) if there exists a function $x^*(t,\tau)$ satisfying Eq. (6.228).

On the other hand, by elementary computations, we find that

$$M_F(x^*, v^*) = \begin{cases} -\infty, & \text{if } x^* \neq A^* v^*, \\ -W_U(B^* v^*), & \text{if } x^* = A^* v^*, \end{cases}$$

where W_U is a support function of U. Thus, in the dual problem in Eq. (6.227), $I^*(x^*,u^*)$ has the form

$$I_*(x^*, u^*) = -\iint_Q [W_U(B^* x^*(t,\tau)) + g^*(u^*(t,\tau), t, \tau)]\mathrm{d}t\,\mathrm{d}\tau - \int_0^1 \langle x^*(0,\tau), \alpha(\tau)\rangle \mathrm{d}\tau,$$

where

$$\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} = A^* x^*(t,\tau) - u^*(t,\tau).$$

6.7 Optimization of the First Boundary Value Problem for Hyperbolic-Type Discrete-Approximation and Differential Inclusions

This section deals with the first boundary value problem for discrete (P_D), the discrete-approximation problem on a uniform grid and differential (P_C) inclusions of hyperbolic type. In the form of the Euler–Lagrange inclusion, necessary and sufficient conditions for optimality are derived for these problems on the basis of new concept of LAMs. In Subsection 4, the results obtained are generalized to the multidimensional case with a second-order elliptic operator (P_M). It must be pointed out that in hyperbolic differential inclusions, the solution is taken in the space of classical solutions.

First, we consider the following optimization problem for discrete hyperbolic inclusions

$$\inf \sum_{t=2,\ldots,T,x=1,\ldots,L-1} g_{t,x}(u_{t,x}) \tag{6.229}$$

$$\text{subject to } \ u_{t+1,x} \in F_{t,x}(u_{t-1,x}, u_{t,x-1}, u_{t,x}, u_{t,x+1}), \tag{6.230}$$

$$u_{t,0} = \alpha_{0t}, \quad u_{t,L} = \alpha_{Lt}, \quad u_{0,x} = \beta_{0x}, \quad u_{1,x} = \beta_{1x} \tag{6.231}$$

$$t = 1, \ldots, T-1; \quad x = 1, \ldots, L-1.$$

where $g_{t,x} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ are real valued functions, $F_{t,x}$ are multivalued mappings, $F_{t,x} : \mathbb{R}^{4n} \to P(\mathbb{R}^n)$ and α_{0t}, α_{Lt}, β_{0x}, β_{1x} are fixed vectors, and T, L are positive integers. We label the problem in Eqs. (6.229)–(6.231) (P_D), and call it the first boundary value problem for a discrete inclusion of hyperbolic type. The problem (P_D) is convex if the mappings $F_{t,x}$ are convex and $g_{t,x}$ are convex proper functions.

A set of points $\{u_{t,x}\}_D = \{u_{t,x} : (t,x) \in D\}$, where

$$D = \{(t,x) : t = 0, \ldots, T; x = 0, \ldots, L, (t,x) \neq (0,0), \ (0,L), \ (T,0), \ (T,L)\}$$

is called a feasible solution for the problem (P_D) if it satisfies the inclusions in Eq. (6.230) and boundary conditions in Eq. (6.231). One can check easily that for any fixed T and L the boundary condition in Eq. (6.231) possesses a feasible solution, and the number of points to be determined is equal to the number of discrete inclusions. In this sense, the name *discrete hyperbolic inclusions* is justified.

The following condition is assumed below for the functions $g_{t,x}$ and the mappings $F_{t,x}$ ($t = 1, \ldots, T-1$; $x = 1, \ldots, L-1$).

Hypothesis H1. Let $\{\tilde{u}_{t,x}\}_D$ be an optimal solution of the problem (P_D) and assume that the cone of tangent directions $K_{\text{gph } F_{t,x}}(\tilde{u}_{t-1,x}, \tilde{u}_{t,x-1}, \tilde{u}_{t,x}, \tilde{u}_{t,x+1}, \tilde{u}_{t+1,x})$ is locally tent. Assume, moreover, that the functions $g_{t,x}$ admit a CUA $h_{t,x}(\overline{u}, \tilde{u}_{t,x})$ at the points $\tilde{u}_{t,x}$, which is continuous with respect to $\overline{u}$.

Hypothesis H2. Let $\{u^0_{t,x}\}_D$ be a feasible solution for convex problem (P_D). Assume that either (a) or (b) is satisfied:

a. $(u^0_{t-1,x}, u^0_{t,x-1}, u^0_{t,x}, u^0_{t,x+1}, u^0_{t+1,x}) \in \text{ri gph } F_{t,x} u_t,$
$x^0 \in \text{ri dom } g_{t,x}, \quad t = 1, \ldots, T-1; \ \ x = 1, \ldots, L-1.$

b. For $u^0_{t,x} \in \text{ri dom } g_{t,x}, t = 1, \ldots, T-1; x = 1, \ldots, L-1$ the indicated points in (a) are interior points of gph $F_{t,x}$, except maybe one of them.

In Subsection 3, we investigate the following problem for hyperbolic differential inclusion:

$$\inf \quad J(u(\cdot,\cdot)) := \iint\limits_R g(u(t,x),t,x)\mathrm{d}x\,\mathrm{d}t + \int_0^1 g_0(u(1,x),x)\mathrm{d}x \tag{6.232}$$

$$\text{subject to} \quad \Box u(t,x) \in F(u(t,x),t,x), \quad (t,x) \in R = [0,1] \times [0,1] \tag{6.233}$$

$$u(0,x) = \varphi^0(x), \quad \frac{\partial u(0,x)}{\partial t} = \varphi^1(x), \tag{6.234}$$

$$u(t,0) = \alpha_0(t), \quad u(t,1) = \alpha_1(t),$$

where $\Box$ is the operator defined as $\Box = (\partial^2/\partial t^2) - (\partial^2/\partial x^2)$. Here $F(\cdot,t,x) : \mathbb{R}^n \to P(\mathbb{R}^n)$, $g : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$, $g_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, and φ^i, α_i $(i = 0,1)$ are continuous functions. We label this continuous problem (P_C) and call it the first boundary value problem for hyperbolic differential inclusions. It is required to find a solution $\tilde{u}(\cdot,\cdot)$ of the boundary value problem in Eqs. (6.233) and (6.234) that minimizes $J(u(\cdot,\cdot))$, where in this context a feasible solution is understood to be a classical solution.

The obtained results are extended to the case of the multidimensional optimal control problem (P_M) for hyperbolic differential inclusions:

$$\inf \quad J(u(\cdot,\cdot)) := \int_0^1 \int_G g(u(t,x),t,x)\mathrm{d}x\,\mathrm{d}t + \int_G g_0(u(1,x),x)\mathrm{d}x \tag{6.235}$$

$$\text{subject to} \quad \frac{\partial^2 u(t,x)}{\partial t^2} - Lu(t,x)) \in F(u(t,x),t,x), \quad (t,x) \in [0,1] \times G \tag{6.236}$$

$$u(0,x) = \varphi_0(x), \quad \frac{\partial u(0,x)}{\partial t} = \varphi_1(x), \quad x \in G \tag{6.237}$$

$$u(t,x) = \alpha(t,x), \quad (t,x) \in B,$$

where $F(\cdot,t,x) : \mathbb{R} \to P(\mathbb{R})$ is a convex closed multivalued mapping for all fixed (t,x), $x = (x_1, \ldots, x_n) \in G$, $D = (0,1) \times G \subset \mathbb{R}^{n+1}$ is a bounded cylinder. Thus, the domain in which Eq. (6.236) is given is a cylinder D of height 1 and base G. The set $B = (0,1) \times S$, where S is a piecewise-smooth boundary of G is the lateral surface of D. G is a domain consisting of arguments $x = (x_1, \ldots, x_n)$, $\{0\} \times G, \{1\} \times G$ are the lower and upper bases, respectively. $g : \mathbb{R} \times [0,1] \times G \to \mathbb{R}$, $g_0 : \mathbb{R} \times G \to \mathbb{R}$ are functions that are continuous and convex in u, and $\varphi_0 : G \to \mathbb{R}^n$, $\varphi^1 : G \to \mathbb{R}^n$, and α are continuous. L is a second-order elliptic operator and $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2, \ldots, \mathrm{d}x_n$.

A function $u(\cdot,\cdot)$ from $C^2(D) \cap C^1[D \cup B \cup (\{0\} \times \overline{G})]$, satisfying the inclusion in Eq. (6.236) in D and the initial boundary condition in Eq. (6.237) is called a classical solution of the problem (P_M), where $C^2(D)$ is the space of functions $u(\cdot,\cdot)$ having continuous second-order derivatives $(\partial^2 u/\partial x_i \partial x_j)$, i, $j = 1, \ldots, n$. It is required to find a classical solution $\tilde{u}(\cdot,\cdot)$ of the initial boundary value problem (P_M) that minimizes $J(u(\cdot,\cdot))$.

1 Necessary and Sufficient Conditions for First Boundary Value Problems of Discrete Hyperbolic Inclusions

First, we formulate necessary and sufficient conditions for the convex problem (P_D).

Theorem 6.55. Let $\{u^0_{t,x}\}_D$ be a feasible solution and $g_{t,x}$ be convex proper functions continuous at the points of $\{u^0_{t,x}\}_D$. Moreover, assume that $F_{t,x}$, $t = 1, \ldots, T-1$; $x = 1, \ldots, L-1$ are convex multivalued mappings. Then for $\{\tilde{u}_{t,x}\}_D$ to be an optimal solution of the first boundary value problem for the discrete inclusion of hyperbolic type (P_D), it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $\{\psi^*_{t,x}\}, \{\eta^*_{t,x}\}, \{\xi^*_{t,x}\}, \{u^*_{t,x}\}$, not all equal to zero, such that:

i. $(\psi^*_{t,x}, \xi^*_{t,x}, u^*_{t,x}, \eta^*_{t,x},) \in F^*_{t,x}(u^*_{t,x}, (\tilde{u}_{t-1,x}, \tilde{u}_{t,x-1}, \tilde{u}_{t,x}, \tilde{u}_{t,x+1}, \tilde{u}_{t+1,x})) + \{0\} \times \{0\} \times \{\psi^*_{t+1,x} + \xi^*_{t,x+1} + \eta^*_{t,x-1} - \lambda \partial g_{t,x}(\tilde{u}_{t,x})\} \times \{0\}$,
ii. $\psi^*_{T,x} = 0, \quad -u^*_{T-1,x} \in \lambda \partial g_{T,x}(\tilde{u}_{T,x}), \; x = 1, \ldots, L-1; \quad \eta^*_{t,0} = 0, \quad \xi^*_{t,L} = 0, \; t = 1, \ldots, T-1.$

Under Hypothesis H2, the conditions (i) and (ii) are also sufficient for the optimality of $\{\tilde{u}_{t,x}\}_D$.

□ In order to use the convex programming result, we form the $m = 2n(L-1) + n(T-1)(L+1)$-dimensional vector $w = (u_0, \ldots, u_T)$, where for any $t = 1, \ldots, T-1$, $u_t = (u_{t,0}, \ldots, u_{t,L}) \in \mathbb{R}^{m(L+1)}$ is an $n(L+1)$-dimensional vector and $u_0 = (u_{0,1}, \ldots, u_{0,L-1}) \in \mathbb{R}^{n(L-1)}$, $u_T = (u_{T,1}, \ldots, u_{T,L-1}) \in \mathbb{R}^{n(L-1)}$.

Setting

$$M_{t,x} = \{w = (u_0, u_1, \ldots, u_T) : (u_{t-1,x}, u_{t,x-1}, u_{t,x}, u_{t,x+1}, u_{t+1,x}) \in \operatorname{gph} F_{t,x}\},$$
$$t = 1, \ldots, T-1; \quad x = 1, \ldots, L-1,$$

$$N = \{w = (u_0, \ldots, u_T) : u_{1,x} = \varphi^1_x, x = 1, \ldots, L-1\},$$

$$M_1 = \{w = (u_0, \ldots, u_T) : u_{0,x} = \varphi^0_x, x = 1, \ldots, L-1\},$$

$$M^L = \{w = (u_0, \ldots, u_T) : u_{t,L} = \alpha^L_t, t = 1, \ldots, T-1\},$$

$$M_2 = \{w = (u_0, \ldots, u_T) : u_{t,0} = \alpha^0_t, t = 1, \ldots, T-1\}.$$

We can formulate a convex minimization problem in the space $\mathbb{R}^m$, which is equivalent to the problem (P_D):

$$\inf \quad g(w) \text{ subject to } w \in N = \left(\bigcap_{\substack{t=1,\ldots,T-1 \\ x=1,\ldots,L-1}} M_{t,x} \right) \cap N \cap M_1 \cap M^L \cap M_2, \tag{6.238}$$

where $g(w) = \sum_{t=2,\ldots,T;x=1,\ldots,L-1} g_{t,x}(u_{t,x})$.

We shall use the results obtained for the convex minimization problem in Eq. (6.238) in the next section. Thus, it is necessary to calculate the dual cones $K^*_{M_{t,x}}(w), K^*_N(w), K^*_{M_1}(w), K^*_{M^L}(w), K^*_{M_2}(w),\ w \in N$.

By Lemma 6.3, we can write

$$\begin{aligned} K^*_{M_{t,x}}(w) = \{ & w^* = (u_0^*, \ldots, u_T^*) : (u^*_{t-1,x}, u^*_{t,x-1}, u^*_{t,x}, u^*_{t,x+1}, u^*_{t+1,x}) \\ & \in K^*_{\text{gph } F_{t,x}}(u_{t-1,x}, u_{t,x-1}, u_{t,x}, u_{t,x+1}, u_{t+1,x}), \quad u^*_{i,j} = 0, \quad (i,j) \neq (t-1,x), (t,x-1), \\ & (t,x), (t,x+1), (t+1,x)\}, t = 1, \ldots, T-1; \quad x = 1, \ldots, L-1. \end{aligned} \tag{6.239}$$

On the other hand,

$$\begin{aligned} K^*_N(w) &= \{w^* = (u_0^*, \ldots, u_T^*) : u_t^* = 0, t \neq 1, u^*_{1,0} = u^*_{1,L} = 0\}, \\ K^*_{M_1}(w) &= \{w^* = (u_0^*, \ldots, u_T^*) : u_t^* = 0, t = 1, \ldots, T\}, \\ K^*_{M^L}(w) &= \{w^* = (u_0^*, \ldots, u_T^*) : u^*_{t,x} = 0, t = 1, \ldots, T-1, x \neq L, u_0^* = u_T^* = 0\}, \\ K^*_{M^2}(w) &= \{w^* = (u_0^*, \ldots, u_T^*) : u^*_{t,x} = 0, t = 1, \ldots, T-1, x \neq 0, u_0^* = u_T^* = 0\}. \end{aligned} \tag{6.240}$$

Obviously, $g(\cdot)$ is continuous at the point $w^0 = (u_0^0, \ldots, u_T^0)$, and by the hypothesis of theorem, $\tilde{w} = (\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_T)$ is a solution of the convex minimization problem in Eq. (6.238). By Theorem 3.4, there exist vectors $w^*(t,x) \in K^*_{M_{t,x}}(\tilde{w})$, $\overline{w}^* \in K^*_N(\tilde{w})$, $w^{1*} \in K^*_{M_1}(\overline{w})$,

$w^{L^*} \in K^*_{M^L}(\tilde{w})$, $w^{2*} \in K^*_{M_2}(\tilde{w})$, $\quad w^{0*} \in \partial_w g(\overline{w})$, and a number $\lambda \in \{0,1\}$, not all equal to zero, such that

$$\sum_{\substack{t=1,\ldots,T-1 \\ x=1,\ldots,L-1}} w^*(t,x) + w^{1*} + \overline{w}^* + w^{L^*} + w^{T^*} = \lambda w^{0*}. \tag{6.241}$$

Remember that $[w^*]_{t,x}$ denotes the components of the vector w^* for the given pair (t,x). Then using Eq. (6.239), we derive that

$$\left[\sum_{\substack{t=1,\ldots,T-1 \\ x=1,\ldots,L-1}} w^*(t,x) + \overline{w}^* + w^{1*} + w^{L^*} + w^{2*} \right]_{t,x} \tag{6.242}$$

$$= u^*_{t,x}(t+1,x) + u^*_{t,x}(t,x+1) + u^*_{t,x}(t,x) + u^*_{t,x}(t,x-1) + u^*_{t,x}(t-1,x) = \lambda u^{0*}_{t,x},$$
$$t=1,\ldots,T-1,\ x=1,\ldots,L-1$$
$$u^*_{t,1}(t,0)=0,\quad u^*_{t,L-1}(t,L)=0,\ t=1,\ldots,T,\quad u^*_{T-1,x}(T,x)=0,\ x=1,\ldots,L-1$$
$$u^*_{T,x}(T-1,x)=\lambda u^{0*}_{T,x}, x=1,\ldots,L-1,\quad u^{0*}_{t,x}\in\partial g_{t,x}(\tilde{u}_{t,x}),\quad [w^{0*}]_{t,x}=u^{0*}_{t,x}. \tag{6.243}$$

Moreover, using Eq. (6.239) and the definition of a LAM, we have

$$\begin{aligned}&(u^*_{t-1,x}(t,x), u^*_{t,x-1}(t,x), u^*_{t,x}(t,x), u^*_{t,x+1}(t,x))\\ &\in F^*_{t,x}(-u^*_{t+1,x}(t,x), (\tilde{u}_{t-1,x}, \tilde{u}_{t,x-1}, \tilde{u}_{t,x}, \tilde{u}_{t,x+1}, \tilde{u}_{t+1,x}))\\ &\qquad t=1,\ldots,T-1;\quad x=1,\ldots,L-1.\end{aligned} \tag{6.244}$$

Hence, introducing the new notations $u^*_{t-1,x}(t,x)=\psi^*_{t,x}$, $u^*_{t,x-1}(t,x)=\xi^*_{t,x}$, $u^*_{t,x+1}(t,x)=\eta^*_{t,x}$, $-u^*_{t+1,x}(t,x)=u^*_{t,x}$ from Eqs. (6.243) and (6.244), we derive the validity of the first part of the theorem. It should be pointed out that by Hypothesis H2, the relation in Eq. (6.241) holds with $\lambda=1$ for the point $w^{0*}\in\partial_w g(\tilde{w})\cap K^*_N(\tilde{w})$. Therefore, by Theorem 3.3, the conditions (i) and (ii) are sufficient for the optimality of $\{\tilde{u}_{t,x}\}_D$. ■

Then taking into account Theorem 2.1, we obtain the result in Corollary 6.4.

Corollary 6.4. If the conditions of Theorem 6.55 are satisfied and if in addition $F(u_1,u_2,u_3,u_4)$ is a closed convex set for every (u_1,u_2,u_3,u_4), then for the optimality of $\{\tilde{u}_{t,x}\}_D$, it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors $\{\psi^*_{t,x}\}$, $\{\eta^*_{t,x}\}$, $\{\xi^*_{t,x}\}$, $\{u^*_{t,x}\}$ not all equal to zero, such that

$$u^*_{t,x}\in\partial_{v^*}H_{F_{t,x}}(\tilde{u}_{t-1,x},\tilde{u}_{t,x-1},\tilde{u}_{t,x},\tilde{u}_{t,x+1},u^*_{t,x}),$$
$$(\psi^*_{t,x},\xi^*_{t,x},u^*_{t-1,x},\eta^*_{t,x})\in\partial_u H_{F_{t,x}}(\tilde{u}_{t-1,x},\tilde{u}_{t,x-1},\tilde{u}_{t,x},\tilde{u}_{t,x+1},u^*_{t,x})$$
$$+\{0\}\times\{0\}\times\{\psi^*_{t+1,x}+\xi^*_{t,x+1}+\eta^*_{t,x-1}-\lambda\partial g_{t,x}(\tilde{u}_{t,x})\}\times\{0\},$$
$$\psi^*_{T,x}=0,\quad -u^*_{T-1,x}\in\lambda\partial g_{T,x}(\tilde{u}_{T,x}),\ x=1,\ldots,L-1;\quad \eta^*_{t,0}=0,\ \xi^*_{t,L}=0,\ t=1,\ldots,T-1.$$

In addition, if Hypothesis H2 is satisfied, the conditions (i) and (ii) are sufficient for optimality.

Theorem 6.56. Assume Hypothesis H1 for the nonconvex problem (P_D). Then for the optimality of $\{\tilde{u}_{t,x}\}_D$, it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors $\{\psi^*_{t,x}\}$, $\{\eta^*_{t,x}\}$, $\{\xi^*_{t,x}\}$, $\{u^*_{t,x}\}$ not all equal to zero, satisfying conditions (i) and (ii) of Theorem 6.55.

□ In the present case, Hypothesis H1 guarantees the relation in Eq. (6.241) for the nonconvex problem (P_D) and so for Eq. (6.238). Then we derive the necessary condition as in Theorem 6.55 by starting from the relation in Eq. (6.241)) written out for the nonconvex problem (P_D). ■

2 Approximation of the Continuous Problem and Necessary Conditions for the Discrete-Approximation Problem

In this section, we use difference operators to approximate the problem (P_C) and apply Theorems 6.55 and 6.56 to obtain a necessary and sufficient condition for optimality. For the given natural numbers N_1, N_2 we choose steps δ and h on the t- and x-axes, respectively, and use the grid functions $u_{t,x} = u_{\delta h}(t,x)$ on a uniform grid on R.

Denoting $\square u = Au - Bu$, where $Au = (\partial^2 u/\partial t^2)$, $Bu = (\partial^2 u/\partial x^2)$, it can easily be seen that the following difference operators are defined on the three-point models [32]; i.e., each of the operators Au, Bu is approximated by the $\tilde{A}u$ and $\tilde{B}u$:

$$\tilde{A}u(x) : = \frac{u(t+\delta,x) - 2u(t,x) + u(t-\delta,x)}{\delta^2};$$
$$\tilde{B}u(x) : = \frac{u(t,x+h) - 2u(t,x) + u(t,x-h)}{h^2}.$$

Thus, we have the following difference boundary value problem approximating (P_C):

$$\inf \quad J_{\delta h}(u(t,x)) : = \sum_{\substack{t=0,\ldots,1-\delta \\ x=0,\ldots,1-h}} \delta h g(u(t,x),t,x)$$

$$(P_A) \quad \text{subject to } \tilde{A}u(x) - \tilde{B}u(x) \in F(u(t,x),t,x), \quad (t,x) \in \overline{\omega}_{\delta h},$$
$$u(0,x) = \varphi^0(x), \quad u(\delta,x) - u(0,x) = \delta\varphi^1(x), \quad x = h, 2h, \ldots, 1-h,$$
$$u(t,0) = \alpha_0(t), \quad u(t,1) = \alpha_1(t), \quad t = \delta, 2\delta, \ldots, 1-\delta.$$

First, for simplicity assume that (P_A) is a discrete-approximation for problem (P_C), where $\mathrm{R} = (0,1) \times (0,1)$, so that $\overline{\omega}_{\delta h} = \{(t,x) : t = 0, \delta, \ldots, 1; x = 0, h, \ldots, 1,$ $(t,x) \neq (0,0), (0,1), (1,0), (1,1)\}$. In order to reduce the problem (P_A) to a problem of the form (P_D), we introduce the multivalued mapping $Q(\cdot,t,x) : \mathbb{R}^{4n} \to P(\mathbb{R}^n)$, defined as

$$Q(u_1,u_2,u_3,u_4,t,x) : = 2(1-\theta)u_3 - u_1 + \theta(u_4+u_2) + \delta^2 F(u_3,t,x), \quad \theta = \frac{\delta^2}{h^2}. \tag{6.245}$$

Then the problem (P_A) can be formulated as follows:

$$\inf \quad J_{\delta h}(u(\cdot,\cdot)), \tag{6.246}$$

$$\text{subject to } u(t+\delta,x) \in Q(u(t-\delta,x), u(t,x-h), u(t,x), u(t,x+h), t, x), \quad (t,x) \in \overline{\omega}_{\delta h}, \tag{6.247}$$

$$u(0,x)=\varphi^0(x), \quad u(\delta,x)=\varphi^0(x)+\delta\varphi^1(x), \ \ x=h,2h,\ldots,1-h,$$

$$u(t,0)=\alpha_0(t), \quad u(t,1)=\alpha_1(t), \ \ t=\delta,2\delta,\ldots,1-\delta.$$

Applying Theorem 6.55, we see that for the optimality of $\{\tilde{u}(t,x)\}, (t,x)\in\overline{\omega}_{\delta h}$ in the problem in Eqs. (6.246) and (6.247), it is necessary that there exist vectors $\{u^*(t,x)\}, \{\psi^*(t,x)\}, \{\xi^*(t,x)\}, \{\eta^*(t,x)\}$ and a number $\lambda=\lambda_{\delta h}\in\{0,1\}$, not all zero, such that

$$\begin{aligned}
&(\psi^*(t,x),\xi^*(t,x),u^*(t-\delta,x),\eta^*(t,x))\\
&\in Q^*(u^*(t,x);(\tilde{u}(t-\delta,x),\tilde{u}(t,x-h),\tilde{u}(t,x),\tilde{u}(t,x+h),\tilde{u}(t+\delta,x),t,x)\\
&+\{0\}\times\{0\}\times\{\psi^*(t+\delta,x)+\xi^*(t,x+h)+\eta^*(t,x-h)-\lambda\partial g(\tilde{u}(t,x),t,x)\}\times\{0\},\\
&\psi^*(1,x)=0, \quad -u^*(1-\delta,x)\in\lambda h\partial g_0(\tilde{u}(1,x),1,x), \ \ x=h,\ldots,1-h,\\
&\eta^*(t,0)=0, \quad \xi^*(t,1)=0, \ \ t=\delta,\ldots,1-\delta.
\end{aligned} \tag{6.248}$$

Now we must express the LAM Q^* in terms of F^*.

Theorem 6.57. Let the cone of tangent directions $K_{\text{gph } Q(\cdot,t,x)}(u_1,u_2, u_3,u_4,v)$, $(u_1, u_2, u_3,u_4,v)\in\text{gph } Q(\cdot,t,x)$ be a local tent. Then

$$K_{\text{gph } Q(.,t,x)}\left(u_3,\frac{v+u_1-\theta(u_2+u_4)-2(1-\theta)u_3}{\delta^2}\right)$$

is a local tent to gph $F(\cdot,t,x)$ and the following inclusions are equivalent:

a. $(\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)\in K_{\text{gph } Q(.,t,x)}(u_1,u_2,u_3,u_4,v)$,

b. $\left(\overline{u}_3,\frac{\overline{v}+\overline{u}_1-\theta(\overline{u}_2+\overline{u}_4)-2(1-\theta)\overline{u}_3}{\delta^2}\right)\in K_{\text{gph } F(.,t,x)}\left(u_3,\frac{v+u_1-\theta(u_2+u_4)-2(1-\theta)u_3}{\delta^2}\right)$.

□ The proof is similar to the one for Theorem 6.31. ■

The next step for the optimization of the discrete-approximation problem in Eqs. (6.246) and (6.247) is an equivalence result for LAM.

Theorem 6.58. Let the cones of tangent directions $K_{\text{gph } Q(\cdot,t,x)}(u_1,u_2, u_3,u_4,v)$ be locally tent. Then the following inclusions are equivalent under the conditions that $v^*+u_1^*=0, \ u_2^*=u_4^*=\theta\psi^*$:

a. $(u_1^*,u_2^*,u_3^*,u_4^*)\in Q^*(v^*,(u_1,u_2,u_3,u_4,v),t,x)$,

b. $\frac{u_3^*-2(1-\theta)\psi^*}{\delta^2}\in F^*\left(v^*;\left(u_3,\frac{v+u_1-\theta(u_2+u_4)-2(1-\theta)u_3}{\delta^2}\right),t,x\right)$.

□ With the use of Theorem 6.57, the proof is elementary (see also Theorem 6.32). ■

Remark 6.23. For the convex problem (P_D), the results of Theorem 6.58 can be obtained by a direct calculation of the subdifferential $\partial_u H_F(u,v^*,t,x)$, $u=(u_1,u_2,u_3,u_4)$ and expressing it via LAM of multivalued mapping F.

Consequently, by the equivalence in Theorem 6.58, the condition in Eq. (6.248) has the form

$$\frac{u^*(t-\delta,x)-\psi^*(t+\delta,x)-\xi^*(t,x+h)-\eta^*(t,x-h)-2(1-\theta)u^*(t,x)}{\delta^2}$$
$$\in F^*(u^*(t,x),(\tilde{u}(t,x),\tilde{A}\tilde{u}(t,x)-\tilde{B}\tilde{u}(t,x)),t,x)-\lambda\,\partial g(\tilde{u}(t,x),t,x), \tag{6.249}$$

$$u^*(t,x)=-\psi^*(t,x),\ \xi^*(t,x)=\eta^*(t,x)=\theta u^*(t,x)$$
$$\theta=\frac{\delta^2}{h^2},\quad t=\delta,2\delta,\dots,1-\delta;\ \ x=h,2h,\dots,1-h. \tag{6.250}$$

Clearly, by using Eq. (6.250), the left-hand side of the inclusion in Eq. (6.249) can be written as

$$\frac{1}{\delta^2}[u^*(t-\delta,x)+u^*(t+\delta,x)-\theta(u^*(t,x+h)+u^*(t,x-h))-2(1-\theta)u^*(t,x)]$$
$$=\frac{u^*(t+\delta,x)-2u^*(t,x)+u^*(t-\delta,x)}{\delta^2}+\frac{u^*(t,x+h)-2u^*(t,x)+u^*(t,x-h)}{h^2}. \tag{6.251}$$

Moreover, from the boundary conditions in Eqs. (6.248) and (6.250), we have

$$u^*(t,0)=0,\quad u^*(t,1)=0,\ t=\delta,\dots,1-\delta,\quad u^*(1,x_2)=0,\ x=h,2h,\dots,1-h$$
$$-u^*(1-\delta,x)\in\lambda h\partial g_0(\tilde{u}(1,x),1,x). \tag{6.252}$$

Then denoting $u^*(t,x)/\mu$, $\psi^*(t,x)/\mu$, $\xi^*(t,x)/\mu$, $\eta^*(t,x)/\mu$ $(\mu=h/\delta)$ by $u^*(t,x)$, $\psi^*(t,x)$, $\xi^*(t,x)$, $\eta^*(t,x)$, respectively, and taking into account the relations in Eqs. (6.249), (6.251), and (6.252), we have proved the following result.

Theorem 6.59. Let $g(\cdot,t,x)$ be a convex proper function and continuous at the points on some feasible solution $\{u^0(t,x)\},(t,x)\in\overline{\omega}_{\delta h}$. Then for the optimality of the solution $\{\tilde{u}(t,x)\}$ in the convex problem (P_A), it is necessary that there exist a number $\lambda=\lambda_{\delta h}\in\{0,1\}$ and grid functions $\{u^*(t,x)\},(t,x)\in\overline{\omega}_{\delta h}$, not all equal to zero, such that

i. $\tilde{A}u^*(t,x)-\tilde{B}u^*(t,x)\in F^*(u^*(t,x),(\tilde{u}(t,x),\tilde{A}\tilde{u}(t,x)-\tilde{B}\tilde{u}(t,x)),t,x)-\lambda\,\partial g(\tilde{u}(t,x),t,x)$,

ii. $u^*(t,0)=u^*(t,1)=0,\quad t=\delta,\dots,1-\delta,$
$u^*(1,x)=0,\quad \dfrac{u^*(1,x)-u^*(1-\delta,x)}{\delta}\in\lambda\partial g_0(\tilde{u}(1,x),1,x),\ \ x=h,\dots 1-h.$

In addition, if the condition in Hypothesis H2 is satisfied, then these conditions are also sufficient for the optimality of $\{\tilde{u}(t,x)\}, (t,x) \in \overline{\omega}_{\delta h}$.

Remark 6.24. Conditions (i) and (ii) of Theorem 6.59 are necessary for optimality in the nonconvex case of the problem (P_A) under Hypothesis H1.

3 Sufficient Conditions for Optimality for Differential Inclusions of Hyperbolic Type

Applying the results of Subsection 2, we next formulate a sufficient condition for optimality for a first boundary value problem of hyperbolic differential inclusions (P_C). Thus, by passing to the formal limit in conditions (i) and (ii) of Theorem 6.59 as $\delta, h \to 0$ and setting $\lambda = 1$, we have

a. $\square u^*(t,x) \in F^*(u^*(t,x), (\tilde{u}(t,x), \square \tilde{u}(t,x)), t, x) - \partial g(\tilde{u}(t,x), t, x) \quad (t,x) \in \mathrm{R}$,
b. $u^*(1,x) = 0, \quad u_t^*(1,x) \in \partial g_0(\tilde{u}(1,x), 1, x)$,
c. $u^*(t,0) = 0$, $u^*(t,1) = 0$.
Furthermore, we formulate the following condition, ensuring that the LAM $F^*(\cdot,\cdot,t,x)$ is nonempty for every fixed $(t,x) \in \mathrm{R}$ (Theorem 2.1):
d. $\square \tilde{u}(t,x) \in F(\tilde{u}(t,x), u^*(t,x), t, x)$.

Theorem 6.60. Let the functions g and g_0 be continuous and convex with respect to u and $F(\cdot,t,x)$ be a convex mapping for all fixed (t,x). Then for the optimality of a solution $\tilde{u}(\cdot,\cdot)$ in first boundary value problem for hyperbolic differential inclusions (P_C), it is sufficient that there exists a classical solution $u^*(\cdot,\cdot)$ such that conditions (a) and (d) hold.
$\square$ By Theorem 2.1, $F^*(v^*,(u,v),t,x) = \partial_u H_F(u,v^*,t,x)$, $v \in F(u;v^*,t,x)$. Then applying Theorem 1.28 (Moreau–Rockafellar) from condition (a), we obtain

$$\square u^*(t,x) \in \partial_u [H_F(\tilde{u}(t,x), u^*(t,x), t, x) - g(\tilde{u}(t,x), t, x)], \quad (t,x) \in \mathrm{R}.$$

Then

$$\begin{aligned} &H_F(u(t,x), u^*(t,x), t, x) - H_F(\tilde{u}(t,x), u^*(t,x), t, x) \\ &- g(u(t,x), t, x) + g(\tilde{u}(t,x), t, x) \le \langle \square u^*(t,x), u(t,x) - \tilde{u}(t,x) \rangle. \end{aligned}$$

In view of condition (d) of Theorem 6.60, we derive that

$$\begin{aligned} \iint_{\mathrm{R}} [g(u(t,x), t, x) - g(\tilde{u}(t,x), t, x)] \mathrm{d}x\, \mathrm{d}t - \iint_{\mathrm{R}} \left\langle \square (u(t,x) - \tilde{u}(t,x)), u^*(t,x) \right\rangle \mathrm{d}x\, \mathrm{d}t \\ + \iint_{\mathrm{R}} \left\langle u(t,x) - \tilde{u}(t,x), \square u^*(t,x) \right\rangle \mathrm{d}x\, \mathrm{d}t \ge 0. \end{aligned} \tag{6.253}$$

Since $u(t,x), \tilde{u}(t,x)$ are feasible solutions (see Eq. (6.234)) and by condition (b), $u^*(1,x)=0$ it is easy to see that

$$\begin{aligned}
&\iint\limits_{R}\left\langle u(t,x)-\tilde{u}(t,x),\frac{\partial^2 u^*(t,x)}{\partial t^2}\right\rangle \mathrm{d}t\,\mathrm{d}x-\iint\limits_{R}\left\langle\frac{\partial^2(u(t,x)-\tilde{u}(t,x))}{\partial t^2},u^*(t,x)\right\rangle \mathrm{d}t\,\mathrm{d}x\\
&\quad=\iint\limits_{R}\frac{\partial}{\partial t}\left\langle u(t,x)-\tilde{u}(t,x),\frac{\partial u^*(t,x)}{\partial t}\right\rangle \mathrm{d}t\,\mathrm{d}x-\iint\limits_{R}\left\langle\frac{\partial(u(t,x)-\tilde{u}(t,x))}{\partial t},\frac{\partial u^*(t,x)}{\partial t}\right\rangle \mathrm{d}t\,\mathrm{d}x\\
&\quad\quad-\iint\limits_{R}\frac{\partial}{\partial t}\left\langle\frac{\partial(u(t,x)-\tilde{u}(t,x))}{\partial t},u^*(t,x)\right\rangle \mathrm{d}t\,\mathrm{d}x+\iint\limits_{R}\left\langle\frac{\partial(u(t,x)-\tilde{u}(t,x))}{\partial t},\frac{\partial u^*(t,x)}{\partial t}\right\rangle \mathrm{d}t\,\mathrm{d}x\\
&\quad=\iint\limits_{R}\frac{\partial}{\partial t}\left\langle u(t,x)-\tilde{u}(t,x),\frac{\partial u^*(t,x)}{\partial t}\right\rangle \mathrm{d}t\,\mathrm{d}x-\iint\limits_{R}\frac{\partial}{\partial t}\left\langle\frac{\partial(u(t,x)-\tilde{u}(t,x))}{\partial t},u^*(t,x)\right\rangle \mathrm{d}t\,\mathrm{d}x\\
&\quad=\int_0^1[\langle u(1,x)-\tilde{u}(1,x),u_t^*(1,x)\rangle-\langle u(0,x)-\tilde{u}(0,x),u_t^*(0,x)\rangle]\mathrm{d}x\\
&\quad\quad-\int_0^1[\langle u_t(1,x)-\tilde{u}_t(1,x),u^*(1,x)\rangle-\langle u_t(0,x)-\tilde{u}_t(0,x),u^*(0,x)\rangle]\mathrm{d}x\\
&\quad=\int_0^1\langle u(1,x)-\tilde{u}(1,x),u_t^*(1,x)\rangle\mathrm{d}x.
\end{aligned}\tag{6.254}$$

By virtue of $u^*(t,0)=u^*(t,1)=0$ and the boundary conditions in Eq. (6.234), by analogy it can be proved that

$$\iint\limits_{R}\left\langle\frac{\partial^2(u(t,x)-\tilde{u}(t,x))}{\partial t^2},u^*(t,x)\right\rangle \mathrm{d}t\,\mathrm{d}x-\iint\limits_{R}\left\langle u(t,x)-\tilde{u}(t,x),\frac{\partial^2 u^*(t,x)}{\partial t^2}\right\rangle \mathrm{d}t\,\mathrm{d}x=0.\tag{6.255}$$

For convenience, in the next calculations, let us denote the difference of the second and third integrals in Eq. (6.253) by Ω. Then summing Eqs. (6.254) and (6.255), we have

$$\Omega=\int_0^1\langle u(1,x)-\tilde{u}(1,x),u_t^*(1,x)\rangle\mathrm{d}x$$

and Eq. (6.253) yields

$$\iint\limits_{R}[g(u(t,x),t,x)-g(\tilde{u}(t,x),t,x)\mathrm{d}t\,\mathrm{d}x+\Omega\geq 0.\tag{6.256}$$

On the other hand, condition (b) implies that

$$g_0(u(t,x),x) - g_0(\tilde{u}(t,x),x) \geq \langle u_t^*(1,x), u(1,x) - \tilde{u}(1,x)\rangle.$$

By integrating this inequality over the interval [0,1], we have

$$\int_0^1 [g_0(u(t,x),x) - g_0(\tilde{u}(t,x),x)]\mathrm{d}x \geq \Omega. \tag{6.257}$$

Consequently, adding Eqs. (6.256) and (6.257), we have that for any feasible solution $u(\cdot,\cdot)$, $J(u(t,x)) \geq J(\tilde{u}(t,x))$; i.e., $\tilde{u}(\cdot,\cdot)$ is an optimal solution. ■

Corollary 6.5. Let the conditions of Theorem 6.60 be satisfied and let $F(\cdot,t,x)$ be a closed mapping. Then conditions (a) and (d) of Theorem 6.60 can be rewritten as follows:

i. $\square u^*(t,x) \in \partial_u H_F(\tilde{u}(t,x), u^*(t,x), t, x) - \partial g(\tilde{u}(t,x), t, x)$,
ii. $\square \tilde{u}(t,x) \in \partial_{v^*} H_F(\tilde{u}(t,x), u^*(t,x), t, x)$.

Thus, conditions (i) and (ii) are equivalent to conditions (a) and (d) of Theorem 6.60.

Theorem 6.61. Let us consider the nonconvex problem (P_C). Then for optimality of $\tilde{u}(\cdot,\cdot)$ in this nonconvex problem, it is sufficient that there is a classical solution $u^*(\cdot,\cdot)$ satisfying the following conditions:

i. $\square u^*(t,x) + u^*(t,x) \in F^*(u^*(t,x); (\tilde{u}(t,x), \square \tilde{u}(t,x)), t, x)$,
ii. $g(u(t,x),t,x) - g(\tilde{u}(t,x),t,x) \geq \langle u^*(t,x), u(t,x) - \tilde{u}(t,x)\rangle$ for all u,
iii. $\langle u^*(t,x), \square \tilde{u}(t,x)\rangle = H_F(\tilde{u}(t,x), u^*(t,x), t, x)$,

where the LAM $F^*(\cdot,\cdot,t,x)$ is given by the Hamiltonian function H_F.
□ By virtue of the LAM defined by a Hamiltonian function, it follows from condition (i) that for all feasible solution $u(\cdot,\cdot)$, we have

$$\begin{aligned} &H_F(u(t,x), u^*(t,x), t, x) - H_F(\tilde{u}(t,x), u^*(t,x), t, x) \\ &\quad \leq \langle \square u^*(t,x) + u^*(t,x), u(t,x) - \tilde{u}(t,x)\rangle, \quad (t,x) \in \mathrm{R}, \end{aligned}$$

which by condition (iii) implies that

$$\langle \square (u(t,x) - \tilde{u}(t,x)), u^*(t,x)\rangle \leq \langle \square u^*(t,x) + u^*(t,x), u(t,x) - \tilde{u}(t,x)\rangle. \tag{6.258}$$

Thus, for any feasible solution $u(\cdot,\cdot)$, from condition (ii) of theorem and the inequality in Eq. (6.258), it easily can be seen that

$$\begin{aligned} &g(u(t,x),t,x) - g(\tilde{u}(t,x),t,x) - \langle \square (u(t,x) - \tilde{u}(t,x)), u^*(t,x)\rangle \\ &\quad + \langle u(x) - \tilde{u}(x), \Delta u^*(x)\rangle \geq 0, \quad (t,x) \in \mathrm{R}. \end{aligned}$$

Now, by integrating this inequality over R, we see that the obtained inequality has the form in Eq. (6.253). Then in view of Eq. (6.253), it is easy to show (see Theorem 6.60) that $\tilde{u}(\cdot,\cdot)$ is optimal. ■

Now we consider the example

$$\inf \quad J(u(\cdot,\cdot)) = \iint_{R} g(u(t,x),t,x)\mathrm{d}x\,\mathrm{d}t + \int_0^1 g_0(u(1,x),x)\mathrm{d}x \tag{6.259}$$

$$\text{subject to} \quad \Box u(t,x) = Au(t,x) + Bw(t,x), \quad w(t,x) \in V,$$

where A is an $n \times n$ matrix, B is a rectangular $n \times r$ matrix, $V \subset \mathbb{R}^r$ is a closed convex set, and g is a continuously differentiable function on (t,x). It is required to find a controlling parameter $w(t,x) \in V$ such that the feasible solution corresponding to it minimizes $J(u(\cdot,\cdot))$.

By using the formula of LAM calculated for the problem in Eq. (6.176) and Theorem 6.60, we have

$$\begin{aligned}
&\Box u^*(t,x) = A^*u^*(t,x) - g'(\tilde{u}(t,x),t,x), \quad (t,x) \in R \\
&u^*(t,x) = 0, \quad u_t^*(0,x) = g_0'(u(1,x),x), \quad u^*(t,0) = 0, \quad u^*(t,1) = 0
\end{aligned} \tag{6.260}$$

$$\langle B\tilde{u}(t,x), u^*(t,x)\rangle = \sup_{w \in V} \langle Bu, u^*(t,x)\rangle.$$

Thus, the feasible solution $\tilde{u}(\cdot,\cdot)$ corresponding to the control $\tilde{w}(\cdot,\cdot)$ minimizes $J(u(\cdot,\cdot))$ in the problem in Eq. (6.259) if there exists a classical solution $u^*(\cdot,\cdot)$ satisfying the conditions in Eq. (6.260).

4 Multidimensional Optimal Control Problems for Hyperbolic Differential Inclusions

In this section, we study the problem (P_M) defined with elliptic operator L.

Theorem 6.62. Let g and g_0 be continuous functions and convex with respect to u. Moreover, let $F(\cdot,t,x)$ be a convex closed mapping for every fixed $(t,x) \in D$. Then a solution $\tilde{u}(\cdot,\cdot)$ minimizes the functional $J(u(\cdot,\cdot))$ in the problem (P_M) if there exists a classical solution $u^*(\cdot,\cdot)$ of the following boundary value problem:

i. $\frac{\partial^2 u^*(t,x)}{\partial t^2} - L^*u^*(t,x) \in F^*(u^*(t,x); \tilde{u}(t,x), L\tilde{u}(t,x), t, x) - \partial g(\tilde{u}(t,x),t,x),$

ii. $u^*(1,x) = 0, \quad u_t^*(1,x) \in \partial g_0(\tilde{u}(1,x),x), \quad x \in G$
$u^*(t,x) = 0, \quad (t,x) \in B,$

iii. $\frac{\partial^2 \tilde{u}(t,x)}{\partial t^2} - L\tilde{u}(t,x) \in F(\tilde{u}(t,x);\ u^*(t,x), t, x), \quad (t,x) \in D,$

where L^* is the operator adjoint to L.

□ Analogously for Theorem 6.60, condition (i) implies that

$$H_F(u(t,x),u^*(t,x),t,x)-H_F(\tilde{u}(t,x),u^*(t,x),t,x)\le g(u(t,x),t,x)-g(\tilde{u}(t,x),t,x)$$
$$+u_{tt}(t,x)-L^*u^*(t,x)(u(t,x)-\tilde{u}(t,x)).$$

Since by condition (ii)

$$H_F(\tilde{u}(t,x),u^*(t,x),t,x)=u^*(t,x)(u_{tt}(t,x)-L\tilde{u}(t,x)),$$

we have

$$\int_0^1\int_G[g(u(t,x),t,x)-g(\tilde{u}(t,x),t,x)]\mathrm{d}x\,\mathrm{d}t\ge\int_0^1\int_G u^*(t,x)[u_{tt}(t,x)-\tilde{u}_{tt}(t,x)$$
$$-L(u(t,x)-\tilde{u}(t,x))]\mathrm{d}x\,\mathrm{d}t-\int_0^1\int_G[u^*_{tt}(t,x)-L^*u^*(t,x)](u(t,x)-\tilde{u}(t,x))\mathrm{d}x\,\mathrm{d}t. \tag{6.261}$$

According to Eq. (6.254), we find that

$$\int_0^1\int_G u^*(t,x)(u_{tt}(t,x)-\tilde{u}_{tt}(t,x))\mathrm{d}x\,\mathrm{d}t-\int_0^1\int_G u^*_{tt}(t,x)(u(t,x)-\tilde{u}(t,x))\mathrm{d}x\,\mathrm{d}t$$
$$=\int_G\left\{\int_0^1[u^*(t,x)(u_{tt}(t,x)-\tilde{u}_{tt}(t,x))-u^*_{tt}(t,x)(u(t,x)-\tilde{u}(t,x))]\mathrm{d}t\right\}\mathrm{d}x$$
$$=-\int_G u^*_t(1,x)(u(1,x)-\tilde{u}(1,x))\mathrm{d}x. \tag{6.262}$$

Further, using the boundary conditions, we deduce from Green's formula that in the multidimensional case

$$\int_G[(u(t,x)-\tilde{u}(t,x))L^*u^*(t,x)-u^*(t,x)L(u(t,x)-\tilde{u}(t,x))]\mathrm{d}x=0. \tag{6.263}$$

Thus, because of the conditions in Eqs. (6.261)–(6.263), we derive

$$\int_0^1\int_G[g(u(t,x),t,x)-g(\tilde{u}(t,x),t,x)]\mathrm{d}x\,\mathrm{d}t\ge-\int_G u^*_t(1,x)(u(1,x)-\tilde{u}(1,x))\mathrm{d}x. \tag{6.264}$$

On the other hand, by integrating the inequality following from condition (ii), we have

$$\int_G [g_0(u(1,x),x) - g_0(\tilde{u}(1,x),x)]\mathrm{d}x \geq -\int_G u_t^*(1,x)(u(1,x) - \tilde{u}(1,x))\mathrm{d}x. \tag{6.265}$$

Thus, adding the inequalities in Eqs. (6.264) and (6.265), we conclude that for all feasible $u(\cdot,\cdot)$, $J(u(t,x)) \geq J(\tilde{u}(t,x))$; i.e., $\tilde{u}(t,x)$ is optimal. ■

Remark 6.25. In addition to the assumptions in Theorem 6.62, suppose that $F(\cdot,t,x)$ is a closed mapping. Then conditions (i)−(iii) can be rewritten as follows (see Corollary 6.5).

i. $u_{tt}^{*''}(t,x) - L^* u^*(t,x) \in \partial_u M(\tilde{u}(t,x), u^*(t,x), t, x) - \partial g(\tilde{u}(t,x), t, x)$,
ii. $\tilde{u}_{tt}^{''}(t,x) - L\tilde{u}(t,x) \in \partial_{v^*} M(\tilde{u}(t,x), u^*(t,x), t, x)$.

Replacing the operator $(\partial^2/\partial x^2)$ with an elliptic operator L and extending the proof of Theorem 6.61 to the problem (P_M) for a nonconvex case, it is not hard to get Theorem 6.63.

Theorem 6.63. Let $\tilde{u}(\cdot,\cdot)$ be some feasible solution of the nonconvex problem (P_M) and suppose $u^*(\cdot,\cdot)$ is a classical solution satisfying the following conditions:

i. $\dfrac{\partial^2 u^*(t,x)}{\partial t^2} - L^* u^*(t,x) + u^*(t,x) \in F^*(u^*(t,x); (\tilde{u}(t,x), L\tilde{u}(t,x)), t, x)$,
$u^*(1,x) = 0, \quad u_t^*(1,x) \in \partial g_0(\tilde{u}(1,x), x), \quad x \in G$,
ii. $\tilde{u}_{tt}(t,x) - L\tilde{u}(t,x) = H_F(\tilde{u}(t,x), u^*(t,x), t, x), \quad u^*(t,x) = 0, \quad (t,x) \in B$,
iii. $g(u,t,x) - g(\tilde{u}(t,x),t,x) \geq u^*(t,x)(u - \tilde{u}(t,x))$ for all u,

where the LAM $F^*(\cdot,\cdot,t,x)$ is given by Hamiltonian function. Then $\tilde{u}(\cdot,\cdot)$ is optimal.

Suppose now that we have the problem (P_M) with homogeneous boundary conditions and let $H^1(D)$ be the Hilbert space consisting of the elements $u(\cdot,\cdot) \in L_2(D)$ having square-integrable generalized derivatives on D, where the inner product and norm are defined as in Remark 6.22.

By analogy with the classical theory of the Dirichlet problem for an elliptic equation [164], we call a function $u(\cdot,\cdot) \in H(D)$ a generalized solution of our problem if it satisfies the integral identity

$$\int_0^1 \int_G (-u_t \eta_t + a_{ij} u_{x_i} \eta_{x_j} + b_i u_{x_i} \eta + cu\eta)\mathrm{d}x\,\mathrm{d}t - \int_G \varphi_1 \eta(0,x)\mathrm{d}x = \int_0^1 \int_G g\eta\,\mathrm{d}x\,\mathrm{d}t$$

for all $\eta(\cdot,\cdot) \in H^{1^\circ}(D)$ (for a more detailed study see, for example, Refs. [131, 178]). Here, $g = g(u,t,x)$ is an arbitrary measurable selection of the multivalued mapping F.

5 Duality in Problems for Hyperbolic Differential Inclusions

As in the previous sections, we can establish the duality problems for primary problems (P_C) and (P_M). Construction of (P_M), the dual problem for (P_C), is based on duality for discrete-approximation problem and infimal convolution of convex functions, where $\varphi^1(x) \equiv 0$, $\alpha_0(t) \equiv 0$, $\alpha_1(t) \equiv 0$. In the considered case,

$$J_*(u^*(t,x), p^*(t,x)) = \iint\limits_{\mathrm{R}} [M_F(\Box u^*(t,x) - p^*(t,x), u^*(t,x)) - g^*(p^*(t,x), t, x)]\mathrm{d}t\,\mathrm{d}x$$
$$- \int_0^1 [g_0^*(- u_t^*(1,x), x) - \langle u^*(0,x), \varphi^0(x)\rangle]\mathrm{d}x,$$

where $u^*(\cdot,\cdot) \in C^2(\mathrm{R})$, $p^*(\cdot,\cdot) \in C(\mathrm{R})$. The following maximization problem under the indicated constraints of feasible solutions is just the desired dual problem:

$$(P_{CD}) \qquad \sup_{\substack{u^*(t,x), p^*(t,x), (t,x) \in \mathrm{R}, \\ u^*(t,0) = 0, u^*(t,1) = 0.}} J_*(u^*(t,x), p^*(t,x)).$$

The dual problem for hyperbolic multidimensional problem (P_M) ($\varphi_1(x) \equiv 0$, $x \in G$; $\alpha_0(t,x) \equiv 0$, $\alpha_1(t,x) \equiv 0, (t,x) \in B$) is defined similarly, where

$$(P_{\mathrm{MD}}) \qquad \sup_{\substack{u^*(t,x), p^*(t,x), \\ u^*(t,x) = 0, (t,x) \in B, u^*(1,x) = 0}} J_*(u^*(t,x), p^*(t,x))$$

and

$$J_*(u^*(t,x), p^*(t,x)) = \int_0^1 \int_G [M_F(u_{tt}^*(t,x) - L^* u^*(t,x) - p^*(t,x), u^*(t,x))$$
$$-g^*(p^*(t,x), t, x)]\mathrm{d}t\,\mathrm{d}x - \int_G [g_0^*(-u_t^*(1,x), x) - \langle u^*(0,x), \varphi^0(x)\rangle]\mathrm{d}x.$$

Here u^* and p^* are classical solutions; $u^*(\cdot,\cdot) \in C^2(D), p^*(\cdot,\cdot) \in C(D)$.

The duality theorems for problems (P_{CD}) and (P_{MD}) are true in the same formulations of the preceding sections.

Remark 6.26. Notice that results analogous to those that we have seen in Chapters 4–6 for different discrete and differential inclusions may be stated in the vector-optimization form with many cost functionals. To do this, it is necessary to introduce the notion of a weakly efficient solution for multicriteria optimization in more general form. Let $f=(f_1, \ldots, f_m)$ be a convex vector-function defined on $\mathbb{R}^n$; i.e., $f:\mathbb{R}^n \to \mathbb{R}^m$. We call x_0 a weakly efficient solution of P_M, the minimization problem of f over $\mathbb{R}^n$, if there are no $x \in \mathbb{R}^n$ such that $f(x) > f(x_0)$ [74,193]. Let f be continuous at some point x_1 and $\overline{M} = \left\{\mu = (\mu_1, \ldots, \mu_m) : \sum_{i=1}^m \mu_i = 1, \mu_i \geq 0\right\}$.

Lemma 6.4. In order for x_0 to be a weakly efficient solution, it is necessary and sufficient that there exists $\mu \in \overline{M}$ such that $0 \in \sum_{i=1}^m \mu_i \partial f_i(x_0)$.
□ By Hurwitz's theorem [74,193] for the weak efficiency of $x_0 \in \mathbb{R}^n$, it is necessary and sufficient that there is a vector $\mu \in \overline{M}$ for which

$$\langle \mu, f(x_0) \rangle = \min \langle \mu, f(x) \rangle.$$

It follows that $\langle \mu, \ f(x) - f(x_0) \rangle \geq 0$ or $\varphi(x) - \varphi(x_0) \geq 0$, where $\varphi(x) = \sum_{i=1}^m \mu_i f_i(x)$. Thus, $0 \in \partial\varphi(x_0)$. It remains to observe that since the condition of P_M (Theorem 1.29, Moreau–Rockafellar) is satisfied, $\partial\varphi(x_0) = \sum_{i=1}^m \mu_i \partial f_i(x_0)$. ■

The following result is immediate from Theorem 3.1 and Lemma 6.4.

Lemma 6.5. Let $Q \subset \mathbb{R}^n$ be a convex set and $x_1 \in Q$ be a point at which $f=(f_1, \ldots, f_m)$ is continuous. Then in order for $x_0 \in Q$ to be a weakly efficient solution, it is necessary and sufficient that there is $\mu \in \overline{M}$ such that

$$K_Q^*(x_0) \cap \sum_{i=1}^m \mu_i \partial f_i(x_0) \neq \varnothing.$$

Then in the corresponding convex minimization problems we should use Lemma 6.5. Therefore, in the adjoint differential inclusions for optimization problems, where the number of functionals is m; i.e., J_i, $i=1, \ldots, m$ (and consequently, g_i, $i=1, \ldots, m$) the subdifferential $\partial g(\tilde{x})$ is replaced by $\sum_{i=1}^m \mu_i \partial g_i(\tilde{x})$.

References

[1] R.P. Agarwal, D. O'Regan, Fixed-point theory for weakly sequentially upper-semicontinuous maps with applications to differential inclusions, Nonlinear Oscillat.. 5(3) (2002) 277–286.

[2] A.A. Agrachev, Y.L. Sachkov, Control Theory from the Geometric Viewpoint, Control Theory and Optimization II, Springer Verlag, Berlin, 2004.

[3] N.U. Ahmed, X. Xiang, Necessary conditions of optimality for differential inclusions in Banach spaces, Nonlinear Anal. 30 (1997) 5437–5445.

[4] O. Aleksandra, On solutions of the Dirichlet problem for a class of partial differential inclusions with superlinear nonlinearities, Numeric. Funct. Anal. and Optim. 23(33–34) (2002) 367–381.

[5] V.M. Alekseev, V.M. Tikhomirov, S.V. Fomin, Optimal Control, Consultants Bureau, New York, 1987.

[6] H. Amann, P. Quittner, Elliptic boundary value problems involving measures: existence, regularity and multiplicity, Adv. Differ. Equat. 3 (1998) 753–813.

[7] N. Arada, Minimax Dirichlet boundary control problem with state constraints, Nonlinear Anal. 46 (2001) 653–673.

[8] Z. Artstein, First order approximations for differential inclusions, Set-Valued Anal. 2 (1994) 7–17.

[9] Z. Artstein, A calculus for set-valued maps and set-valued evolution equations, Set-Valued Anal. 3 (1995) 213–261.

[10] A.V. Arutyunov, S.M. Aseev, V.I. Blagodatskikh, Necessary contitions of the first order in the problem of optimal control of a differential inclusion with phase constraints, Sb Math. 79 (1994) 117–139.

[11] S.M. Aseev, Smooth approximations of differential inclusions and time-optimality problem, Proc. Steklov Inst. Math. 200 (1991) 27–34.

[12] S.M. Aseev, A method of smooth apporximations in the theory of necessary optimality contitions for differential inclusions, Izvestia Math. 61 (1997) 235–258.

[13] J.P. Aubin, Boundary-value problems for systems of first order partial differential inclusions, Nonlinear Differ. Equat. Appl. 7(1) (2004) 67–90.

[14] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, MA, 1990.

[15] J.-P. Aubin, A. Cellina, Differential Inclusions, Springer, Berlin, 1984.

[16] J.P. Aubin, G. Da Prato, Stochastic viability and invariance, Ann. Scuola Norm. di Pisa 27 (1980) 595–694.

[17] J.P. Aubin, H. Frankowska, Set-valued solutions to the Cauchy problem for hyperbolic system of partial differential inclusion, Nonlinear Differ. Equat. Appl. 4 (1996) 149–168.

[18] J.-P. Aubin, H. Frankowska, On inverse function theorems for set-valued maps, J. Math. Pures Appl. 66 (1987) 71–89.

[19] J.-P. Aubin, I. Ekeland, Applied Nonlinear Analysis, Wiley, New York, 1984.

[20] J.P. Aubin, A. Cellina, Differential Inclusions, Springer Verlag, Grundlagen der Math., Wiss., 1984.

[21] J.P. Aubin, H. Frankowska, Hyperbolic systems of partial differential inclusion, Ann. Scuola Norm. di Pisa 18 (1991) 541–562.
[22] R.J. Aumann, Integrals of set-valued functions, J. Math. Anal. Appl. 12 (1965) 1–12.
[23] A. Auslender, J. Mechler, Second order viability problems for differential inclusions, J. Math. Anal. Appl. 181 (1994) 205–218.
[24] A.V. Balakrishnan, Optimal control problems in Banach spaces, SIAM J. Control Optim. 3 (1965) 152–180.
[25] A.V. Balakrishnan, Applied Functional Analysis, second ed., Springer, New York, 1981.
[26] S. Bandyopadhyay, A.C. Barroso, B. Dacorogna, J. Matias, Differential inclusions for differential forms, Calc. Var. Partial Diff. Equat. 28(4) (2007) 449–469.
[27] V. Barbu, Necessary conditions for distributed control problems governed by parabolic variational inequalities, SIAM J. Control Optim. 19 (1981) 64–86.
[28] V. Barbu, The time optimal control of variational inequalities, dynamic programming and the maximum principle, Recent Mathematical Methods in Dynamic Programming (Rome, 1984), Lecture Notes in Math, 1119, Springer-Verlag, Berlin, 1985, pp. 1–119.
[29] V. Barbu, I. Lasiechka, R. Triggiani, Tangential Boundary Stabilization of Navier–Stokes Equations, American Mathematical Society, Providence, RI, USA, 2006.
[30] R.J. Bellman, K.L. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.
[31] M. Benchohra, S.K. Ntouyas, On second order boundary value problems for functional differential inclusions in Banach spaces, Appl. Math. 28 (2001) 293–300.
[32] M. Benchohra, J. Henderson, S.K. Ntouyas, Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces, J. Math. Anal. Appl. 263 (2001) 763–780.
[33] M. Benchohra, J. Henderson, S.K. Ntouyas, On a boundary value problem for second order impulsive functional differential inclusions in Banach spaces, Nonlinear Differ. Equat. (2008)Vol. to appear.
[34] M. Benchohra, S.K. Ntouyas, Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces, J. Math. Anal. Appl. 258 (2001) 573–590.
[35] V.I. Blagodatskikh, The maximum principle for differential inclusions, Proc. Steklov Inst. Math. 166 (1984) 23–43.
[36] V.I. Blagodatskikh, A.F. Filippov, Differential inclusions and optimal control, Proc. Steklov Inst. Math. 169 (1986) 199–259.
[37] V.G. Boltyanskii, Optimal Control of Discrete Systems, Nauka, Moskow, 1973.
[38] V.G. Boltyanskij, Tent's method in theory of extremal problems, Uspekhi Matematicheskikh Nauk 30 (1975) 1–55.
[39] Y.G. Borisovich, B.D. Gel'man, A.D. Myshkis, V.V. Obukhovskii, Multivalued Mappings, Itogi Nauki i Tekniki: Math. Anal. 19 (1982) 127–230.
[40] J.M. Borwein, D.M. Zhuang, Verifiable necessary and sufficient conditions for openness and regularity of set-valued and single-valued maps, J. Math. Anal. Appl. 134 (1988) 441–459.
[41] J.M. Borwein, Q.J. Zhu, A survey of subdifferential calculus with applications, Nonlinear Anal. 38 (1999) 687–773.
[42] D. Borwien, J.M. Borwein, X. Wang, Approximate subgradients and coderivatives in Rn, Set-Valued Anal. 4 (1996) 375–398.
[43] J.M. Borwien, W.B. Moors, X. Wang, Generalized subdifferentials: a Baire categorical approach, Trans. Amer. Math. Soc. 353 (2001) 3875–3893.

[44] F. Bucci, A Dirichlet boundary control problem for the strongly damped wave equation, SIAM J. Control Optim. 30 (1992) 1092−1100.
[45] A.G. Butkovskii, Theory of Optimal Control by Systems with Distributed Parameters, "Nauka", Moscow, English trans. Distributed Control Systems, Amer. Elsevier, New York, 1965.
[46] L. Caffarelli, L. Nirenberg, J. Spruck, Nonlinear second-order elliptic equations, V. The Dirichlet problem for Weingarten hypersurfaces, Comm. Pure Appl. Math. 41 (1988).
[47] A. Cellina, On the set of solutions to Lipschitzian differential inclusions, Differ. Integ. Equat. 1 (1988) 459−500.
[48] A. Cernea, Some second-order necessary conditions for nonconvex hyperbolic differential inclusion problem, J. Math. Anal. Appl. 253 (2001) 616−639.
[49] A. Cernea, On the set of solution of some non-convex non-closed hyperbolic differential inclusions, Czechoslovak Math. J. 52 (2002) 215−224.
[50] F.H. Clarke, The Euler−Lagrange differential inclusion, J. Differ. Equat. 19 (1975) 80−90.
[51] F.H. Clarke, Optimal solutions to differential inclusions, J. Optim. Theor. Appl. 19 (1976) 469−478.
[52] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[53] F.H. Clarke, Hamiltonian analysis of the generalized problem of Bolza, Trans. Amer. Math. Soc. 301 (1987) 385−400.
[54] F.H. Clarke, Methods of Dynamic and Nonsmooth Optimization, SIAM Publications, Philadelphia, PA, 1989.
[55] F.H. Clarke, G.G. Watkins, Necessary conditions, controllability and the value function for differential-difference inclusions, Nonlinear Anal. 10 (1986) 1155−1179.
[56] F.H. Clarke, P.R. Wolenski, The sensitivity of optimal control problems to time delay, SIAM J. Control Optim. 29 (1991) 1176−1215.
[57] F.H. Clarke, P.R. Wolenski, Necessary conditions for functional differential inclusions, Appl. Math. Optim. 34 (1996) 34−51.
[58] F.H. Clarke, Y.S. Ledyaev, M.L. Radulescu, Approximate invariance and differential inclusions in Hilbert spaces, J. Dynam. Control Syst. 3(4) (1997) 493−518.
[59] F.H. Clarke, Y.S. Ledyaev, R.J. Stern, P.R. Wolenski, Qualitative proporties of trajectories of control systems: a survey, J. Dynam. Control Syst. 1 (1995) 1−47.
[60] F.H. Clarke, Y.S. Ledyaev, R.J. Stern, P.R. Wolenski, Nonsmooth Analysis and Control Theory, Springer, New York, 1998.
[61] V. Coverstone-Carrol, C.A. Hartman, A.L. Herman, D.B. Spencer, Optimal spacecraft trajectories via higher order differential inclusions, spaceflight mechanics, Proceeding of the AAS/AIAA Space Flight Mechanics Meeting, 7−10 February, Breckenridge, CO, Pt. 1, 1999, pp. 377−395.
[62] F.S. De Blasi, G. Pianigiani, V. Staciu, On the solution of some non-convex hyperbolic differential inclusions, Czechoslovak Math. J. 45 (1995) 107−116.
[63] M.D.R. De Pinco, M.M.A. Ferreira, F.A.C.C. Fontes, An Euler−Lagrange inclusion for optimal control problems with state constraints, J. Dynam. Control Syst. 8 (2002) 23−45.
[64] M.S. de Queiroz, Michael Malisoff, Peter Wolenski, Optimal Control, Stabilization and Nonsmooth Analysis, Springer, Berlin/Heidelberg, 2004.
[65] V.F. Demyanov, L.V. Vasilev, Nondifferentiable Optimization, Nauka, Moscow, 1981.
[66] V.F. Demyanov, A.M Rubinov, Constructive Nonsmooth Analysis, Pater Lang, Frankfurt, Germany, 1995.
[67] P. Diamond, V Opoitsev, Stability of a class of differential inclusions, Dynam. Control 11(3) (2001) 229−242.

[68] S. Domachawski, Boundary value problems for nonconvex differential inclusions, Math. Nachr. 239 (2002) 28−41.
[69] T. Donchev, Lower semicontinuous differential inclsions: one-sided Lipschitz approach, Coll. Math. 74 (1997) 177−184.
[70] T. Donchev, B.S. Mordukhovich, E. Farkhi, Discrete approximations, relaxation and optimization of one-sided Lipschitzian differential inclusions in Hilbert Spaces, J. Differential Equations 243 (2007), 301−328.
[71] A.D. Dontchev, E.M. Farkhi, Error estimates for discretized differential inclusions, Computing 41 (1989) 349−358.
[72] A.L. Dontchev, F. Lempio, Difference methods for differential inclusions: a survey, SIAM Rev. 34 (1992) 263−294.
[73] A.L. Dontchev, T. Zolezzi, Well-posed optimization problems, Springer, New York, 1993.
[74] J. Dutta, C. Tammer, Lagrangian conditions for vector optimization on Banach spaces, Mathematical Methods of Operations Research 3 (2006), 521−540.
[75] I. Ekaland, R. Temam, Analyse Convexe et Problemes Variationelles, Dunod and Gauthier Villars, Paris, 1974.
[76] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, North-Holland, Amsterdam, The Netherlands, 1976.
[77] M. Fabian, P.D. Loewen, B.S. Morudkhovich, Subdifferential calculus in Asplund generated spaces, J. Math. Anal. Appl. 322 (2005) 787−795.
[78] H.O. Fattorini, The maximum principle for control systems described by linear parabolic equations, J. Math. Anal. Appl. 259 (2001) 630−651.
[79] R.P. Fedorenko, Maximum principle for differential inclusions (necessity), USSR Comput. Math. Math. Phys. 11 (1970) 885−893.
[80] A.F. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Kluwer, Dordrecht, The Netherlands, 1988.
[81] E. Fornosini, G. Marchesini, Doubly indexed dynamical systems, Math. Syst. Theory 12(1) (1978).
[82] H. Frankowska, Necessary conditions for the Bolza problem, Math. Oper. Res. 10 (1985) 361−366.
[83] H. Frankowska, The maximum principle for an optimal control to a differential inclusion with endpoint constraints, SIAM J. Control Optim. 25 (1987) 145−157.
[84] H. Frankowska, A priori estimates for operational differential inclusions, J. Differ. Equat. 84 (1990) 100−128.
[85] H. Frankowska, Some inverse mappings theorems, Ann. Inst. H. Poincare Analyse Non Lineaire 7 (1990) 183−234.
[86] H. Frankowska, Lower semicontinuous solutions of Hamilton−Jacobi−Bellman equations, SIAM J. Control Optim. 31 (1993) 257−272.
[87] H. Frankowska, Optimal synthesis via superdifferentials of value functions, Control Cybernet 34(3) (2005).
[88] H. Frankowska, The maximum principle for a differential inclusion problem, Lect. Notes Control 62 (1984) 517−531.
[89] H. Frankowska, F. Rampazzo, Filippov's and Filippov−Wazewski's theorems on closed domains, J. Differ. Equat. 161 (2000) 449−478.
[90] H. Frankowska, M. Plaskacz, Semicontinuous solutions of Hamilton−Jacobi equations with state constrains, in: J. Andress, L. Gorniewicz, P. Nistri (Eds.), Differential Inclusion and Optimal Control, Lecture Notes in Nonlinear Analysis, vol. 2, J. Schauder Center for Nonlinear Studies, 1998, pp. 145−161.

[91] A. Friedman, Optimal control for hereditary processes, J. Math. Anal. Appl. 15 (1964) 396–414.

[92] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[93] M. Frigon, On a critical point theory for multivalued functionals and application to partial differential nonlinear analysis, Theory Method. Appl. 31(5–6) (1998) 735–753.

[94] R. Gabasov, F.M. Kirillova, The Maximum Principle in the Optimal Control Theory, Nauka i Tekhnika, Minsk, Belarus, 1974.

[95] R. Gabasov, F.M. Kirillova, Qualitative Theory of Optimal Processes, Marcel Dekker, New York, 1976.

[96] E. Gatsori, S.K. Ntouyas, Y.G. Sficas, On a nonlocal Cauchy problem for differential inclusions, Abstr.Appl. Anal. 5 (2004) 425–434.

[97] S.R. Grace, R.P. Agarwal, D. O'Regan, A selection of oscillation criteria for second order differential inclusions, Appl. Math. Lett. 22(2) (2008) 153–158.

[98] M.L. Gusakova, On necessary optimality conditions for systems of neutral type, J. Differ. Equat. 810 (1974) 1894–1897.

[99] J. Hale, Theory of Functional Differential Equations, Springer Verlag, New York, 1977.

[100] T. Haddad, M. Yarou, Existence of solutions for nonconvex second-order differential inclusions in the infinite dimensional space, Electron. J. Differ. Equat. 2006(33) (2006) 1–8.

[101] H. Halkin, A satisfactory treatment of equality and operator constraints in Dubovitskii–Milyutin optimizations formalism, J. Optim. Theory Appl. 6 (1970) 138–149.

[102] H. Halkin, Implicit functions and optimization problems without continuous differentiability of the data, SIAM J. Control Optim. 12 (1974) 229–236.

[103] H. Halkin, Necessary Conditions for Optimal Control Problems Differentiable and Nondifferentiable Data, Springer, Berlin, 1978.

[104] E. Hebey, Nonlinear Analysis on Manifold: Sobolev Spaces and Inequalities, Americian Mathematical society, Providence, RI, USA, 2000.

[105] J.-B. Hiriart-Urruty, C. Lemarechal, Convex Analysis and Minimization Algorithms I, Springer Verlag, Berlin/Heidelberg/New York, 1996.

[106] R. Horst, P.M. Pardalos, N.V. Thoani, Introduction to Global Optimization, Kluwer, Dordrecht, The Netherlands, 2000.

[107] A.G. Ibrahim, K.S. Alkulaibi, On existence of monotone solutions for secondorder nonconvex differential inclusions in infinite dimensional spaces, Portugaliae Mathematica 61(2) (2004) 231–248.

[108] A.D. Ioffe, Codirectional compactness, metric regularity and subdifferential calculus, in: M. Thera (Ed.), *Constructive, Experimental and Nonlinear Analysis*, Canad. Math. Soc. Conf. Proc., vol. 27, 2000, pp. 123–164.

[109] A.D. Ioffe, J.-P. Penot, Subdifferentials of performane functions and calculus of coderivatives of set-valued mappings, Serdica Math. J. 22 (1996) 359–384.

[110] A.D. Ioffe, V.M. Tikhomirov, Extensions of variational problems, Trans. Moscow Math. Soc. 18 (1968) 207–273.

[111] A.D. Ioffe, V.M. Tikhomirov, Theory of Extremal Problems, North-Holland, Amsterdam, The Netherlands, 1979.

[112] D. Jiang, D. O'Regan, R.P. Agarwal, Optimal existence theory for single and multiple positive periodic solitions to functional difference, Appl. Math. Comput. 161(2) (2005) 441–462.

[113] T. Kaczorek, Two Dimensional Linear Systems, Springer, Springer-Verlag, New York, 1985.
[114] W. Kaplan, Maxima and Minima with Applications Practical Optimization and Duality, Nauka, Moscow, 1999.
[115] G.L. Kharatishvili, The maximum principle in the theory of optimal processes with a delay, Sovieth Math. Dokl. 2 (1961) 28−32.
[116] G.L. Kharatishvili, T.A. Tadumadze, A nonlinear optimal control problem with variable delays, nonfixed initial moment, and piecewise continuous prohistory, Proc. Steklov Inst. Math. 220 (1998) 233−252.
[117] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer, Dordrecht, The Netherlands, 1991.
[118] V.B. Kolmanovskii, V.R. Nosov, Neutral-type systems with aftereffects, Autom. Remote Control 45 (1984) 1−28.
[119] V.B. Kolmanovskii, L.E. Shaikhet, Control of Systems with Aftereffect, Nauka, Moscow, 1996.
[120] N.N. Krasovskii, A.I. Subbotin, Game-Theoretical Control Problems, Springer, New York, 1988.
[121] A.Y. Kruger, Generalized differentials of nonsmooth functions and necessary conditions for an extremum, Siberian Math. J. 26 (1985) 370−379.
[122] A.V. Kryazhimskii, Convex optimizations via feedbacks, SIAM J. Control Optim. 37(1) (1998) 278−302 .
[123] A.V. Kryazhimskii, Y.S. Osipov, On evolutionary-differential games, Proc. Steklov Inst. Math. 211 (1995) 257−287.
[124] A. Kurzhanski, Set-Valued Analysis and Differential Inclusions, Springer Verlag, New York, LLC, 1993.
[125] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[126] I. Lasiecka, R. Triggiani, Dirichlet boundary control problems for parabolic equations with quadratic cost: analyticity and Riccati feedback synthesis, SIAM J. Control Optim. 21 (1983) 41−67.
[127] I. Lasiecka, R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control, I: Riccati's feedback synthesis and regularity of optimal solutions, Appl. Math. Optim. 21 (1987) 41−67.
[128] I. Lasiecka, R. Triggiani, Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions, II: General boundary data, J. Differ. Equat. 94 (1991) 112−164.
[129] P.J. Laurent, Approximation et Optimization, Hermann, Paris, 1972.
[130] F. Lempio, V.M. Veliov, Discrete approximations to differential inclusions, Mitteilungen der GAMM 21 (1998) 101−135.
[131] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, Berlin, 1991.
[132] P.D. Loewen, Optimal Control via Nonsmooth Analysis, American Mathematical Society, Providence, Rhode Islands, 1993.
[133] P.D. Loewen, R.T. Rockafellar, The adjoint arc in nonsmooth optimization, Trans. Am. Math. Soc. 325(1) (1991) 39−72.
[134] P.D. Loewen, R.T. Rockafellar, Optimal control of unbounded differential inclusions, SIAM J. Control Optim. 32 (1994) 442−470.
[135] P.D. Loewen, F.H. Clarke, R.B. Vinter, Differential inclusions with free time, Ann. Inst. H. Poincare Analyse Non Lineaire 5 (1988) 573−593.

[136] P.D. Loewen, R.T. Rockafellar, Bolza problems with general time constraits, SIAM J. Control Optim. 35(6) (1997) 2050−2069 .
[137] V. Lupulescu, A viability result for nonconvex second order differential inclusions, Electron. J. Differ. Equat. 76 (2002) 1−12.
[138] V. Lupulescu, Viable solutions for second order nonconvex functional differential inclusions, Electron. J. Differ. Equat. 110 (2005) 1−11.
[139] L.A. Lysternik, V.I. Sobolev, Elemente der Funktional Analysis, Akademie Verlag, Berlin, 1968.
[140] E.N. Mahmudov, B.N. Pshenichnyi, Necessary condition of extremum and evasion problem, Preprint, Institute Cybernetcis of Ukraine SSR, Kiev, 1978, pp. 3−22.
[141] E.N. Mahmudov, B.N. Pshenichnyi, Polyhedral mappings, Izv. Acad. Sci. Azerbaijan 2 (1979) 10−14.
[142] E.N. Mahmudov, Necessary conditions of an extremum for nonautonomous differential inclusions, Izv. Acad. Sci. Azerbaijan 6 (1979) 3−7.
[143] E.N. Mahmudov, Polyhedral differential inclusions and linear problem of optimal control on linear manifolds, Dep. VINITI, Moscow, 1979, No: 3121-79.
[144] E.N. Mahmudov, Duality in optimal control problems of optimal control described by convex discrete and differential inclusions with delay, Automat. Remote Control 48(2) (1987) 149−159.
[145] E.N. Mahmudov, Duality in problems from the theory of convex difference inclusions with aftereffect, Differ. Equat. 23(8) (1987) 886−893.
[146] E.N. Mahmudov, Duality in the problems of optimal-control for systems described by convex differential inclusions with delay, *Problems Control Informat. Theor.* (*Problemy Upravleniya i Teorii Informatsii*) 16(6) (1987) 411−422.
[147] E.N. Mahmudov, Duality in problems described by convex multivalued mappings, Cybernetics 91(2) (1987) 238−244.
[148] E.N. Mahmudov, On duality in problems of theory of convex difference inclusions with after effect, Differents Uravn. 23(8) (1987) 886−893.
[149] E.N. Mahmudov, A polyhedral optimization problem for polyhedral discrete and differential inclusions, and duality, Cybernetics 24(3) (1988) 324−336.
[150] E.N. Mahmudov, Necessary conditions for problems of optimal control that are describable by differential inclusions with distributed parameters, *Dokl. Akad. Nauk SSSR* (*translation in Soviet Math.Dokl.*) 303(1) (1988) 29−33.
[151] E.N. Mahmudov, Necessary and sufficient conditions for an extremum for discrete and differential inclusions with distributed parameters, Siberian. Mat. J. 30(2) (1989) 266−278.
[152] E.N. Mahmudov, Optimization of discrete inclusions with distributed parameters, Optimization 21(2) (1990) 197−207.
[153] E.N. Mahmudov, Some extremal problems for discrete and differential inclusions with distributed parameters of elliptic type, and duality, Cybernetics 26(2) (1990) 215−226.
[154] E.N. Mahmudov, Problems on the extremum for discrete and differential inclusions with distributed parameters of hyperbolic type, Ukrainian Math. J. 42(2) (1990) 1476−1483.
[155] E.N. Mahmudov, Sufficient conditions for optimality on a nonfixed time interval for differential inclusions with delay and duality, Differ. Equat. 27(6) (1991) 675−680.
[156] E.N. Mahmudov, A two-parameter optimal control problem for systems of discrete inclusions, Automat. Remote Control 52(3) (1991) 353−362.

[157] E.N. Mahmudov, Optimization of differential inclusions with a phase constraint, and duality, Mat. Notes 50(6) (1991) 1280−1287.
[158] E.N. Mahmudov, Optimization on a nonfixed time interval of differential inclusions with delay, *Soviet Math. J.* (Iz. VUZ) 35(11) (1991) 43−48.
[159] E.N. Mahmudov, B.N. Pshenichnyi, An optimality principle for discrete and differential inclusions with distributed parameters of parabolic type, and duality, Izv. Russian. Academy of Sciences Ser. Mat 57(2) (1994) 91−112.
[160] E.N. Mahmudov, Mathematical Analysis and Applications, Papatya, Istanbul, 2000.
[161] E.N. Mahmudov, G. Çiçek, Optimization of differential inclusions of Bolza type with state constarints and Duality, Math. J. 7(2) (2005) 21−38.
[162] E.N. Mahmudov, On duality in problems of optimal control described by convex differential inclusions of Goursat−Darboux type, J. Math. Anal. Appl. 307(2) (2005) 628−640.
[163] E.N. Mahmudov, The optimality principle for discrete and first order partial differential inclusions, J. Math. Anal. Appl. 308(2) (2005) 605−619.
[164] E.N. Mahmudov, Necessary and sufficient conditions for discrete and differential inclusions of elliptic type, J. Math. Anal. Appl. 323(2) (2006) 768−789.
[165] E.N. Mahmudov, Optimization of discrete and differential inclusions of Goursat−Darboux type with state constraints, Adv. Difference Equat. Art. ID 41962 (2006) 1−16.
[166] E.N. Mahmudov, Locally adjoint mappings and optimization of the first boundary value problem for hyperbolic type discrete and differential inclusions, Nonlinear Anal. 67(10) (2007) 2966−2981.
[167] E.N. Mahmudov, G. Çicek, Optimization of differential inclusions of Bolza type and dualities, Appl. Comput. Math. 6(1) (2007) 88−96.
[168] E.N. Mahmudov, Duality in the problems of optimal control described by first order partial differential inclusions, J. Math. Progr. Oper. Res. 59(4) (2008) 589−599.
[169] E.N. Mahmudov, Optimal control of higher order differential inclusions of Bolza type with varying time interval, Nonlinear Anal. 69(5−6) (2008) 1699−1709.
[170] E.N. Mahmudov, Sufficient conditions for optimality for differential inclusions of parabolic type and duality, J. Global Optim. 41(1) (2008) 31−42.
[171] E.N. Mahmudov, Optimal control of Cauchy problem for first order discrete and partial differential inclusions, J. Dynam. Control Syst. 15(4) (2009) 587−610.
[172] E.N. Mahmudov, Ö. Değer, Optimal control of the elliptic type differential inclusions with Dirichlet and Neumann boundary conditions, Journal of Dynamical and Control Systems 17(2) (2011) 163−185.
[173] E.N. Mahmudov, Ö. Değer, Optimal control of the elliptic type differential inclusions with Dirichlet and Neumann boundary conditions, J. Dyn. Control Syst. 17(2) (2011) 163−185.
[174] V.L. Makarov and A.M. Rubinov, The Mathematical Theory of Economic Dynamics and Equilibrium, Nauka, Moscow, 1973, English transl., Springer Verlag, Berlin, (1977).
[175] L.M. Madan, Matrix Theory: Selected Topics and Useful Results, Delhi, (1997).
[176] M. Marcus, K. Minc, A Survey of Matrix Theory and Matrix Inequalities, Dover Publications, New York, (1992).
[177] L. Marco, J.A. Murillo, Lyapunov functions for second-order differential inclusions: a viability approach, J. Math. Anal. Appl. 262(1) (2001) 339−354.
[178] N.G. Medhin, On optimal control of functional-differential systems, J. Optimization Theory Appl. 85 (1995) 363−376.

[179] I.V. Melnikova, The Cauchy problem for a differential inclusion in Banach spaces and distribution spaces, Siberian Math. J. 42(4) (2001) 751−765.
[180] V.P. Mikhailov, Partial Differantial Equations, Nauka, Moscow, 1976.
[181] A.A. Milyutin, Convex-valued Lipschitz differential inclusions and Pntryagin's maximum pricniple, in: R.V. Gamkrelidge (Ed.), Optimal Control, vol. 4, 1999, pp. 175−187.
[182] L.I. Minchenko, Necessary optimality conditions for differential-difference inclusions, Nonlinear Anal. 35 (1999) 307−322.
[183] L.I. Minchenko, A.A. Volosevich, Value function and necessary optimality conditions in optimal control porblems for differential-difference inclusions, Nonlinear Anal. 53 (2003) 407−424.
[184] B.S. Mordukhovich, Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems, Soviet Math. Dokl. 27 (1980) 526−530.
[185] B.S. Mordukhovich, On the duality principle in the controllability and observability theory for functional differential equations of neutral type, Dokl. Akad. Nauk BSSR (1983) 110−113.
[186] B.S. Mordukhovich, Duality theory in systmes with aftereffect, J. Appl. Math. Mech. 48 (1984) 440−447.
[187] B.S. Mordukhovich, Approximation and optimization of differential inclusions, Cybernetics 24 (1988) 781−788.
[188] B.S. Mordukhovich, Approximation Methods in Problems of Optimization and Control, Nauka, Moscow, 1988.
[189] B.S. Mordukhovich, Maximum Principle for Nonconvex Finite Difference Systems, Springer, Berlin, 1990.
[190] B.S. Mordukhovich, On variational analysis of differential inclusions, in: A. Ioffe, L. Marcus, S. Reich (Eds.), Optimization and Nonlinear Analysis, Pitman Research Notes Math. Ser., vol. 244, 1992, pp. 199−213.
[191] B.S. Mordukhovich, Sensitivity analysis for constraint and variational systems by means of set-valued differentiation, Optimization 31 (1994) 13−46.
[192] B.S. Mordukhovich, Discrete approximations and refined Euler−Lagrange conditions for nonconvex differential inclusions, SIAM J. Control Optim. 33 (1995) 882−915.
[193] B.S. Mordukhovich, Optimization and Finite Difference Approximations of Nonconvex Differential Inclusions with Free Time, Springer, New York, 1996.
[194] B.S. Mordukhovich, Coderivatives of set-valued mappings: calculus and applications, Nonlinear Anal. 30 (1997) 3059−3070.
[195] B.S. Mordukhovich, Optimal control of nonconvex discrete and differential inclusions, Sociedad Matematica Mexicana, Mexico, 1998.
[196] B.S. Mordukhovich, Optimal control of discrete, differential, and delay-differential inclusions, Surveys in Modern Mathematics and its Applications, Nonsmooth Analysis and Optimization, 61, VINITI, Moscow, 1999, pp. 33−65
[197] B.S. Mordukhovich, Optimal control of difference, differential, and differential-difference inclusions, J. Math. Sci. 100 (2000) 2612−2632.
[198] B.S. Mordukhovich, Coderivative calculus and robust Lipschitzian stability for variational systems, J. Convex Anal. 13 (2006) 799−822.
[199] B.S. Mordukhovich, A.M. Sasonkin, Duality and optimality conditions in control problems for functional-differential systems of neutral-type, J. Differ. Equat. 21 (1985) 532−540.
[200] B.S. Mordukhovich, D. Wang, Optimal control of semilinear evolutions inclusions via discrete approximations, Control Cybernet 34 (2005) 849−870.

[201] B.S. Mordukhovich, D. Wang, Optimal control of semilinear unbounded differential inclusions, Nonlinear Anal. 63 (2005) 847−853.
[202] B.S. Mordukhovich, J.V. Outrata, Second-order subdifferentials and their applications, SIAM J. Optim. 12 (2001) 139−169.
[203] B.S. Mordukhovich, J.-P. Raymond, Neumann boundary control of hyperbolic equations with pointwise statae constraints, SIAM J. Control Optim. 43 (2004) 1354−1372.
[204] B.S. Mordukhovich, L. Wang, Optimal control of hereditary differential inclusions, in: Proc. 41st IEEE Conference on Decisive Control, vols. 1107−1112, 2002.
[205] B.S. Mordukhovich, L. Wang, Optimal control of constrained delay-differential inclusions with multivalued initial conditions, Control Cybernet 32 (2003) 585−609.
[206] B.S. Mordukhovich, L. Wang, Optimal control of differential-algebraic inclusions, in: M. De Queiroz, M. Malisoff and P. Wolenski (Eds.), Optimal Control, Stabilization and Nonsmooth Analysis, Lecture Notes Cont. Inf. Sci. 301, 2004, pp. 73−83.
[207] B.S. Mordukhovich, L. Wang, Optimal control of neutral functional-differential inclusions, SIAM J. Control Optim. 43 (2004) 111−136.
[208] B.S. Mordukhovich, L. Wang, Optimal control of nonautonomous functional-differential inclusions of neutral type, Nonlinear Anal. 63(5−7) (2005) 840−846.
[209] B.S. Mordukhovich, N.M. Nam, Variational stability and margian functions via generalized differentiation, Math. Oper. Res. 30 (2005) 1−18.
[210] B.S. Mordukhovich, R. Trubnik, Stability of discrete approximations and necessary optimality conditions for delay-differential inclusions, Ann. Oper. Res. 101 (2001) 149−170.
[211] B.S. Mordukhovich, I.A. Shvartsman, Discrete maximum principle for nonsmooth optimal control problems with delays, Cybernet. Systems Anal. 38 (2002) 255−264.
[212] B.S. Mordukhovich, K. Zhang, Minimax control of parabolic systems with Dirichlet boundary conditions and state constraints, Appl. Math. Optim. 5(1) (1997) 323−360.
[213] B.S. Mordukhovich, L. Wang, Optimal control of neutral functional-differential inclusions linear in velocities, TEMA Tend. Mat. Appl. Comput. 22 (2004) 1−15.
[214] B.S. Mordukhovich, Variational Analysis and Generalised Differentiation, vols. I and II, Springer, Springer-Verlag Berlin Heidelberg, 2006.
[215] J. Munoz, P. Pedregal, Explicit solutions of nonconvex variational problems in dimension one, Appl. Math. Optim. 41 (2000) 129−140.
[216] K. Nitka-Styczen, Optimal Periodic Control of Hereditary Processes. Theory, Algorithms and Applications, Technical University of Wroclaw, Wroclaw, Poland, 1999.
[217] O. Olech, By existence theory in optimal control problems, in: Control Theory and Topics in Functional Analysis, Vol. 1, 1976, pp. 291−328.
[218] J. Outrata, B.S. Mordukhovich, Coderivative analysis of quasivariational inequalities with applications to stability and optimization, SIAM J. Optim. 18 (2007) 389−412.
[219] N.S. Papageorgiou, Existence of solutions for hyperbolic differential inclusions in Banach spaces, Arch. Math. (Brno) 28 (1992) 205−213.
[220] N.S. Papageorgiou, N. Shahzad, Properties of the solution set of nonlinear evolution inclusions, Appl. Math. Optim. 36 (1997) 1−20.
[221] E.S. Polovinkin, G.V. Smirnov, An approach to differentiation of multifunctions and necessary optimality conditions for differential inclusions, J. Differ. Equat. 22 (1986) 660−668.
[222] E.S. Polovinkin, G.V. Smirnov, Time-optimal problem for differential inclusions, J. Differ. Equat. 22 (1986) 940−952.

[223] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, John Wiley & Sons Inc., New York, 1964.
[224] B.N. Pshenichnyi, Necessary conditions for an extremum for differential inclusions, Kibernetika 12 (1976) 60−73.
[225] B.N. Pshenichnyi, On necessary extremality conditions for nonsmooth functions, Kibernetika 13 (1977) 92−96.
[226] B.N. Pshenichnyi, Convex Analysis and Extremal Problems, Nauka, Moscow, 1980.
[227] L. Richard, C.H. Yung, Optimality conditions and duality for a non-linear time delay control problem, Optim. Control Appl. Method. 18(5) (1997) 327−340.
[228] R.T. Rockafellar, Convex Analysis, Second Printing, Princeton University, Princeton, New Jersey, 1972.
[229] R.T. Rockafellar, Generalized directional derivatives and subgradients of nonconvex functions, Canad. J. Math. 32 (1980) 157−180.
[230] R.T. Rockafellar, Second-order convex analysis, J. Nonlinear Convex Anal. 1 (2000) 1−16.
[231] R.T. Rockafellar, Variational Analysis and its Applications, Preface to a special issue of Set-Valued Anal. 12 (2004) 1−4.
[232] R.T. Rockafellar, D. Zagrodny, A derivative−coderivative inclusion in second-order nonsmooth analysis, Set-Valued Anal. 5 (1997) 1−17.
[233] R.T. Rockafellar, P.R. Wolenski, Convexity in Hamilton Jocobi theory, I. Dynamics and duality, SIAM J. Control Optim. 39 (2000) 1323−1350.
[234] R.T. Rockafellar, P.R. Wolenski, Convexity in Hamilton Jocobi theory, II. Envelope representataions, SIAM J. Control Optim. 39 (2000) 1351−1372.
[235] J.D.L Rowland, R.B. Vinter, Dynamic optimization problems with free time and active state constraints, SIAM J. Control Optim. 31 (1993) 677−691.
[236] A.M. Rubinov, X. Yang, Lagrange-Type Functions in Constrained Non-convex Optimization, Kluwer Academic Publishers, Norwell, 2003.
[237] A.M. Rubinov, Superlinear Multivalued Mappings and Their Applications to Problems in Mathematical Economics, Nauka, Leningrad, 1980.
[238] A.M. Rubinov, Abstract Convexity and Global Optimization, Kluwer, Dordrecht, The Netherlands, 2000.
[239] G.V. Smirnov, Discrete approximations and optimal solutions to differential inclusions, Cybernetics 27 (1991) 101−107.
[240] G.V. Smirnov, Introduction to the Theory of Differential Inclusions, American Mathematical Society, Providence, Rhode Islands, 2001.
[241] V.A. Srochko, Optimality conditions of the maximum principle type in Goursat−Darboux systems, Siberian J. Math. 25 (1984) 126−132.
[242] A.I. Subbotin, Generalized Solutions of First-Order PDEs, Birkhauser, Boston, MA, 1995.
[243] Ha. Xuan DucTruong, Some Variants of the Ekeland Variational Principle for a Set-Valued Map, J. Optim. Theory Appl. 124(1) (2005) 187−206.
[244] D. Tiba, Optimality conditions for distributed control problems with nonlinear state equation, SIAM J. Control Optim. 23 (1985) 85−110.
[245] V.M. Tikhomirov, Elements of the Theory of Extrema, Tinbergen Institute, TI 97-048/4, Amsterdam, The Netherlands, 1997.
[246] A.N. Tikhonov, A.A. Samarskii, The Equations of Mathematical Physics (third ed.), Nauka, Moscow (1966) English transl. of second ed., vols. 1, 2, Holden–Day, San Francisco, CA (1964) 1967.

[247] A.A. Tolstonogov, Differential Inclusions in a Banach Spaces, Kluwer, Dordrecht, The Netherlands, 2000.
[248] R. Trigiani, I. Lasiecka, Further results on exact controllability of the Euler−Bernoulli equation with control in the Dirichlet/Neumann boundary conditions, Stabilization of Flexible Structures Vol. 147/1990 (1990) 226−234.
[249] R. Trigiani, I. Lasiecka, Control Theory for Partial Differential Equations: Continuous and Approximation Theories I, II, Cambridge University Press, New York, 2000.
[250] H.D. Tuan, Continget and intermediate tangent cones in hyperbolic differential inclusions and necessary optimality conditions, J. Math. Anal. Appl. 185 (1994) 86−106.
[251] H.D. Tuan, On controllability and extremality in nonconvex differential inclusions, J. Optim. Theory Appl. 85 (1995) 435−472.
[252] H.D. Tuan, On solution sets of nonconvex Darboux problems and applications to optimal control with endpoint constraints, J. Austral. Math. Soc. Ser. B 37 (1996) 354−391.
[253] R. Vinter, H.H. Zheng, Necessary conditions for free end-time measurably time dependent optimal control problems with state constraints, Set-Valued Anal. 8 (2000) 11−29.
[254] R. Vinter, Optimal Control, Birkhäuser, Boston, MA, 2000.
[255] R.B. Vinter, H. Zheng, Necessary conditions for optimal control problems with state constraints, Trans. Am. Math. Soc. 350(3) (1998) 1181−1204 .
[256] L. Wang, Discrete approximations to optimization of neutral functional differential inclusions, J. Math. Anal. Appl. 309 (2005) 474−488.
[257] L. Wang, Discrete approximations and necessary optimality conditions for functional-differential inclusions of neutral type, Research Report #1, in: Proc. 43rd CDC, vol. 1, 2004, p. 8.
[258] L. Wang, Optimal control of neutral functional-differential inclusions, SIAM J. Control Optim. 43 (2004) 111−136.
[259] J. Warga, Necessary optimality conditions without differentiability assumptions in optimal control, J. Differ. Equat. 15 (1975) 41−62.
[260] K. Wilfred, Maxima and Minima with Applications, Practical Optimization and Duality, John Wiley & Sons, Inc., New York, 1999.
[261] P.R. Wolenski, The exponential formula for the reachable set of a Lipschitz differential inclusion, SIAM J. Control Optim. 28 (1990) 1148−1161.
[262] H.H. Zheng, Second-order necessary conditions for differential inclusion problems, Appl. Math. Optim. 30(1) (1994) 1−14.
[263] H.H. Zheng, Necessary conditions for optimal control problems with state constraints, Trans. Am. Math. Soc. 350 (1998) 1181−2004.
[264] H.H. Zheng, Necessary conditions for free end-time measurably time dependent optimal control problems with state constraints, Set-Valued Anal. 8 (2000) 11−29.
[265] X.Y. Zheng, K.F. Ng, The Fermat rule for multifunctions in Banach spaces, Math. Progr. 104 (2005) 69−90.
[266] W.-X. Zhong, Duality system in applied mechanics and optimal control, Adv. Mech. Math. 5 (1992).
[267] Q.J. Zhu, Necessary optimality conditions for nonconvex differential inclusion with endpoint constraints, J. Differ. Equat. 124 (1996) 186−204.

Glossary of Notations

Operations and Symbols

$\forall$	universal quantifier, "for every"
$\Rightarrow$	sign of implication, "... implies ..."
$\square$	proof beginning
$\blacksquare$	end of proof
$\langle x, x^* \rangle$	scalar(or inner) product of elements $x \in X$, $x^* \in X^*$
$:\equiv$	identically equal
$:=$	equal by definition
$\\| \cdot \\|$	Norm
lim inf (lim sup)	lower (upper) limit
HODI	higher-order differential inclusion
HOADI	higher-order adjoint differential inclusion
DSI and DFI	discrete and differential inclusions, respectively
A^*	adjoint (transposed) matrix of A
Δ	Laplace's operator
$\square$	difference of second-order partial derivatives with respect to t and x, respectively
L	second-order elliptic operator
$\\| a_{ij}(x) \\|$	positively definite matrix
L_s	sth order differential expression
L_s^*	sth order adjoint differential expression
$y^J(x)$	vertex defined by the index set $J \subset I = \{1, \ldots, m\}$ the number of which exactly is n.
L^*	operator adjoint to elliptic operator L.
$\tilde{A}$	difference operator defined on the three-point models
$\sum_{[0,1]}(0, N_0)$	family of solutions to ordinary differential inclusion (CP) with the initial condition $x(0) \in N_0$ defined on [0,1]
$S(\tau, 0, N_0)$	section of family of solutions
div $w^*(x)$	divergence operation

Spaces

$\mathbb{R} = (-\infty, \infty)$	real line
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}_+^m$	positive orthant of $\mathbb{R}^m$
ℓ_2	infinity dimensional coordinate-wise Hilbert space

$C(\overline{G})$ and $C^1(\overline{G})$	spaces of continuous functions and functions having a continuous derivative in G, respectively
$C^2(D)$	space of functions $u(\cdot,\cdot)$ having second-order continuous derivatives
$C^{1,2}(D)$	space of functions u, having first- and second-order continuous derivatives in first and second arguments, respectively
$H^1(G)$	Hilbert space consisting of the elements $u(x) \in L_2(G)$ having square-integrable generalized derivatives on G
$H^{2,1}(D)$	Hilbert space of functions having generalized derivatives on D with corresponding inner product
$W^n_{1,s}([t_0, t_1])$	Banach space, endowed with the corresponding norms
$W^{1,1}(Q)$	Sobolev space

Sets

conv M	convex hull of a set M
Lin M	linear hull for a set M
Aff M	affine hull of a set M
int M and ri M	interior and relative interior of a set M, respectively
$\overline{M}$	closure of M
$0^+ M$	recession cone of a set M
M^{O}	polar of a set M
K^*	dual cone to the cone K
cone M	cone generated by M
$P(Y)$	the family of all subsets of Y
$K^*_{\mathrm{gph}\ F}(x, y)$	dual to cone $K_{\mathrm{gph}\ F}(x,y) = \mathrm{cone}(\mathrm{gph}\ F - z)$
$K_A(x)$	cone of tangent directions of the set A at a point $x \in A$
D	nondegeneracy domain of polytope $F(x)$

Functions

$f : X \to Y$	single-valued mapping from X to Y
$r_M(x)$	Minkovski's (or gauge) function
$f_1 \oplus f_2$	infimal convolution of convex functions f_1, f_2
f^*	conjugate of function f
epi f	epigraph of f
dom f	domain of f
f_0^+	recession function of f
$H_M(\cdot)$	support function of a nonempty set M
$\mathrm{conv}\{f_i : i \in I\}$	convex hull of the pointwise infimum of the collection I
$\bar{f}$	closure of f
$f'(x_0,p)$	directional derivative of f at x_0 with respect to a vector $p \in X = \mathbb{R}^n$
$\partial f(x_0)$	subdifferential of f at x_0

$\delta_A(\cdot)$	indicator function of a set A
$\rho(A,B)$	Hausdorff metric between A and B
CUA $h(\bar{x},x)$	convex upper approximation of f at a point x
$H_F(x,y^*)$	Hamiltonian function of F
$L(x,y^*)$	Lagrangian of convex programming

Mappings

$F : X \to P(Y)$	multivalued (set-valued) mappings from X to Y
F^*	adjoint multivalued mapping (AM) from Y^* to X^*
$F^*(y^*;z)$	locally adjoint mapping (LAM) to F at a point $z \in \mathrm{gph}\, F$
gph F	graph of F
dom F	domain of F
$F^{-1}: Y \to P(X)$	inverse multivalued mapping to F
$F_1 + F_2$	sum of multivalued mappings of F_1,F_2
$F_1 \times F_2$	cartesian product of multivalued mappings of F_1 and F_2
$F_1 \circ F_0$	composition multivalued mapping of F_0 and F_1
$F(x;y^*)$	argmaximum set of $F(x)$ at a given y^*
$F^*(y^*,x)$	adjoint mapping (AM) to F at a point $x \in \mathrm{dom}\, F$